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FUZZY SET CONNECTED FUNCTIONS

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Abstract. The purpose of this paper is to introduce the concept of fuzzy set connected functions and investigate their properties.

1. Introduction

Throughtout this paper, I will denote the closed unit interval $[0, 1]$. X and Y will be non-empty sets. Zadeh [15] generalized characteristic functions into fuzzy sets and Chang [5] introduced the topological structure in a class of fuzzy sets in a given set.

For X, I^X denotes the collection of all functions from X into I. A member λ of I^X is called a *fuzzy set* of X. We will denote fuzzy sets in X by λ, μ, δ, ξ and etc. The fuzzy null set 0 and the fuzzy whole set 1 denote constant functions taking 0 and 1 for each $x \in X$, respectively. If there are confusions in using 1, we will use the whole set X instead of 1. A fuzzy set λ is said to be contained in a fuzzy set μ (denoted by $\lambda \leq \mu$) iff $\lambda(x) \leq \mu(x)$ for each $x \in X$. The complement $1 - \lambda$ of a fuzzy set λ of X is defined by $(1 - \lambda)(x)$ for each $x \in X$.

If λ is a fuzzy set of X and δ is a fuzzy set of Y, then $\lambda \times \delta$ is a fuzzy set of $X \times Y$, defined by $(\lambda \times \delta)(x, y) = \min\, \{\lambda(x), \delta(y)\}$ for each $(x, y) \in X \times Y$.

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A *fuzzy point* x_β of X is a fuzzy set of X such that it takes the value 0 for all $y \in X$ except x, that is, it is defined as

$$x_\beta(y) = \begin{cases} \beta, & \text{if } y = x \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in$ X. A fuzzy point x_β is said to be contained in a fuzzy set λ(denoted by $x_\beta \in \lambda$) if $\beta \leq \lambda(x)$. Let $f : X \to Y$ be a function and λ be a fuzzy set of X. Then a fuzzy set $f(\lambda)$ of Y is defined as

$$f(\lambda)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \lambda(x) \ , & \text{if } f^{-1}(y) \neq \emptyset \\ 0 \ , & \text{otherwise} \end{cases}$$

for each $y \in Y$, and if μ is a fuzzy set of Y, then a fuzzy set of X $f^{-1}(\mu)$ is defined as

$$f^{-1}(\mu)(x) = (\mu \circ f)(x) = \mu(f(x))$$

for each $x \in$ X.

2. Definition and Terminology

From now on, $f : (X, \tau) \to (Y, \tau^*)$ denotes a function from a fuzzy topological space (X, τ) into a fuzzy topological space (Y, τ^*), and in this section we describe definitions and theorems without their proofs, which we need in the last section. For concepts which is not defined here, we refer to [4], [5], [8], [10], [12] and [13].

Let (X, τ) be a fuzzy topological space(written as fts). For a fuzzy set λ of X, its closure Cl(λ) and its interior Int(λ) are defined by Cl(λ) $= \bigwedge\{\nu : \lambda \leq \nu,\ 1 - \nu \in \tau(X)\}$ and Int(λ) $= \bigvee\{\nu : \nu \leq \lambda,\ \nu \in \tau(X)\}$, respectively. A fuzzy set which is both fuzzy open and fuzzy closed is called a fuzzy clopen set, and a class of all fuzzy clopen set of a fts X will be denoted by FCO(X).

DEFINITION 2.1. A fts X is said to be *fuzzy connected* if 1 and 0 are the only fuzzy subsets of X which are both fuzzy open and fuzzy closed.

THEOREM 2.2. A fts X is not fuzzy connected if there are fuzzy clopen sets $\lambda(\neq 1)$ and $\delta(\neq 1)$ in X such that $\lambda \vee \delta = 1$ and $\lambda \wedge \delta = 0$.

In the above theorem, the condition of $\lambda(\neq 1)$ and $\delta(\neq 1)$ can not be dropt, as shown by the following example.

EXAMPLE 2.3. For a fts (X, τ) where $\tau = \{0, 1\}$. Taking $\lambda = 1$ and $\delta = 0$, $\lambda, \delta \in \mathrm{FCO}(X)$, $\lambda \vee \delta = 1$ and $\lambda \wedge \delta = 0$. But (X, τ) is fuzzy connected.

