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INTERVAL-VALUED FUZZY STRONG SEMI-OPEN SETS

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Abstract. The notions of IVF strongly semiopen (semiclosed) sets and IVF (strong) semi-interior (IVF (strong) semi-closure) are introduced, and several examples are provided. Related properties are investigated. In particular, characterizations of an IVF strongly semiopen set are discussed.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [14], several researchers were concerned about the generalizations of the notion of fuzzy sets, e.g., fyzzy set of type n [15], intuitionistic fuzzy sets [1] and interval-valued fuzzy sets [5]. The concept of interval-valued fuzzy sets was introduced by Gorzałczany [5], and recently there has been progress in the study of such sets by several researchers (see [4], [7], [8], [9], [10], [11], [13]). Azad [2] introduced fuzzy semiopen (semiclosed) sets and fuzzy regular open (closed) sets, and then considered generalizations of semicontinuous mapping, semiopen mapping, semiclosed mapping, almost continuous mapping, and weakly continuous mapping in fuzzy setting. In [7], the topology of interval-valued fuzzy sets is defined, and some of its properties are discussed, and then Mondal et al. [8] studied the connectedness in the topology of interval-valued fuzzy sets. Using the concept of interval-valued fuzzy

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(IVF) sets, Jun et al. [6] introduced the notion of IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets, and then they investigated relationships between IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets. They also introduced the notion of IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF α-open mappings, and then they provided relationships between IVF open mappings, IVF preopen mappings, IVF semiopen mappings and IVF α-open mappings. In this paper, we introduce the notions of IVF strongly semiopen (semiclosed) sets and IVF (strong) semi-interior (IVF (strong) semi-closure), and then we investigate several properties. We provide characterizations of an IVF strongly semiopen set.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D[0,1])\ (M = N \Leftrightarrow M^L = N^L,\ M^U = N^U)$.
(ii) $(\forall M, N \in D[0,1])\ (M \leq N \Leftrightarrow M^L \leq N^L,\ M^U \leq N^U)$.

For every $M \in D[0,1]$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

Let X be a nonempty set. A mapping $\mathscr{A} : X \to D[0,1]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. For each $x \in X$, $\mathscr{A}(x)$ is a closed interval whose lower and upper end points are denoted by $\mathscr{A}(x)^L$ and $\mathscr{A}(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $a \in [0,1]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by $[a,b]_p$. In particular, if $b=a$, then it is also denoted by a_p. Denote by $IVF(X)$ the set of all IVF sets in X.

For every $\mathscr{A},\mathscr{B}\in IVF(X)$, we define

$$\mathscr{A}=\mathscr{B} \Leftrightarrow (\forall x\in X)\,([\mathscr{A}(x)]^L=[\mathscr{B}(x)]^L \quad \text{and} \quad [\mathscr{A}(x)]^U=[\mathscr{B}(x)]^U),$$

$$\mathscr{A}\subseteq\mathscr{B} \Leftrightarrow (\forall x\in X)\,([\mathscr{A}(x)]^L\leq[\mathscr{B}(x)]^L \quad \text{and} \quad [\mathscr{A}(x)]^U\leq[\mathscr{B}(x)]^U).$$

The *complement* $\mathscr{A}^c$ of $\mathscr{A}$ is defined by

$$[\mathscr{A}^c(x)]^L=1-[\mathscr{A}(x)]^U \quad \text{and} \quad [\mathscr{A}^c(x)]^U=1-[\mathscr{A}(x)]^L$$

for all $x\in X$. For a family of IVF sets $\{\mathscr{A}_i \mid i\in\Lambda\}$ where Λ is an index set, the *union* $G=\bigcup_{i\in\Lambda}\mathscr{A}_i$ and the *intersection* $F=\bigcap_{i\in\Lambda}\mathscr{A}_i$ are defined by

$$(\forall x\in X)\,([G(x)]^L=\sup_{i\in\Lambda}[\mathscr{A}_i(x)]^L,\ [G(x)]^U=\sup_{i\in\Lambda}[\mathscr{A}_i(x)]^U),$$

$$(\forall x\in X)\,([F(x)]^L=\inf_{i\in\Lambda}[\mathscr{A}_i(x)]^L,\ [F(x)]^U=\inf_{i\in\Lambda}[\mathscr{A}_i(x)]^U),$$

respectively.

