East Asian Math. J. **23** (2007), No. 1, pp. 111–122

DECOMPOSITION SERIES AND SUPRATOPOLOGICAL SERIES OF NEIGHBORHOOD SPACES

Sang Ho Park

Abstract. In this paper, we will show some relations between decomposition series $\{\nu^\alpha : \alpha \text{ is an ordinal }\}$ and supratopological series $\{\sigma_\alpha \nu : \alpha \text{ is an ordinal }\}$ for a neighborhood structure ν and the formular $\sigma_\alpha \nu = \nu^{(\omega^\alpha)}$, where ω is the first limit ordinal.

1. Introduction and Preliminaries

A convergence structure [1] is a correspondence between the filters on a given set X and the subsets of X which specifies which filters converge to which points of X. Also, for a given convergence structure q on a set X, [3] introduced the associated decomposition series $\{\pi^\alpha q : \alpha \text{ is an ordinal }\}$.

A *supratopology*, [5], is defined to be a collection of subsets of a set X (called *supraopen sets*) which contains X and is closed under arbitrary unions, but, unlike a topology, is not required to be closed under finite intersections. If (X, τ) is a topological space the collections of semi-open, preopen, and semi-preopen sets relative to τ each form supratopologies on X derived from τ.

Received February 28, 2007. Revised May 7, 2007.

2000 Mathematics Subject Classification: 54A05, 54A10, 54A20.

Key words and phrases: p-stack, neighborhood structure (space), supratopological neighborhood structure (space), decomposition series, supratopological series, neighborhood quotient maps.

In 2002, Kent and Min [4] defined “neighborhood p-stack $\nu(x)$”, “neighborhood structure ν” on X, which $\nu(x) \leq \dot{x}$ for all $x \in X$ and the pair (X, ν) as a “neighborhood space”, where $\dot{x}$ denotes the fixed ultrafilter generated by $\{x\}$.

In order to develop the theory of neighborhood spaces, it is necessary to introduce a new vehicle “p-stack” for describing convergence. Given a set X, a collection $\mathcal{C}$ of subsets of X is called a *stack* if $A \in \mathcal{C}$ whenever $B \in \mathcal{C}$ and $B \subseteq A$; a stack which is closed under finite intersections and does not contain the empty set is called a *filter*. The concept that we need is intermediate in generality between a stack and a filter.

Definition 1.1. ([4]). A stack $\mathcal{H}$ on a set X is called a *p-stack* if it satisfies the following condition:

$$(p) \quad A, B \in \mathcal{H} \ \textit{implies} \ A \cap B \neq \emptyset.$$

Condition (p) is called the *pairwise intersection property* (P.I.P.) which is strictly weaker than the well-known *finite intersection property* (F.I.P.). A collection $\mathcal{B}$ of subsets of X with the P.I.P. is called a *p-stack base*. For any collection $\mathcal{B}$, we denote by $\langle \mathcal{B} \rangle = \{A \subseteq X : \exists B \in \mathcal{B}$ such that $B \subseteq A\}$ the stack generated by $\mathcal{B}$, and if $\mathcal{B}$ is p-stack base, then $\langle \mathcal{B} \rangle$ is a p-stack. If $\mathcal{B}$ is a p-stack base with the F.I.P., then $\mathcal{B}$ is a *filter subbase*, and in this case $\mathcal{B}$ generates the filter $[\mathcal{B}] = \{A \subseteq X : \exists B_1, \cdots, B_n \in \mathcal{B}$ such that $\cap_{i=1}^{n} B_i \subseteq A\}$.

Let $pS(X)$ (respectively, $S(X)$ $F(X)$) denote the set of all p-stacks (respectively, stacks, filters) on X, partially ordered by inclusion. The maximal elements in $pS(X)$ (respectively, $F(X)$) are called *ultrapstacks* (respectively, *ultrafilters*). One may easily verify that every ultrafilter is an ultrapstack, and (via Zorn's Lemma) that every p-stack (respectively, filter) is contained in an ultrapstack (respectively, ultrafilter).

