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COMPUTATIONAL METHOD TO CHARACTERIZE S_3-ALGEBRAS

Farhat Nisar and Shaban Ali Bhatti

Abstract. In this paper we give a concrete computational method to characterize S_3-algebras.

1. Introduction

In [10] K. Iseki introduced the notion of BCI-algebra and establish some of its properties. Onwards so many eminent researchers contributed a lot to the development of the discipline. S.K. Goel [4] calculated the number of BCI-algebras of order 3 and partially BCI-algebras of order 4. In [2] S.A. Bhatti, M.A. Chaudhry and A.H. Zaidi calculated proper BCI-algebras of order 5 up to isomorphism and posed an open problem stated as follows:

How many proper BCI-algebras of order n exist?

Almost 14 years have gone, this problem is still unsolved. In [1] S.A. Bhatti, M.A. Chaudhry and B. Ahmad classified BCI-algebras into S_i-algebras, i=1, 2, 3, 4 and investigated some properties of S_3-algebras and S_4-algebras. S_4-algebras coincide with abelian groups and hence their characterization falls in the area of group theory and is answered there at. An S_3-algebras is a type of proper BCI-algebra. We answer the problem completely in it as follows:

"Let X be a S_3-algebra of order n with $o(M) = m$, then number of S_3-algebras is LR, where L = number of BCK-algebras of order m and R = number of distinct p-semisimple BCI-algebras of order n-m+1."

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2. Preliminaries

DEFINITION 1 ([10]). A BCI-algebra X is an abstract algebra $(X, *, o)$ of type (2, 0), where * is a binary operation, o is a constant which is the smallest element in X, satisfying the following conditions; for all $x, y, z \in X$,

1 $((x * y) * (x * z)) * (z * y) = o$
2 $(x * (x * y)) * y = o$
3 $x * x = o$
4 $x * y = o = y * x \Rightarrow x = y$

where $x * y = o \Leftrightarrow x \leq y$.

If $o * x = o$ holds for all $x \in X$, then X is a BCK-algebra [9]. Moreover, the following properties hold in every BCK/BCI-algebra ([10]):

5 $x * o = x$
6 $(x * y) * z = (x * z) * y$

In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ is called the BCK-part of X. A BCI-algebra X is called proper if $X - M \neq \phi$. In a BCI-algebra X, $X - M = \{x \in X : o * x \neq o\}$ is known as the pure BCI-part of X.

7 Let X be a BCI-algebra. If $M = o$, then X is called a p-semisimple BCI-algebra [11].
8 Let X be a p-semisimple BCI-algebra. If we define $x + y = x * (o * y)$, then $(X, +, o)$ is an abelian group [3, 11].

DEFINITION 2 ([1]). Let X be a BCI-algebra, for $x, y \in X$, x, y are said to be comparable if $x \leq y$ or $y \leq x$. Similarly in BCK-algebras, if $x * y = o$ or $y * x = o$, the x and y are comparable.

DEFINITION 3 ([1]). An element $x_o \in X$ is said to be an initial element in X, if $x \leq x_o \Rightarrow x = x_o$. Obviously o is an initial element.

DEFINITION 4 ([1]). Let I_x denote the set of all initial elements of X. We call it the center of X. The reason for calling I_x as the center of X is that each branch (defined below) originates from a unique point of this subset. The cardinality of the center is the same as that as the set of branches of X.

DEFINITION 5 ([1]). Let X be a BCI-algebra with I_x as its center. Let $x_o \in I_x$, then the set $A(x_o) = \{x \in X : x_o * x = o\}$. $A(x_o)$ is known as the branch of X determined by x_o. Each branch $A(x_o)$ is nonempty because by property (3), $x_o * x_o = o \Rightarrow x_o \in A(x_o)$.

We note that $A(x_o)$ consists of all those elements of X which succeed x_o. If $A(x_o) = \{x_o\}$, then $A(x_o)$, the branch determined by x_o, is known as a uniary comparable.

DEFINITION 6 ([1]). A proper BCI-algebra X with $M \neq o$, is S_3 -algebra if each branch $A(x_o)$ in $X - M$ is uniary comparable i.e for all $x \in X - M$, $A(x) = \{x\}$.

9 Let X be a S_3-algebra with M as its BCK-part. Then for $x, y \in M$, $z \in X - M$, $x * z = y * z$ [5].

10 Let X be a S_3-algebra with M as its BCK-part. Then $G = \{o\} \cup (X - M)$ is p-semisimple [5].

11 Let X be a S_3-algebra with M as its BCK-part. Then for $x \in M$, $y \in X - M$, $y * x = y$ [1].

DEFINITION 7 ([6]). Let X be a BCI-algebra. An element $x_o \in X$ is said to be a Semi-neutral element in X if and only if for all $x \neq x_o$, $x * x_o = x$ and $x_o * x = x_o$.

Moreover in [6], we show that the set of all Semi-neutral elements is denoted as $S(X)$ and is known as the Semi-neutral part of the BCK-algebra X. Obviously $S(X)$ is nonempty, because X is a BCK-algebra, therefore $o * x = o$ and $x * o = x$. So, $o \in S(X)$.