Let $f:X\to Y$ be a mapping and let $\mathscr{A}$ be an IVF set in X. Then the *image* of $\mathscr{A}$ under f, denoted by $f(\mathscr{A})$, is defined as follows:

$$[f(\mathscr{A})(y)]^L=\begin{cases}\sup\limits_{y=f(x)}[\mathscr{A}(x)]^L, & f^{-1}(y)\neq\emptyset,\\ 0, & \text{otherwise,}\end{cases}$$

$$[f(\mathscr{A})(y)]^U=\begin{cases}\sup\limits_{y=f(x)}[\mathscr{A}(x)]^U, & f^{-1}(y)\neq\emptyset,\\ 0, & \text{otherwise,}\end{cases}$$

for all $y\in Y$. Let $\mathscr{B}$ be an IVF set in Y. Then the it inverse image of $\mathscr{B}$ under f, denoted by $f^{-1}(\mathscr{B})$, is defined as follows:

$$(\forall x\in X)\,([f^{-1}(\mathscr{B})(x)]^L=[\mathscr{B}(f(x))]^L,\ [f^{-1}(\mathscr{B})(x)]^U=[\mathscr{B}(f(x))]^U).$$

Definition 2.1. [7] A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (briefly, *IVF topology*) for X if it satisfies:

(i) $\tilde{0},\tilde{1}\in\tau$,

(ii) $\mathscr{A},\mathscr{B}\in\tau \Rightarrow \mathscr{A}\cap\mathscr{B}\in\tau$,

(iii) $\mathscr{A}_i\in\tau, i\in\Delta \Rightarrow \bigcup_{i\in\Delta}\mathscr{A}_i\in\tau$.

Every member of τ is called an *IVF open set.* An IVF set $\mathscr{A}$ in X is called an *IVF closed set* if the complement of $\mathscr{A}$ is an IVF open set, that is, $\mathscr{A}^c \in \tau$. Moreover, (X, τ) is called an *interval-valued fuzzy topological space* (briefly, *IVF topological space*).

DEFINITION 2.2. [7] Let (X, τ) and (Y, κ) be IVF topological spaces. A mapping $f : X \to Y$ is said to be *continuous* if $f^{-1}(\mathscr{A}) \in \tau$ for all $\mathscr{A} \in \kappa$.

For an IVF set $\mathscr{A}$ in an IVF topological space (X, τ), the *IVF closure* and the *IVF interior* of $\mathscr{A}$, denoted by $\mathscr{A}^-$ and $\mathscr{A}^\circ$, respectively, are defined as

$$\mathscr{A}^- = \cap\{\mathscr{B} \in IVF(X) \mid \mathscr{B} \text{ is IVF closed and } \mathscr{A} \subset \mathscr{B}\},$$

$$\mathscr{A}^\circ = \cup\{\mathscr{B} \in IVF(X) \mid \mathscr{B} \text{ is IVF open and } \mathscr{B} \subset \mathscr{A}\},$$

respectively. Note that $\mathscr{A}^\circ$ is the largest IVF open set which is contained in $\mathscr{A}$, and that $\mathscr{A}$ is IVF open if and only if $\mathscr{A} = \mathscr{A}^\circ$.