Definition 1.2.. ([4]). Let X be a set, and let $\nu \subseteq pS(X)$ be given by $\nu = \{\nu(x) : x \in X\}$, where $\nu(x) \subseteq \dot{x}$ for all $x \in X$ and $\dot{x}$ is the ultrapstack containing $\{x\}$. Then ν is called a *neighborhood structure* on X, $\nu(x)$ is called the *ν-neighborhood stack* at x, and (X, ν) is called a *neighborhood space.* A p-stack $\mathcal{H}$ on X ν-converges to x (written $\mathcal{H} \xrightarrow{\nu} x$) if $\nu(x) \subseteq \mathcal{H}$. For convenience, "neighborhood" will be henceforth abbreviated by "nbd.".

Let $N(X)$ be the set of all nbd. structures on X, partially ordered as follows: $\nu \leq \mu \Leftrightarrow \nu(x) \subseteq \mu(x)$, (denoted by $\nu(x) \leq \mu(x)$), for all $x \in X$ (in which case ν is *coarser* than μ and μ is *finer* than ν). Then $N(X)$ is a complete lattice [4].

Proposition 1.3. ([4]). *If $\mathcal{A} = \{\nu_i : i \in J\} \subseteq N(X), \nu = \inf_{N(X)} \mathcal{A}$, and $\mu = \sup_{N(X)} \mathcal{A}$, then $\nu(x) = \cap\{\nu_i(x) : i \in J\}$ and $\mu(x) = \cup\{\nu_i(x) : i \in J\}$.*

If (X, ν) is a nbd. space and $A \subseteq X$, let

$$I_\nu(A) = \{x \in A : A \in \nu(x)\};$$
$$Cl_\nu(A) = \{x \in X : A \cap V \neq \emptyset, \ forall \ V \in \nu(x)\}.$$

Proposition 1.4. ([4]). *If (X, ν) is a nbd. space and $A \subseteq X$, then:*

(1) $I_\nu(A) = \{x \in A : A \in \mathcal{H}$, for every p-stack $\mathcal{H} \xrightarrow{\nu} x\}$;

(2) $Cl_\nu(A) = X \setminus I_\nu(X \setminus A)$;

(3) $Cl_\nu(A) = \{x \in X : \exists \mathcal{H} \in pS(X)$ such that $\mathcal{H} \xrightarrow{\nu} x$ and $A \in \mathcal{H}\}$.

In 1997 and 1999, [6] and [7] showed comparing properties of *decomposition series* with those of the *topological series*, which is related in convergence space. In this paper, we shall change some results of [2], [3], [6] and [7] related to convergence spaces to nbd. spaces and other sources using more modern notation and terminology. We are mainly interested in changing such properties to nbd. spaces.

2. Decomposition Series in $N(X)$

Let (X,ν) be a nbd. space. Given an ordinal number $\alpha \geq 1$, let I_ν^α and Cl_ν^α denote the αth iterations of interior operator and closure operator for ν, respectively. For $A \subseteq X$, we inductively define:

$$I_\nu^\alpha(A) = \begin{cases} I_\nu(I_\nu^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists;} \\ \cap_{\beta<\alpha} I_\nu^\beta(A), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$
$$Cl_\nu^\alpha(A) = \begin{cases} Cl_\nu(Cl_\nu^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists;} \\ \cup_{\beta<\alpha}(Cl_\nu^\beta(A)), & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$$

where $I_\nu^0(A) = A$ and $Cl_\nu^0(A) = A$.

For any ordinal α and $\mathcal{G} \in pS(X)$, we define the αth ν nbd. p-stack $\nu^\alpha(\mathcal{G})$ and the αth closure p-stack $Cl_\nu^\alpha(\mathcal{G})$, respectively, as follows:

$$\nu^\alpha(\mathcal{G}) = \{A \subseteq X : I_\nu^\alpha(A) \in \mathcal{G}\};$$
$$Cl_\nu^\alpha(\mathcal{G}) = \langle\{Cl_\nu^\alpha(G) : G \in \mathcal{G}\}\rangle.$$

While, we know that $\nu(\dot{x}) = \nu(x)$ and if $\alpha < \beta$, then $\nu^\beta(\mathcal{G}) \leq \nu^\alpha(\mathcal{G}) \leq \mathcal{G}$.