DEFINITION 2.4. A fts X is said to be *fuzzy connected between fuzzy sets* λ and δ [9] if there is no fuzzy clopen set η such that $\lambda \leq \eta$ and $\eta \wedge \delta = 0$.

Let a fts Y be a subspace of a fts X. It is shown [9] that if Y is fuzzy connected between fuzzy sets λ and δ, then X is fuzzy connected between fuzzy sets λ and δ.

THEOREM 2.5. If a fts X is not fuzzy connected between fuzzy sets λ and δ, then there is a $\xi \in \mathrm{FCO(X)}$ such that $\lambda \leq \xi \leq (1 - \delta)$.

The converse of the above theorem may not be true, as shown by the next example.

EXAMPLE 2.6. Let $I = [0, 1]$ and $\tau = \{0, \xi, 1\}$ where $\xi(x) = \frac{1}{2}$ for all $x \in I$. Then $\xi \in \mathrm{FCO}(I)$. Let $\lambda(x) = \frac{1}{8}$ and $\delta(x) = \frac{1}{4}$ for all $x \in I$. Then $\lambda \leq \xi \leq 1 - \delta$, but (I, τ) is fuzzy connected between λ and δ since there is no $\eta \in \mathrm{FCO}(I)$ such that $\lambda \leq \eta$ and $\eta \wedge \delta = 0$.

DEFINITION 2.7. A fts (X, τ) is said to be *fuzzy extremally disconnected* [3] if $\lambda \in \tau$ implies $\mathrm{Cl}(\lambda) \in \tau$.

DEFINITION 2.8. $f : (X, \tau) \to (Y, \tau^*)$ is said to be *fuzzy slightly continuous* if for each $\mu \in \mathrm{FCO}(Y)$, there exists a $\lambda \in \tau$ such that $f(\lambda) \leq \mu$.

THEOREM 2.9. $f : (X, \tau) \to (Y, \tau^*)$ is fuzzy slightly continuous if for each $\mu \in \mathrm{FCO}(Y)$, $f^{-1}(\mu) \in \tau$.

The converse of Theorem 2.9 may not be true, as shown by the follow.

EXAMPLE 2.10. Let $I = [0, 1]$ and $\tau = \{0, \lambda, 1\}$ and $\tau^* = \{0, \mu, 1\}$ where $\lambda(x) = \frac{1}{8}$ and $\mu(x) = \frac{1}{2}$ for all $x \in I$. And let $f : (I, \tau) \to (I, \tau^*)$ be the identity. Then f is fuzzy slightly continuous and $\mu \in \mathrm{FCO}(I)$, but $f^{-1} \notin \tau$ because $f^{-1}(\mu)(x) = \mu(f(x)) = \mu(x)$ for all $x \in X$.

DEFINITION 2.11. $f : (X, \tau) \to (Y, \tau^*)$ is said to be *fuzzy continuous* [5] if for each $\mu \in \tau^*$, $f^{-1}(\mu) \in \tau$.

DEFINITION 2.12. $f : (X, \tau) \to (Y, \tau^*)$ is said to be *fuzzy weakly continuous* [1] if for each $\mu \in \tau^*$, $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Cl}(\mu)))$.

Let λ be a fuzzy set of an fts X. Then λ is said to be *fuzzy regular open* [4] if $\mathrm{Int}(\mathrm{Cl}(\lambda)) = \lambda$.

DEFINITION 2.13. $f : (X, \tau) \to (Y, \tau^*)$ is said to be *fuzzy almost continuous* [1] if for each fuzzy regular open set μ of Y, $f^{-1}(\mu) \in \tau$.

3. Fuzzy set connected functions

In this section X, Y and Z will denote fts' without specification.

DEFINITION 3.1. A function $f : X \to Y$ is said to be *fuzzy set connected* provided that $f(X)$ is fuzzy connected between fuzzy sets $f(\lambda)$ and $f(\delta)$ with respect to relative fuzzy topology if X is fuzzy connected between fuzzy sets λ and δ.