DEFINITION 2.3. [6] An IVF set $\mathscr{A}$ in (X, τ) is called an *IVF semiopen set* in (X, τ) if it satisfies:

$$(\exists \mathscr{B} \in \tau)\,(\mathscr{B} \subseteq \mathscr{A} \subseteq \mathscr{B}^-);$$

and an *IVF semiclosed set* in (X, τ) if it satisfies:

$$(\exists \mathscr{B}^c \in \tau)\,(\mathscr{B}^\circ \subseteq \mathscr{A} \subseteq \mathscr{B}).$$

Denote by $IVFSO(X)$ (resp. $IVFSC(X)$) the set of all IVF semiopen sets (resp. IVF semiclosed sets) in (X, τ).

DEFINITION 2.4. [6] An IVF set $\mathscr{A}$ in (X, τ) is called an *IVF preopen set* in (X, τ) if $\mathscr{A} \subseteq \mathscr{A}^{-\circ}$; and is called an *IVF preclosed set* in (X, τ) if $\mathscr{A}^{\circ -} \subseteq \mathscr{A}$.

Denote by $IVFPO(X)$ (resp. $IVFPC(X)$) the set of all IVF preopen sets (resp. IVF preclosed sets) in (X, τ).

3. Interval-valued fuzzy strongly semiopen sets

In what follows let (X, τ) denote an IVF topological space unless otherwise specified.

Definition 3.1. An IVF set $\mathscr{A}$ in (X,τ) is called

(1) an *IVF strongly semiopen set* of (X,τ) if it satisfies:

$$(\exists \mathscr{B} \in \tau)\,(\mathscr{B} \subseteq \mathscr{A} \subseteq \mathscr{B}^{-\circ});$$

(2) an *IVF strongly semiclosed set* of (X,τ) if there is an IVF closed set $\mathscr{B}$ in X such that $\mathscr{B}^{\circ -} \subseteq \mathscr{A} \subseteq \mathscr{B}$.

Denote by $IVFSSO(X)$ (resp. $IVFSSC(X)$) the set of all IVF strongly semiopen (resp. semiclosed) sets of (X,τ).

Example 3.2. We list some examples of IVF strongly semiopen (semiclosed) sets.

(1) Every IVF open (closed) set is an IVF strongly semiopen (semiclosed) set.
(2) Every IVF strongly semiopen set is not only an IVF semiopen set but also an IVF preopen set.
(3) Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be IVF sets in $I=[0,1]$ defined by

$$\mathscr{A}(x) = \begin{cases} [x, x+\frac{1}{8}], & 0 \le x \le \frac{1}{2}, \\ [1-x, \frac{9}{8}-x], & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\mathscr{B}(x) = \begin{cases} \mathbf{1}, & 0 \le x \le \frac{1}{2}, \\ \mathbf{\frac{5}{8}}, & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mathscr{C}(x) = \begin{cases} [x, x+\frac{1}{8}], & 0 \le x < \frac{7}{8}, \\ \mathbf{1}, & \frac{7}{8} \le x \le 1. \end{cases}$$

The collection $\tau = \{\tilde{0}, \mathscr{A}, \tilde{1}\}$ is an IVF topology for I. The IVF sets $\mathscr{B}$ and $\mathscr{C}$ are IVF strongly semiopen sets in (I,τ), and $\mathscr{B}\cap\mathscr{C}$ is also an IVF strongly semiopen sets in (I,τ).

The following example shows that the converse of Examples 3.2(1) and 3.2(2) are not valid in general. Before providing examples, we first state the following lemma.

Lemma 3.3. [6] *Let A be an IVF set in (X,τ). Then*

(i) $\mathscr{A} \in IVFSO(X) \Leftrightarrow \mathscr{A} \subseteq \mathscr{A}^{\circ -}$,
(ii) $\mathscr{A} \in IVFSC(X) \Leftrightarrow \mathscr{A}^{-\circ} \subseteq \mathscr{A}$.