Let $\{\mathcal{H}_i : i \in J\} \subseteq pS(X)$. Then we can show that $\cap_{i\in J}\mathcal{H}_i \in pS(X)$ and $\nu(\cap_{i\in J}\mathcal{H}_i) = \cap_{i\in J}\nu(\mathcal{H}_i)$. Also, if $\mathcal{H}_i$'s are not disjoint, then $\cup_{i\in J}\mathcal{H}_i \in pS(X)$ and $\nu(\cup_{i\in J}\mathcal{H}_i) = \cup_{i\in J}\nu(\mathcal{H}_i)$.

Also we know that if $\alpha < \beta$, then $\nu^\beta(\mathcal{G}) \leq \nu^\alpha(\mathcal{G}) \leq \mathcal{G} \leq I_\nu^\alpha(\mathcal{G}) \leq I_\nu^\beta(\mathcal{G})$.

Proposition 2.1. ([4]). *For every ordinal α and $A \subseteq X$, $X \setminus Cl_\nu^\alpha(A) = I_\nu^\alpha(X \setminus A)$.*

If (X,ν) is a nbd. space and $\alpha \geq 1$, let ν^α be the nbd. structure on X defined by $A \in \nu^\alpha(x) \iff x \in I_\nu^\alpha(A)$. Then we know that $\beta < \alpha$ implies $I_\nu^\alpha(A) \subseteq I_\nu^\beta(A)$, so $\nu^\alpha \leq \nu^\beta$.

PROPOSITION 2.2. *For any ordinals* α, β, $x \in X$ *and* $A \subseteq X$,
(1) $I_\nu^{\alpha+\beta}(A) = I_\nu^\beta(I_\nu^\alpha(A))$;
(2) $\nu^{\alpha+\beta}(x) = \nu^\alpha(\nu^\beta(x))$.

Proof. (1) Let α be a fixed ordinal. We use transfinite induction on β. If $\beta = 1$, $I_\nu^{\alpha+1} = I_\nu(I_\nu^\alpha(A))$ follows by definition. Next, let β be an arbitrary ordinal.

Case 1. $\exists \beta'$ such that $\beta' + 1 = \beta$. By the induction hypothesis $I_\nu^{\alpha+\beta'}(A) = I_\nu^{\beta'}(I_\nu^\alpha(A))$, and so

$$I_\nu^{\alpha+\beta}(A) = I_\nu^{\alpha+\beta'+1}(A) = I_\nu(I_\nu^{\alpha+\beta'}(A))$$
$$= I_\nu(I_\nu^{\beta'}(I_\nu^\alpha(A))) = I_\nu^{\beta'+1}(I_\nu^\alpha(A))) = I_\nu^\beta(I_\nu^\alpha(A)).$$

Case 2. β is a limit ordinal.

$$I_\nu^{\alpha+\beta}(A) = \cap_{\gamma<\beta} I_\nu^{\alpha+\gamma}(A) = \cap_{\gamma<\beta} I_\nu^\gamma(I_\nu^\alpha(A)) = I_\nu^\beta(I_\nu^\alpha(A)).$$

(2) $A \in \nu^{\alpha+\beta}(x) \iff x \in I_\nu^{\alpha+\beta}(A) \iff x \in I_\nu^\beta(I_\nu^\alpha(A)) \iff I_\nu^\alpha(A) \in \nu^\beta(x) \iff A \in \nu^\alpha(\nu^\beta(x))$. □

COROLLARY 2.3. *For any ordinals* α, β, *and* $\mathcal{F} \in S(X)$,

$$\nu^{\alpha+\beta}(\mathcal{F}) = \nu^\alpha(\nu^\beta(\mathcal{F})).$$

DEFINITION 2.4. *A descending chain* $\{\nu^\alpha : \alpha \geq 1\}$ of nbd. structures on X is called *a decomposition series* of (X, ν), where $\nu^1 = \nu$. Also, if $\nu(\nu(x)) = \nu(x)$ for each $x \in X$, ν is called a *supratopological* nbd. structure.

PROPOSITION 2.5. ([4]). *If* $\mathcal{A} = \{\nu_i : i \in J\} \subseteq N(X)$ *and each* $\nu_i \in \mathcal{A}$ *is supratopological, so is* $\sup_{N(X)} \mathcal{A}$.