Note that any nonzero element x of a BCK-algebra X such that $x \leq y$ for some $y \in X$ (or $y \leq x$ for some $y(\neq o) \in X$ can not be a semi-neutral element of X.

DEFINITION 8 ([6]). A BCK-algebra X is said to be a Semi-neutral BCK-algebra if it satisfies for all $x, y \in X$, $x \neq y \Rightarrow x * y = x$.

12 If X is a Semi-neutral BCK-algebra of finite order then X is unique.

PROPOSITION 1. *A p-semisimple BCI-algebra of order n is unique if n is not divisible by the square of any prime number.*

Proof. Let X be a p-semisimple BCI-algebra of order n such that n is not divisible by the square of any prime number. P-semisimple BCI-algebra coincide with an abelian group (see [3, 11]). It is known that there is a unique abelian group (upto isomorphism) of order n if and only if the order n is not divisible by the square of any prime number (see [8], chapter 11, page 219, Ex. 11, part (f), also see page A15). Thus it follows that X is unique. □

PROPOSITION 2. *Let X be a S_3-algebra with M as its BCK-part. Then for $x \in M$, $z \neq o$, $z \in G = \{o\} \bigcup (X - M)$,*

(i) $x * z = o * z$;
(ii) $z * x = z$.

Proof. Follows from (9).
Follows from (11). □

THEOREM 1. *Let X be a S_3-algebra of order n with $o(M) = m$, then number of S_3-algebras is LR, where L = number of BCK-algebras of order m and R = number of distinct p-semisimple BCI-algebras of order n-m+1.*

Proof. Let X be a S_3-algebra with $o(X) = n$ and $o(M) = m$. Since $o(X) = n$ and $o(M) = m$, therefore $o(X - M) = n - m$. So, $o(G) = o(\{o\} \bigcup (X - M)) = n - m + 1$. Without any loss of generality, we take $M = \{x_1 = o, x_2, ..., x_m\}$ and $G = \{x_1 = o, x_{m+1}, ..., x_n\}$. Because of proposition 2, for each $x_r \in M$ and $x_t \neq o$, $x_t \in G$,

$$x_t * x_r = x_t \tag{1}$$

and

$$x_r * x_t = o * x_t \tag{2}$$

for $r = 1$ to m and $t = m + 1$ to n. □

From equation (1), it follows that the entries in the $(m + 2)^{th}$ to $(n + 1)^{th}$ cells of the 3^{rd} to $(m + 1)^{th}$ columns are the same as in $(m + 2)^{th}$ to $(n + 1)^{th}$ cells of the first column in the multiplication table 1 (given below) representing such S_3-algebra of order n.

Further from equation (2), it follows that the entries in the $(m+2)^{th}$ to $(n+1)^{th}$ cells of the 3^{rd} to $(m+1)^{th}$ rows are the same as in $(m+2)^{th}$

to $(n+1)^{th}$ cells of the 2nd row in the multiplication table 1 (given below) representing such S_3-algebra of order n.

Since M is the BCK-part of S_3-algebra, therefore for each $x_r \in M$,

$$o * x_r = o \tag{3}$$

for $r = 1$ to m.

Now using the properties (3), (5) and equations (1)-(3), the multiplication table representing such S_3-algebra of order n is shown as follows:

Table 1

*	o	x_2	x_3	- -	x_m	x_{m+1}	- -	x_{n-1}	x_n
o	o	o	o	- -	o	$o * x_{m+1}$ =??	- -	$o * x_{n-1}$ =??	$o * x_n$=??
x_2	x_2	o	?	- -	?	-do-	- -	-do-	-do-
x_3	x_3	?	o	- -	?	-do-	- -	-do-	-do-
- -	- -	?	?	- -	?	- -	- -	-do-	-do-
x_m	x_m	?	?	- -	o	-do-	- -	-do-	-do-
x_{m+1}	x_{m+1}	x_{m+1}	x_{m+1}	- -	x_{m+1}	o	- -	??	??
- -	- -	- -	- -	- -	- -	- -	- -	??	??
x_{n-1}	x_{n-1}	x_{n-1}	x_{n-1}	- -	x_{n-1}	??	- -	??	??
x_n	x_n	x_n	x_n	- -	x_n	??	- -	??	??

In the above multiplication table the double dashed columns represent the missing 5^{th} to m^{th} and $(m+3)^{th}$ to $(n-1)^{th}$ columns and the the double dashed rows represent the missing 5^{th} to m^{th} and $(m+3)^{th}$ to $(n-1)^{th}$ rows. The cells for which the values to be computed are denoted as ? and ??, where ? denotes x*y such that x, y $\in$ M and ?? denotes x*y such that x, y $\in$ G

By (10) G is a p-semisimple BCI-algebra. Because of (8) a p-semisimple algebra coincide with an abelian group and therefore the number of p-semisimple algebras (up to isomorphism) of order n-m+1 is equal to the number of abelian groups (up to isomorphism) of order n-m+1. We are given that this number is R. Thus there are R multiplication tables representing such p-semisimple algebra G of order n-m+1. Hence the blank cells filled with ?? of multiplication table 1 can be filled in R distinct ways with the entries from the corresponding cells of R multiplication tables representing G respectively. So, we get R multiplication tables representing S_3-algebras of order n with remaining blank cells filled with ?.