EXAMPLE 3.4. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ be IVF sets in $I = [0,1]$ defined by

$$\mathscr{A}(x) = \begin{cases} [0, \frac{1}{8}], & 0 \le x \le \frac{1}{2}, \\ [\frac{7}{3}x - \frac{7}{6}, \frac{7}{3}x - \frac{25}{24}], & \frac{1}{2} \le x \le \frac{7}{8}, \\ [\frac{7}{3}x - \frac{7}{6}, 1], & \frac{7}{8} \le x \le 1, \end{cases}$$

$$\mathscr{B}(x) = \begin{cases} \mathbf{1}, & 0 \le x < \frac{1}{4}, \\ [-\frac{7}{2}x + \frac{7}{4}, -\frac{7}{2}x + \frac{15}{8}], & \frac{1}{4} \le x \le \frac{1}{2}, \\ \mathbf{0}, & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mathscr{C}(x) = \begin{cases} [0, \frac{1}{8}], & 0 \le x \le \frac{1}{4}, \\ [\frac{7}{5}x - \frac{7}{20}, \frac{7}{5}x - \frac{9}{40}], & \frac{1}{4} \le x \le \frac{7}{8}, \\ [x, 1], & \frac{7}{8} \le x \le 1, \end{cases}$$

$$\mathscr{D}(x) = \begin{cases} [0, \frac{1}{8}], & 0 \le x \le \frac{1}{8}, \\ [\frac{7}{6}x - \frac{7}{48}, \frac{7}{6}x - \frac{1}{48}], & \frac{1}{8} \le x \le \frac{7}{8}, \\ [x, 1], & \frac{7}{8} \le x \le 1. \end{cases}$$

The collection $\tau_1 = \{\tilde{0}, \mathscr{A}, \tilde{1}\}$ is an IVF topology for I. We can verify that $\mathscr{A} \subseteq \mathscr{C} \subseteq \tilde{1} = \mathscr{A}^{-\circ}$ so that $\mathscr{C}$ is an IVF strongly semiopen but it is not IVF open. Now we take an IVF topology $\tau_2 = \{\tilde{0}, \mathscr{A}, \mathscr{B}, \mathscr{A} \cup \mathscr{B}, \mathscr{A} \cap \mathscr{B}, \tilde{1}\}$ on $I = [0,1]$. We can easily verify that $\mathscr{D} \subseteq \mathscr{D}^{\circ -}$ so from Lemma 3.3 that $\mathscr{D}$ is an IVF semiopen set in (I, τ_2). But $\mathscr{D}$ is not IVF strongly semiopen because there is no $\mathscr{E} \in \tau_2$ such that $\mathscr{E} \subseteq \mathscr{D} \subseteq \mathscr{E}^{-\circ}$. Let $\tau_3 = \{\tilde{0}, \mathscr{C}, \tilde{1}\}$ be an IVF topology on $I = [0,1]$. We can easily show that $\mathscr{A} \subseteq \tilde{1} = \mathscr{A}^{-\circ}$ so that $\mathscr{A}$ is an IVF preopen set in (I, τ_3). But it is not an IVF strongly semiopen set in (I, τ_3).

LEMMA 3.5. *For an IVF set $\mathscr{A}$ in (X, τ), we have*

$$(\mathscr{A}^\circ)^c = (\mathscr{A}^c)^- \quad \text{and} \quad (\mathscr{A}^-)^c = (\mathscr{A}^c)^\circ.$$

Proof. Straightforward. □

THEOREM 3.6. *For an IVF set $\mathscr{A}$ in (X, τ), the following assertions are equivalent.*

(i) *$\mathscr{A}$ is an IVF strongly semiopen set.*
(ii) *$\mathscr{A} \subseteq \mathscr{A}^{\circ - \circ}$.*
(iii) *$\mathscr{A}$ is both an IVF semiopen set and an IVF preopen set.*