Let $\sigma_\nu = \{A \subseteq X : I_\nu(A) = A\}$ and $\sigma\nu(x) = \{A \subseteq X : \exists U \in \sigma_\nu$ s.t. $x \in U \subseteq A\}$. Then σ_ν is a supratopology in the sense of [5] and $\sigma\nu(x) = \langle\{A \subseteq X : x \in I_\nu(A) = A,\}\rangle$, so $\sigma\nu$ is the finest supratopological nbd. structure on X coarser than ν. ([4]). Also, we can easily show that $\sigma\nu = \sigma\nu^\alpha$ for each ordinal $\alpha \geq 1$ and

$$\nu \geq \nu^2 \geq \nu^3 \geq \cdots\cdots \geq \nu^\omega \geq \cdots\cdots \geq \sigma\nu,$$

where ω is the first limit ordinal.

3. μ-Supratopological Nbd. Spaces and Supratopological Series

In this section, we shall define "μ-supratopological nbd. structures" and "supratopological series", and change some results of [6] related to convergence spaces to nbd. spaces.

Henceforth ν, μ and η mean nbd. structures on X and ω is the first limit ordinal. Also, (X, ν) means a nbd. space equipped with a second nbd.structure μ.

PROPOSITION 3.1. *Let (X, ν) be a nbd. space and $x \in X$. Then $\nu(x) = \mu(\nu(x))$ iff $\nu(x) = I_\mu(\nu(x))$.*

Proof. We have already known $\mu(\nu(x)) \le \nu(x) \le I_\mu(\nu(x))$.

($\Longrightarrow$) Let $A \in I_\mu(\nu(x))$. Then there exists $B \in \nu(x) = \mu(\nu(x))$ such that $I_\mu(B) \subseteq A$, so $I_\mu(B) \in \nu(x)$, and hence $A \in \nu(x)$. Consequently $\nu(x) = I_\mu(\nu(x))$.

($\Longleftarrow$) Let $A \in \nu(x)$. Then $I_\mu(A) \in I_\mu(\nu(x)) = \nu(x)$, so $A \in \mu(\nu(x))$, and hence $\nu(x) = \mu(\nu(x))$. □

DEFINITION 3.2.. A nbd. space (X, ν) is *μ-supratopological* (or ν is *μ-supratopological*) iff $\nu(x) = \mu(\nu(x))$ for all $x \in X$. In particular, if (X, ν) is ν-supratopological, then (X, ν) is *supratopological* (or ν is *supratopological*) .

PROPOSITION 3.3. *(X, ν) is μ-supratopological iff $\mu(\mathcal{H}) \xrightarrow{\nu} x$, whenever a p-stack $\mathcal{H} \xrightarrow{\nu} x$.*

Proof. ($\Longrightarrow$) Let $\mathcal{H} \xrightarrow{\nu} x$. Then $\nu(x) \le \mathcal{H}$, so $\nu(x) = \mu(\nu(x)) \le \mu(\mathcal{H})$, and hence $\mu(\mathcal{H}) \xrightarrow{\nu} x$.

($\Longleftarrow$) Since $\nu(x) \xrightarrow{\nu} x$, by the assumption, $\mu(\nu(x)) \xrightarrow{\nu} x$, so $\mu(\nu(x)) \ge \nu(x)$, and hence $\mu(\nu(x)) = \nu(x)$. Thus (X, ν) is μ-supratopological. □

PROPOSITION 3.4. *If (X, ν) is μ-supratopological and $\mu \le \eta$, then (X, ν) is η-supratopological.*

Proof. Since (X, ν) is μ-supratopological and $\mu \le \eta$, $\nu(x) = \mu(\nu(x)) \le \eta(\nu(x))$. Thus, $\nu(x) = \eta(\nu(x))$, so (X, ν) is η-supratopological. □

COROLLARY 3.5. *(1) If (X, ν) is μ-supratopological and $\mu \le \nu$, then (X, ν) is supratopological.*

(2) If (X, ν) is supratopological and $\nu \le \mu$, then (X, ν) is μ-supratopological.

THEOREM 3.6. *Let ω be the first limit ordinal. If (X, ν) is μ-supratopological, then:*

(1) $\nu \le \mu^\omega \le \mu$,

(2) $(X, \sigma\nu)$ is μ-supratopological.

Proof. Since (X, ν) is μ-supratopological, $\nu(x) = \mu(\nu(x))$.