Further we are given that there are L distinct BCK-algebras of order m. Thus there are L Iseki tables representing BCK-algebras order m.

Hence the blank cells filled with ? of any of R multiplication tables representing S_3-algebras of order n can be filled in L distinct ways with the entries from the corresponding cells of L Iseki tables respectively. Hence it follows that there are LR distinct S_3-algebras of order n.

COROLLARY 1. *If the BCK-part M of a finite S_3-algebra X is semi-neutral and $o(G = \{o\} \bigcup (X - M))$ is not divisible by the square of any prime number, then X is unique.*

Proof. Let X be a S_3-algebra with M as its BCK-part. Since the BCK-part M is semi neutral BCK-algebra therefore by (12), it is unique. It is given that $o(G = \{o\} \bigcup (X - M)))$ is not divisible by the square of any prime number. By (10) G is a p-semisimple BCI-algebra and by proposition 1, G is unique. Thus it follows that L = 1 and R = 1. Hence LR = 1 which shows that X is unique. □

EXAMPLE 1. Let $X = \{o, a, b, c, d, e, f, g\}$ be a S_3-algebra of order 8 with $M = \{o, a\}$ as its BCK-part. Then the BCI-part $X - M = \{b, c, d, e, f\}$. Thus, it follows $o(M) = 2$ and $o(X - M) = 6$ and $o(G) = o(\{o\} \bigcup \{X - M)) = 7$. By lemma (10) G is a p-semisimple BCI-algebra. Since, order of G is not divisible by the square of any prime number therefore by proposition 1, G is unique. Thus it follows that R = 1. Since a BCK-algebra of order 2 is a Semi-neutral BCK-algebra, therefore by (12) it is unique. So, L = 1. Hence LR = 1 which shows that X is unique. The multiplication table representing such S3-algebra is given as follows:

Table 2

*	o	a	b	c	d	e	f	g
o	o	o	g	f	e	d	c	b
a	a	o	g	f	e	d	c	b
b	b	b	o	g	f	e	d	c
c	c	c	b	o	g	f	e	d
d	d	d	c	b	o	g	f	e
e	e	e	d	c	b	o	g	f
f	f	f	e	d	c	b	o	g
g	g	g	f	e	d	c	b	o

Hence we have a unique S_3-algebra of order 8.

EXAMPLE 2. Let $X = \{o, a, b, c, d, e, f, g\}$ be a S_3-algebra of order 8 with $M = \{o, a, b\}$ as its BCK-part. Then the BCI-part $X - M = \{c, d, e, f, g\}$. Thus, it follows $o(M) = 3$ and $o(X - M) = 5$ and $o(G) = o(\{o\} \bigcup \{X - M)) = 5$. By (10) G is a p-semisimple BCI-algebra. Since, order of G is not divisible by the square of any prime number therefore by proposition 1, G is unique. Thus it follows that R = 1. As there are three BCK-algebras of order 3 (See [7]) so L = 3. Hence LR = 3 which shows that there are 3 such S_3-algebras. The multiplication tables representing such S_3-algebras are given as follows:

Table 3

*	o	a	b	c	d	e	f	g
o	o	o	o	c	e	d	g	f
a	a	o	o	c	e	d	g	f
b	b	a	o	c	e	d	g	f
c	c	c	c	o	g	f	e	d
d	d	d	d	f	o	g	c	e
e	e	e	e	g	f	o	d	c
f	f	f	f	d	c	e	o	d
g	g	g	g	e	d	c	f	o

Table 4

*	o	a	b	c	d	e	f	g
o	o	o	o	c	e	d	g	f
a	a	o	o	c	e	d	g	f
b	b	b	o	c	e	d	g	f
c	c	c	c	o	g	f	e	d
d	d	d	d	f	o	g	c	e
e	e	e	e	g	f	o	d	c
f	f	f	f	d	c	e	o	d
g	g	g	g	e	d	c	f	o

Table 5

*	o	a	b	c	d	e	f	g
o	o	o	o	c	e	d	g	f
a	a	o	a	c	e	d	g	f
b	b	b	o	c	e	d	g	f
c	c	c	c	o	g	f	e	d
d	d	d	d	f	o	g	c	e
e	e	e	e	g	f	o	d	c
f	f	f	f	d	c	e	o	d
g	g	g	g	e	d	c	f	o

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Farhat Nisar
Department of Mathematics
Queen Mary College
Lahore - Pakistan
E-mail: fhtnr2003@yahoo.com

Shaban Ali Bhatti
Department of Mathematics
University of the Punjab
Lahore - Pakistan
E-mail: shabanbhatti@math.pu.edu.pk