(iv) $\mathscr{A}^c$ *is an IVF strongly semiclosed set.*
(v) $(\mathscr{A}^c)^{-\circ-} \subseteq \mathscr{A}^c$.
(vi) $\mathscr{A}^c$ *is both an IVF semiclosed set and an IVF preclosed set.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathscr{A}$ is an IVF strongly semiopen set in X. Then there exists $\mathscr{B} \in \tau$ such that $\mathscr{B} \subseteq \mathscr{A} \subseteq \mathscr{B}^{-\circ}$. Hence $\mathscr{B} \subseteq \mathscr{A}^\circ$, and hence $\mathscr{B}^{-\circ} \subseteq \mathscr{A}^{\circ-\circ}$. Since $\mathscr{A} \subseteq \mathscr{B}^{-\circ}$, it follows that $\mathscr{A} \subseteq \mathscr{A}^{\circ-\circ}$. Hence (ii) is valid.

(ii) $\Rightarrow$ (i) Assume that $\mathscr{A} \subseteq \mathscr{A}^{\circ-\circ}$. Let $\mathscr{B} = \mathscr{A}^\circ$. Then $\mathscr{B} \in \tau$. It follows from (ii) that

$$\mathscr{B} = \mathscr{A}^\circ \subseteq \mathscr{A} \subseteq \mathscr{A}^{\circ-\circ} = \mathscr{B}^{-\circ}$$

so that $\mathscr{A}$ is an IVF strongly semiopen set in X.

(i) $\Rightarrow$ (iii) See Example 3.2(2).

(iii) $\Rightarrow$ (ii) Let $\mathscr{A} \in IVFSO(X) \cap IVFPO(X)$. Since $\mathscr{A} \in IVFPO(X)$, we have $\mathscr{A} \subseteq \mathscr{A}^{-\circ}$. Now $\mathscr{A} \in IVFSO(X)$ implies that $\mathscr{A} \subseteq \mathscr{A}^{\circ-}$. Hence

$$\mathscr{A} \subseteq \mathscr{A}^{-\circ} \subseteq (\mathscr{A}^{\circ-})^{-\circ} = \mathscr{A}^{\circ-\circ}.$$

(i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (v) and (iii) $\Leftrightarrow$ (vi) follows from Lemma 3.5.

(v) $\Rightarrow$ (vi) Assume that (v) is valid. Let $\mathscr{B} = (\mathscr{A}^c)^-$. Then

$$\mathscr{B}^\circ = (\mathscr{A}^c)^{-\circ} \subseteq (\mathscr{A}^c)^{-\circ-} \subseteq \mathscr{A}^c \subseteq (\mathscr{A}^c)^- = \mathscr{B}.$$

Since $\mathscr{B}^c \in \tau$, it follows that $\mathscr{A}^c$ is an IVF semiclosed set in (X, τ). Note that $(\mathscr{A}^c)^{\circ-} \subseteq (\mathscr{A}^c)^{-\circ-} \subseteq \mathscr{A}^c$. Hence $\mathscr{A}^c$ is an IVF preclosed set in (X, τ).

(vi) $\Rightarrow$ (v) Let $\mathscr{A}^c$ be both an IVF semiclosed set and an IVF preclosed set in (X, τ). Then there exists an IVF set $\mathscr{B}$ in X such that $\mathscr{B}^c \in \tau$ and $\mathscr{B}^\circ \subseteq \mathscr{A}^c \subseteq \mathscr{B}$. Moreover, $(\mathscr{A}^c)^{\circ-} \subseteq \mathscr{A}^c$. Since $\mathscr{B}$ is an IVF closed set in (X, τ), it follows that

$$\begin{aligned} \mathscr{A}^c \subseteq \mathscr{B} \;&\Rightarrow\; (\mathscr{A}^c)^- \subseteq \mathscr{B}^- = \mathscr{B} \\ &\Rightarrow\; (\mathscr{A}^c)^{-\circ} \subseteq \mathscr{B}^\circ \subseteq \mathscr{A}^c \\ &\Rightarrow\; (\mathscr{A}^c)^{-\circ} = (\mathscr{A}^c)^{-\circ\circ} \subseteq (\mathscr{A}^c)^\circ \\ &\Rightarrow\; (\mathscr{A}^c)^{-\circ-} \subseteq (\mathscr{A}^c)^{\circ-} \subseteq \mathscr{A}^c \end{aligned}$$