(1) Claim: $\nu(x) \le \mu^\omega(x)$. Let $V \in \nu(x)$. Then $V \in \mu(\nu(x))$, so $I_\mu(V) \in \nu(x)$. By Induction, $I_\mu^n(V) \in \nu(x)$ for all $n < \omega$, so $x \in I_\mu^n(V)$ for all $n < \omega$. Thus $x \in \cap_{n<\omega} I_\mu^n(V) = I_\mu^\omega(V)$, and hence $V \in \mu^\omega(x)$. Thus the Claim is proved, so $\nu \le \mu^\omega$. Also $\mu^\omega \le \mu$ is obvious.

(2) Since $(X, \sigma\nu)$ is $\sigma\nu$-supratopological and $\sigma\nu \le \nu \le \mu$, by Proposition 3.4, $(X, \sigma\nu)$ is μ-supratopological. □

DEFINITION 3.7. For $\nu, \mu \in N(X)$, $\sigma_\mu \nu$ is defined by

$$\sigma_\mu \nu = \sup_{N(X)}\{\eta : \eta \le \nu, \quad \eta \text{ is } \mu\text{-supratopological}\}.$$

By [4],

$$(\sigma_\mu \nu)(x) = \cup\{\eta(x) : \eta \le \nu, \quad \eta \text{ is } \mu\text{-supratopological}\}, \quad \forall x \in X.$$

PROPOSITION 3.8. *Let $\nu, \mu \in N(X)$ and $\mathcal{F} \in pS(X)$. If there exist $\mathcal{G} \xrightarrow{\nu} x$ and $n \in N$ such that $\mathcal{F} \ge \mu^n(\mathcal{G})$, then $\mathcal{F} \xrightarrow{\sigma_\mu \nu} x$.*

Proof. Suppose that there exist $n \in N$ and $\mathcal{G} \xrightarrow{\nu} x$ such that $\mu^n(\mathcal{G}) \leq \mathcal{F}$. Since $\mathcal{G} \xrightarrow{\nu} x$, $\mathcal{G} \xrightarrow{\eta} x$ for any μ-supratopological nbd. structure $\eta \leq \nu$. Since η is μ-supratopological, $\mu^n(\mathcal{G}) \xrightarrow{\eta} x$, so $\mathcal{F} \xrightarrow{\eta} x$, and hence $\eta(x) \leq \mathcal{F}$ for any μ-supratopological nbd. structure $\eta \leq \nu$. Thus $\cup\{\eta(x) : \eta$ is μ-supratopological, $\eta \leq \nu\} = \sigma_\mu\nu(x) \leq \mathcal{F}$. Consequently, $\mathcal{F} \xrightarrow{\sigma_\mu\nu} x$. □

PROPOSITION 3.9. *Let $\nu, \mu \in N(X)$ and Λ be a set of μ-supratopological nbd. structures on X. Then*

$$s = \sup_{N(X)} \Lambda \quad \textit{and} \quad s = inf_{N(X)}\Lambda$$

are μ-supratopological.

Proof. Let $\Lambda = \{\nu_i : i \in J\} \subseteq N(X)$ and $x \in X$. By Proposition 1.3, we know that $s(x) = \cup_{i\in J}\nu_i(x)$ and $d(x) = \cap_{i\in J}\nu_i(x)$. Since ν_i is μ-supratopological, $\mu(\nu_i(x)) = \nu_i(x)$. Thus, $\mu(s(x)) = \mu(\cup_{i\in J}\nu_i(x)) = \cup_{i\in J}\mu(\nu_i(x)) = \cup_{i\in J}\nu_i(x) = s(x)$ and $\mu(d(x)) = \mu(\cap_{i\in J}\nu_i(x)) = \cap_{i\in J}\mu(\nu_i(x)) = \cap_{i\in J}\nu_i(x) = d(x)$. Therefore, both s and d are μ-supratopological. □

COROLLARY 3.10. *For $\nu, \mu \in N(X)$ on X, $(X, \sigma_\mu\nu)$ is μ-supratopological, and so $\sigma_\mu\nu$ is the finest μ-supratopological nbd. structure on X coarser than ν.*