THEOREM 3.2. A function $f : X \to Y$ is fuzzy set connected if and only if for each $\xi \in \mathrm{FCO}(f(X))$, $f^{-1}(\xi) \in \mathrm{FCO}(f(X))$.

Proof. Let X be fuzzy connected between fuzzy sets λ and δ. Suppose $f(X)$ is not fuzzy connected between fuzzy sets $f(\lambda)$ and $f(\delta)$. Then by Theorem 2.5 there exists a $\xi \in \mathrm{FCO}(f(X))$ such that $f(\lambda) \leq \xi \leq (1 - f(\delta))$. By hypothesis, $f^{-1}(\xi) \in \mathrm{FCO}(X)$ and $\lambda \leq f^{-1}(\xi) \leq (1 - \delta)$. Therefore, X is not fuzzy connected between λ and δ. It is a contradiction. Hence f is fuzzy set connected.

To show the converse, let $f : X \to Y$ be fuzzy set connected and $\xi \in \mathrm{FCO}(f(X))$. Suppose $f^{-1}(\xi)$ is not a fuzzy clopen set of X. Then X is fuzzy connected between fuzzy sets $f^{-1}(\xi)$ and $1 - f^{-1}(\xi)$. Therefore, $f(X)$ is fuzzy connected between fuzzy sets $f(f^{-1}(\xi))$ and $f(1 - f^{-1}(\xi))$. But $f(f^{-1}(\xi)) = \xi \wedge f(X) = \xi$ and $f(1 - f^{-1}(\xi)) = f(X) \wedge (1 - \xi) = 1 - \xi$ implies that ξ is not a fuzzy clopen set of $f(X)$. It is a contradiction. □

THEOREM 3.3. IF $f : (X, \tau) \to (Y, \tau^*)$ is fuzzy set connected, then $f^{-1}(\mu) \in \mathrm{FCO}(X)$ for each $\mu \in \mathrm{FCO}(Y)$.

Notice that $f^{-1}(\xi) \in \mathrm{FCO}(X)$ for any $\xi \in \mathrm{FCO}(Y)$, if $f : X \to Y$ is fuzzy set connected.

Theorem 3.4. Every fuzzy continuous function is fuzzy set connected.

The converse of Theorem 3.4 may not be true, as shown by the following example.

Example 3.5. Let $X = \{x, y\}$, $Y = \{a, b\}$ and $\lambda \leq X$, $\mu \leq Y$ defined as follows:

$$\lambda(x) = 0.3,\ \lambda(y) = 0.4$$

$$\mu(a) = 0.6,\ \mu(b) = 0.5$$

Let $\tau = \{0, \lambda, X\}$ and $\tau^* = \{0, \mu, Y\}$. Then the function $f : (X, \tau) \to (Y, \tau^*)$ defined by $f(x) = a$ and $f(y) = b$ is fuzzy set connected, but not fuzzy continuous.

Theorem 3.6. Every function $f : X \to Y$ such that $f(X)$ is fuzzy connected is a fuzzy set connected function.

Proof. Let $f(X)$ be fuzzy connected. Then there is no non-empty proper fuzzy set of $f(X)$ which is clopen. Hence vacuously f is fuzzy set connected. □

Theorem 3.7. Let $f : X \to Y$ be a fuzzy set connected function. If X is fuzzy connected, then $f(X)$ is fuzzy connected.

Proof. Suppose $f(X)$ is fuzzy disconnected. Then there is a non-empty proper closed open fuzzy set ξ of $f(X)$. Since f is fuzzy set connected, by Theorem 3.2 $f^{-1}(\xi)$ is a non-empty proper closed open fuzzy set of X. Consequently X is not fuzzy connected. It contradicts. □

Theorem 3.8. Let $f : X \to Y$ be a fuzzy set connected surjection and $g : Y \to Z$ be a fuzzy set connected function. Then $g \circ f : X \to Z$ is fuzzy set connected.

Proof. Let $\xi \in \mathrm{FCO}(g(Y))$. Then $g^{-1}(\xi) \in \mathrm{FCO}(Y) = \mathrm{FCO}(f(X))$. Thus $f^{-1}(g^{-1}(\xi)) \in \mathrm{FCO}(X)$. Now $(g \circ f)(X) = g(Y)$, and $(g \circ f)^{-1}(\xi) = f^{-1}(g^{-1}(\xi))$. So $g \circ f$ is fuzzy set connected by Theorem 3.2. □

THEOREM 3.9. Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. If g is fuzzy set connected, then f is fuzzy set connected.