which proves (v). □

Lemma 3.7. [6, Lemma 3.4] *For a family of* $\{\mathscr{A}_k \mid k \in \Delta\}$ *of IVF sets in* (X, τ), *we have* $\cup\mathscr{A}_k^- \subseteq (\cup\mathscr{A}_k)^-$ *and* $\cup\mathscr{A}_k^\circ \subseteq (\cup\mathscr{A}_k)^\circ$.

THEOREM 3.8. *Let* $\{\mathscr{A}_k \mid k \in \triangle\}$ *be a collection of IVF strongly semiopen sets in* (X, τ)*. Then* $\bigcup_{k\in\triangle} \mathscr{A}_k$ *is an IVF strongly semiopen set in* (X, τ)*, where* $\triangle$ *is any index set.*

Proof. Let $\mathscr{A}_k \in IVFSSO(X)$ for $k \in \Delta$. Then

$$(\forall k \in \Delta)\,(\exists \mathscr{B}_k \in \tau)\,(\mathscr{B}_k \subseteq \mathscr{A}_k \subseteq \mathscr{B}_k^{-\circ}).$$

It follows from Lemma 3.7 that

$$\bigcup_{k\in\Delta} \mathscr{B}_k \subseteq \bigcup_{k\in\Delta} \mathscr{A}_k \subseteq \bigcup_{k\in\Delta} \mathscr{B}_k^{-\circ} \subseteq (\bigcup_{k\in\Delta} \mathscr{B}_k^{-})^{\circ} \subseteq (\bigcup_{k\in\Delta} \mathscr{B}_k)^{-\circ}$$

so from $\bigcup_{k\in\Delta} \mathscr{B}_k \in \tau$ that $\bigcup_{k\in\Delta} \mathscr{A}_k \in IVFSSO(X)$. □

Similarly we have

THEOREM 3.9. *If* $\{\mathscr{A}_k \mid k \in \triangle\}$ *is a collection of IVF strongly semiclosed sets in* (X, τ)*, then* $\bigcap_{k\in\triangle} \mathscr{A}_k$ *is an IVF strongly semiclosed set in* (X, τ)*, where* $\triangle$ *is any index set.*

The intersection (resp. union) of two IVF strongly semiopen (resp. semiclosed) sets need not be an IVF strongly semiopen (semiclosed) set as seen in the following example.

EXAMPLE 3.10. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be IVF sets in $I = [0, 1]$ defined by

$$\mathscr{A}(x) = \begin{cases} [0.3, 0.4], & 0 \le x \le \frac{1}{2}, \\ [0.2, 0.3], & \frac{1}{2} < x < \frac{3}{4}, \\ [0.7, 0.8], & \frac{3}{4} \le x \le 1, \end{cases}$$

$$\mathscr{B}(x) = \begin{cases} [0.8, 0.9], & 0 \le x \le \frac{1}{2}, \\ [0.9, 1], & \frac{1}{2} < x < \frac{3}{4}, \\ [0.4, 0.5], & \frac{3}{4} \le x \le 1, \end{cases}$$

$$\mathscr{C}(x) = \begin{cases} [0.3, 0.4], & 0 \le x \le \frac{1}{2}, \\ [0.2, 0.3], & \frac{1}{2} < x < \frac{3}{4}, \\ [0.8, 0.8], & \frac{3}{4} \le x \le 1, \end{cases}$$

$$\mathscr{D}(x) = \begin{cases} [0.8, 0.9], & 0 \leq x \leq \frac{1}{2}, \\ [0.9, 1], & \frac{1}{2} < x < \frac{3}{4}, \\ [0.5, 0.6], & \frac{3}{4} \leq x \leq 1. \end{cases}$$