DEFINITION 3.11. Let $\nu \in N(X)$ and $\alpha \geq 0$ ordinal number. The *supratopological series* for ν is the descending ordinal sequence $\{\sigma_\alpha\nu\}$ on X defined recursively as follows:

$\sigma_0\nu = \nu$
$\sigma_1\nu = \sup_{N(X)}\{\eta :\ \eta$ is ν-supratopological$\}$;
$\sigma_2\nu = \sup_{N(X)}\{\eta :\ \eta$ is $\sigma_1\nu$-supratopological$\}$;
$\sigma_3\nu = \sup_{N(X)}\{\eta :\ \eta$ is $\sigma_2\nu$-supratopological$\}$;
:
:
$\sigma_\alpha\nu = \sup_{N(X)}\{\eta :\ \eta$ is $\sigma_{\alpha'}\nu$-supratopological$\}$, if $\alpha = \alpha' + 1$;

$\sigma_\alpha \nu = \inf_{N(X)}\{\sigma_\beta \nu : \beta < \alpha\}$, if α ia a limit ordinal.

By Propositions 3.4, 3.9 and Corollary 3.10, we know that $\sigma_\beta \nu$ is ν-supratoplogical for any ordinal $\beta \geq 1$, $\sigma_1 \nu = \sigma_\nu \nu$, $\sigma_2 \nu = \sigma_{\sigma_1 \nu} \nu = \sigma_{\sigma_\nu \nu} \nu$, $\cdots$, etc., and

$$\nu \geq \sigma_1 \nu \geq \sigma_2 \nu \geq \sigma_3 \nu \geq \cdots\cdots \geq \sigma_\omega \nu \geq \cdots\cdots .$$

4. Decomposition Series and Supratopological Series of Nbd. Spaces

In this section, for given nbd. structure ν on X, we will show relations between decomposition series and supratopological series of ν.

Definition 4.1. For $\nu \in N(X)$, there exists $\widetilde{\nu}$ which is *the finest ν-supratopological nbd. structure* on X, that is,

$$\widetilde{\nu} = \sup_{N(X)}\{\eta : \eta \text{ is } \nu\text{-supratopological}\},$$

and by [4],

$$\widetilde{\nu}(x) = \cup\{\eta(x) : \eta \text{ is } \nu\text{-supratopological}\}, \quad \forall x \in X.$$

Proposition 4.2. *If $\mathcal{G} \xrightarrow{\nu} x$, then $\nu^{n+1}(x) \leq \nu^n(\mathcal{G})$, where $n < \omega$.*

Proof. $V \in \nu^{n+1}(x) \implies x \in I_\nu^{n+1}(V) \implies x \in I_\nu(I_\nu^n(V)) \implies I_\nu^n(V) \in \nu(x) \implies I_\nu^n(V) \in \mathcal{G}$, since $\mathcal{G} \xrightarrow{\nu} x \implies \mathcal{G} \geq \nu(x)$. Thus $V \in \nu^n(\mathcal{G})$. □

Proposition 4.3. *$\widetilde{\nu} = \sigma_1 \nu$.*

Proof. It follows from Theorem 3.6, Definition 3.7, Definition 3.11 and Definition 4.1. □

Let α and β be any ordinals, and ω the first(least) limit ordinal, and Ω be the first(least) uncountable limit ordinal. Then recall that $\alpha + \Omega = \Omega$ for any countable ordinal α, and $\alpha + \beta = \beta$ iff $\beta \geq \alpha \cdot \omega$. By using this fact, we obtain the following.

Proposition 4.4. *(1) If $\alpha+\beta=\beta$, then (X,ν^{β}) is ν^{α}-supratopological.*

(2) If (X,μ) is ν^{α}-supratopological and ν^{β}-supratopological, then (X,μ) is $\nu^{\alpha+\beta}$-supratopological.

(3) (X,ν^{ω}) is ν-supratopological and so ν^{n}-supratopological for each ordinal $n<\omega$.

(4) (X,ν^{ω^2}) is ν^{ω}-supratopological and so $\nu^{\omega\cdot n}$-supratopological for each ordinal $n<\omega$.

(5) (X,ν^{Ω}) is ν^{α}-supratopological for each countable ordinal α.

(6) If $0<\alpha<\beta$, then $(X,\nu^{\omega^{\beta}})$ is $\nu^{\omega^{\alpha}}$-supratopological.