Proof. Let ξ any fuzzy clopen set of the subspace $f(X)$ of Y. Then $X \times \xi$ is a fuzzy clopen set of the subspace $X \times f(X)$ of the fuzzy product space $X \times Y$. Since $g(X)$ is a subset of $X \times f(X)$, $(X \times \xi) \wedge g(X)$ is a fuzzy clopen set of the subspace $g(X)$ of $X \times Y$. By Theorem 3.2, $g^{-1}((X \times \xi) \wedge g(X)) \in \mathrm{FCO}(X)$. It follows from $g^{-1}((X \times \xi) \wedge g(X)) = g^{-1}(X \times \xi) = f^{-1}(\xi)$ that $f^{-1}(\xi) \in \mathrm{FCO}(X)$. Hence f is fuzzy set connected. □

THEOREM 3.10. If a surjection $f : (X, \tau) \to (Y, \tau^*)$ is fuzzy weakly continuous, then f is fuzzy set connected.

Proof. Let $\mu \in \mathrm{FCO}(Y)$. Since f is fuzzy weakly continuous, $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Cl}(\mu))) = \mathrm{Int}(f^{-1}(\mu))$. Hence $f^{-1}(\mu) \in \tau$. Moreover, we obtain $\mathrm{Cl}(f^{-1}(\mu)) \leq f^{-1}(\mathrm{Cl}(\mu)) = f^{-1}(\mu)$ from [14, Theorem 4.6]. This shows that $f^{-1}(\mu)$ is a closed fuzzy set of X. Since f is surjective, by Theorem 3.2 we obtain that f is a fuzzy set connected function. □

The converse of Theorem 3.10 may be false, as shown by the following example.

EXAMPLE 3.11. Let $X = \{x, y\}$, $Y = \{a, b\}$ and $\lambda \leq X$, $\mu \leq Y$ defined as follows:

$$\lambda(x) = 0.3,\ \lambda(y) = 0.4$$
$$\mu(a) = 0.4,\ \mu(b) = 0.5$$

Let $\tau = \{0, \lambda, X\}$ and $\tau^* = \{0, \mu, Y\}$. Then the function $f : (X, \tau) \to (Y, \tau^*)$ defined by $f(x) = f(y) = a$ is fuzzy set connected, but not fuzzy weakly continuous.

THEOREM 3.12. Let Y be fuzzy extremally disconnected. If $f : X \to Y$ is fuzzy set connected and surjective, then f is fuzzy almost continuous.

Proof. Let x_p be a fuzzy point of X and μ be a fuzzy open set of Y containing $f(x_p)$. Since Y is fuzzy extremally disconneted, $\mathrm{Cl}(\mu) \in \mathrm{FCO}(Y)$. Thus $\mathrm{Cl}(\mu) \leq f(X)$ and $\mathrm{Cl}(\mu) \in \mathrm{FCO}(f(X))$. Putting $f^{-1}[\mathrm{Cl}(\mu) \wedge f(X)] = \lambda$, it follows from Theorem 3.2 that $\lambda \in \mathrm{FCO}(X)$

because f is fuzzy set connected. Clearly $x_p \in \lambda$ and $f(\lambda) \leq \text{Cl}(\mu) \leq \text{Int}(\text{Cl}(\mu))$. Hence by [14, Theorem 4.12], f is fuzzy almost continuous. □

Corollary 3.13. Let Y be fuzzy extremally disconnected. If a function $f : X \to Y$ is fuzzy set connected and is surjective, then f is fuzzy weakly continuous.

Theorem 3.14. Let Y be fuzzy extremally disconnected and $f : X \to Y$ be surjective. Then the follows are equivalent:

(1) f is fuzzy set connected.
(2) f is fuzzy almost continuous.
(3) f is fuzzy weakly continuous.

Proof. This is an immediate consequence of Theorem 3.10, Theorem 3.12 and Corollary 3.13. □

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