The collection $\tau = \{\tilde{0}, \mathscr{A}, \mathscr{B}, \mathscr{A} \cap \mathscr{B}, \mathscr{A} \cup \mathscr{B}, \tilde{1}\}$ is an IVF topology for I. We can verify that $\mathscr{C}$ and $\mathscr{D}$ are IVF strongly semiopen sets in I. But $\mathscr{C} \cap \mathscr{D}$ is not an IVF strongly semiopen set in I because

$$\begin{aligned} &\tilde{0} \subseteq \mathscr{C} \cap \mathscr{D} \nsubseteq \tilde{0} = \tilde{0}^{-\circ}, \\ &\mathscr{A} \nsubseteq \mathscr{C} \cap \mathscr{D} \subseteq \mathscr{A}^{-\circ} = \tilde{1}, \\ &\mathscr{B} \nsubseteq \mathscr{C} \cap \mathscr{D} \subseteq \mathscr{B}^{-\circ} = \tilde{1}, \\ &\mathscr{A} \cap \mathscr{B} \subseteq \mathscr{C} \cap \mathscr{D} \nsubseteq (\mathscr{A} \cap \mathscr{B})^{-\circ} = \mathscr{A} \cap \mathscr{B}, \\ &\mathscr{A} \cup \mathscr{B} \nsubseteq \mathscr{C} \cap \mathscr{D} \subseteq (\mathscr{A} \cup \mathscr{B})^{-\circ} = \tilde{1}. \end{aligned}$$

We have a question: Is the intersection (resp. union) of an IVF strongly semiopen (resp. semiclosed) set with an IVF open (resp. closed) set an IVF strongly semiopen (resp. semiclosed) set? But the answer is negative as seen in the following example.

Example 3.11. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be IVF sets in $I = [0, 1]$ which are described in Example 3.10. Consider an IVF topology $\tau = \{\tilde{0}, \mathscr{A}, \mathscr{B}, \mathscr{A} \cap \mathscr{B}, \mathscr{A} \cup \mathscr{B}, \tilde{1}\}$ in I. Then $\mathscr{A}$ is an IVF open set in I, and $\mathscr{D}$ is an IVF strongly semiopen set (since $\mathscr{B} \subseteq \mathscr{D} \subseteq \tilde{1} = \mathscr{B}^{-\circ}$) in I which is not an IVF open set in I. We note that

$$\begin{aligned} &\tilde{0} \subseteq \mathscr{A} \cap \mathscr{D} \nsubseteq \tilde{0} = \tilde{0}^{-\circ}, \\ &\mathscr{A} \nsubseteq \mathscr{A} \cap \mathscr{D} \subseteq \mathscr{A}^{-\circ} = \tilde{1}, \\ &\mathscr{B} \nsubseteq \mathscr{A} \cap \mathscr{D} \subseteq \mathscr{B}^{-\circ} = \tilde{1}, \\ &\mathscr{A} \cap \mathscr{B} \subseteq \mathscr{A} \cap \mathscr{D} \nsubseteq (\mathscr{A} \cap \mathscr{B})^{-\circ} = \mathscr{A} \cap \mathscr{B}, \\ &\mathscr{A} \cup \mathscr{B} \nsubseteq \mathscr{A} \cap \mathscr{D} \subseteq (\mathscr{A} \cup \mathscr{B})^{-\circ} = \tilde{1}. \end{aligned}$$

Hence $\mathscr{A} \cap \mathscr{D}$ is not an IVF strongly semiopen set in I.