Proof. (1)-(4) These are obvious by Proposition 2.2 (2).

(5) It follows from $\alpha+\Omega=\Omega$ for each countable ordinal α.

(6) Let $0<\alpha<\beta$. Then $\alpha+1\leq\beta$ and so $\omega^{\alpha}\cdot\omega=\omega^{\alpha+1}\leq\omega^{\beta}$. Thus $\omega^{\alpha}+\omega^{\beta}=\omega^{\beta}$. □

Recall that for $\nu\in N(X)$, $\nu^{\alpha}(x)$ is the nbd. p-stack for ν^{α}, the αth term in the decomposition series for ν and the first term in the supratoplogical series for ν is $\sigma_1\nu=\widetilde{\nu}$.

Also $\sigma_1\nu$ is the finest ν-supratopological nbd. structure on X and the lower ν-supratopological modification of ν, since $\sigma_1\nu\leq\nu$.

While, by Theorem 3.6, if (X,μ) is ν-supratopological, then $\mu\leq\nu$. Thus, if $\nu<\mu$, then (X,μ) is not ν-supratopological. This implies ν has no upper ν-supratopological modification unless ν is supratopological. We next show that that $\sigma_2\nu$ is related to $\sigma_1\nu$ exactly as $\sigma_1\nu$ is related to ν. Note that the lower $\sigma_1\nu$-supratopological modification of $\sigma_1\nu$ is $\widetilde{\sigma_1\nu}$ defined by:

$$\widetilde{\sigma_1\nu}=\sup_{N(X)}\{\eta:\eta \text{ is } \sigma_1\nu\text{-supratopological}\},$$

Proposition 4.5. *For any $\nu\in N(X)$ and the first limit ordinal ω,*

(1) $\widetilde{\nu}=\nu^{\omega}$;

(2) $\sigma_2\nu=\widetilde{\sigma_1\nu}$;

(3) more generally, $\sigma_{\alpha}\nu=\widetilde{\sigma_{\alpha'}\nu}$ for $\alpha=\alpha'+1$.

Proof. (1) Since ν^ω is ν-supratopological, $\widetilde{\nu} \geq \nu^\omega$. By Proposition 4.3, $\widetilde{\nu} = \sigma_1\nu$. Since $(X, \sigma_1\nu)$ is ν-supratopological, by Theorem 3.6, $\sigma_1\nu \leq \nu^\omega$, Thus $\widetilde{\nu} = \nu^\omega$.

(2) Note that $\sigma_2\nu = \sigma_{\sigma_1\nu}\nu$ and $\widetilde{\sigma_1\nu} = \sigma_{\sigma_1\nu}(\sigma_1\nu)$.

Since $\sigma_1\nu \leq \nu$, by the definitions, $\sigma_2\nu \geq \widetilde{\sigma_1\nu}$. While, since $\sigma_2\nu$ is $\sigma_1\nu$-supratopological, $\sigma_2\nu \leq \widetilde{\sigma_1\nu}$. Thus $\sigma_2\nu = \widetilde{\sigma_1\nu}$.

(3) The proof is equal to replace $\sigma_2\nu$ and $\sigma_1\nu$ by $\sigma_\alpha\nu$ and $\sigma_{\alpha'}\nu$, respectively, in the proof of (2). $\square$

PROPOSITION 4.6. *$\sigma_1\nu = \nu^\omega$ and $\sigma_\alpha\nu = \nu^{\omega^\alpha}$ for a non-limit ordinal α, where ν^{ω^α} means $\nu^{(\omega^\alpha)}$*

Proof. The first equality follows from the Proposition 4.3 and 4.5. The second equality is proved by the induction hypothesis. Suppose that $\sigma_{\alpha'}\nu = \nu^{\omega^{\alpha'}}$ for $\alpha = \alpha' + 1$. Then $\sigma_\alpha\nu = \widetilde{\sigma_{\alpha'}\nu} = (\sigma_{\alpha'}\nu)^\omega = (\nu^{\omega^{\alpha'}})^\omega = \nu^{\omega^{\alpha'}\cdot\omega} = \nu^{\omega^{\alpha'+1}} = \nu^{\omega^\alpha}$. $\square$

PROPOSITION 4.7. *If α is a limit ordinal, $\nu^\alpha(x) = \cap_{\beta<\alpha}\nu^\beta(x)$.*

Proof. $A \in \nu^\alpha(x) \iff x \in I_\nu^\alpha(A) = \cap_{\beta<\alpha} I_\nu^\beta(A) \iff x \in I_\nu^\beta(A),\ \forall\beta < \alpha \iff A \in \nu^\beta(x),\ \forall\beta < \alpha \iff A \in \cap_{\beta<\alpha}\nu^\beta(x)$. $\square$

PROPOSITION 4.8. *If α is a limit ordinal, then $\sigma_\alpha\nu = \nu^{\omega^\alpha}$.*

Proof. Recall that Definition 3.7, $(\sigma_\alpha\nu)(x) = \cap\{(\sigma_\beta\nu((x) : \beta < \alpha\}$, for all $x \in X$. We will used the induction hypothesis. Assume that $\sigma_\beta\nu = \nu^{\omega^\beta}$ for $\beta < \alpha$. Then $(\sigma_\alpha\nu)(x) = \cap\{(\sigma_\beta\nu)(x) : \beta < \alpha\} = \cap_{\beta<\alpha}\nu^{\omega^\beta}(x) = \nu^{\omega^\alpha}(x)$. $\square$

Let $l_D\nu$ be the length of decomposition series and $l_T\nu$ the supratopological series for $\nu \in N(X)$, defined by:

$l_D\nu$=inf$\{\lambda : \lambda$ is an ordinal such that $\nu^\lambda = \nu^{\lambda+1}\}$;

$l_T\nu$ =inf$\{\lambda : \lambda$ is an ordinal such that $\sigma_\lambda\nu = \sigma_{\lambda+1}\nu\}$.

Then we know that $l_D\nu$=inf$\{\lambda : \lambda$ is an ordinal such that $I_\nu^\lambda(A) = I_\nu^{\lambda+1}(A),\ \forall A \subseteq X\}$=inf$\{\lambda : \lambda$ is an ordinal such that $\nu^\lambda = \sigma\nu\}$. Finally, we obtain the following relation between them.

Proposition 4.9. *For $\nu \in N(X)$ and an ordinal α,*
(1) if $l_T\nu \leq \alpha$, then $\sigma_\alpha\nu = \sigma\nu$;
(2) if $l_T\nu \leq \alpha$, then $l_D\nu \leq \omega^\alpha$.

Proof. (1) Let $\lambda = l_T\nu$. Then $\sigma_\lambda\nu = \sigma_{\lambda+1}\nu = \sigma\nu$. Since $\lambda \leq \alpha$, $\sigma_\lambda\nu \geq \sigma_\alpha\nu \geq \sigma\nu$. Thus $\sigma_\alpha\nu = \sigma\nu$.

(2) Since $l_T\nu \leq \alpha$, $\sigma_\alpha\nu = \nu^{\omega^\alpha} = \sigma\nu$. Thus $l_D\nu \leq \omega^\alpha$. □

REFERENCES

[1] D. C. Kent, *Convergence functions and their related topologies*, Fund. Math. **54** (1964), 125–133.

[2] D. C. Kent, *Convergence quotient maps*, Fund. Math. **65** (1969), 197–205.

[3] D. C. Kent and G. D. Richardson, *The deomposition series of a Convergence space*, Czechoslovak Mathematical Journal **23** (1973), 437–446.

[4] D.C. Kent and Won Keun Min, *Neighborhood Spaces*, International Journal of Mathematics and Mathematical Sciences **32** (2002), 387–399.

[5] A. S. Mashhourr, A. A. Allam, F. S. Mahmoud, and F.H. Khedr, *On Supratopological Spaces*, Indian J. Pure Appl. Math. **14** (1983), 502–510.

[6] Scott Alan Wilde, *Study and Characterization of the p-supratopological Convergence Spaces*, 14, 1997.

[7] Scott A. Wilde and D.C. Kent, *p-supratopological and p-Regular: Dual Notions in Convergence Theory*, International Journal of Mathematics and Mathematical Sciences **22** (1999), 1–12.

Sang Ho Park
Department of Mathematics and
Research Institute of Natural Science,
Gyeongsang National University
Jinju 660-701, Korea
E-mail: sanghop@gnu.ac.kr