Definition 3.12. Let $\mathscr{A}$ be an IVF set in (X, τ) and define the following sets:

$$\mathscr{A}^{\ominus} = \cup\{\mathscr{B} \mid \mathscr{B} \subseteq \mathscr{A}, \ \ \mathscr{B} \text{ is IVF semiopen}\}$$

$$\mathscr{A}^{\sim} = \cap\{\mathscr{B} \mid \mathscr{A} \subseteq \mathscr{B}, \ \ \mathscr{B} \text{ is IVF semiclosed}\}.$$

We call $\mathscr{A}^{\ominus}$ and $\mathscr{A}^{\sim}$ the *IVF semi-interior* and *IVF semi-closure* of $\mathscr{A}$, respectively.

REMARK 3.13. Obviously, $\mathscr{A}^{\ominus}$ is the greatest IVF semiopen set which is contained in $\mathscr{A}$ and $\mathscr{A}^{\sim}$ is the least IVF semiclosed set which contains $\mathscr{A}$.

DEFINITION 3.14. Let $\mathscr{A}$ be an IVF set in (X, τ). We define the *IVF strong semi-interior* and the *IVF strong semi-closure* of $\mathscr{A}$, written by $\mathscr{A}_{\circ}$ and $\mathscr{A}_{-}$, respectively, by

$$\mathscr{A}_{\circ} = \cup\{\mathscr{B} \mid \mathscr{B} \in IVFSSO(X),\, \mathscr{B} \subseteq \mathscr{A}\},$$

$$\mathscr{A}_{-} = \cap\{\mathscr{B} \mid \mathscr{B}^{c} \in IVFSSO(X),\, \mathscr{A} \subseteq \mathscr{B}\},$$

respectively.

EXAMPLE 3.15. Let $\mathscr{A}$ and $\mathscr{B}$ be IVF sets in $I = [0, 1]$ defined by

$$\mathscr{A}(x) = \begin{cases} [\frac{1}{8}, \frac{1}{4}], & 0 \le x \le \frac{1}{2}, \\ [\frac{1}{4}, \frac{3}{8}], & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mathscr{B}(x) = \begin{cases} [\frac{3}{8}, \frac{1}{2}], & 0 \le x \le \frac{1}{2}, \\ [\frac{1}{4}, \frac{3}{8}], & \frac{1}{2} < x \le 1. \end{cases}$$

The collection $\tau = \{\tilde{0}, \mathscr{A}, \mathscr{B}, \tilde{1}\}$ is an IVF topology for I. Let $\mathscr{D}$ be an IVF set in I defined by

$$\mathscr{D}(x) = \begin{cases} [\frac{1}{4} + \frac{3}{4}x, \frac{5}{16} + \frac{3}{4}x], & 0 \le x \le \frac{1}{2}, \\ [\frac{7}{8} - \frac{1}{2}x, \frac{15}{16} - \frac{1}{2}x], & \frac{1}{2} \le x \le 1. \end{cases}$$

(1) We know that $IVFSO(I) = \{\tilde{0}, \tilde{1}, \mathscr{C} \mid \mathscr{C} \in IVF(I),\, \mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{B}^{c}\}$ and $IVFSC(I) = \{\tilde{0}, \tilde{1}, \mathscr{C} \mid \mathscr{C} \in IVF(I),\, \mathscr{B} \subseteq \mathscr{C} \subseteq \mathscr{A}^{c}\}$. Then the IVF semi-interior $\mathscr{D}^{\ominus}$ and the IVF semi-closure $\mathscr{D}^{\sim}$ of $\mathscr{D}$ are given by

$$\mathscr{D}^{\ominus}(x) = \begin{cases} [\frac{1}{4} + \frac{3}{4}x, \frac{5}{16} + \frac{3}{4}x], & 0 \le x \le \frac{1}{3}, \\ [\frac{1}{2}, \frac{5}{16} + \frac{3}{4}x], & \frac{1}{3} \le x \le \frac{5}{12}, \\ [\frac{1}{2}, \frac{5}{8}], & \frac{5}{12} \le x \le \frac{1}{2}, \\ [\frac{7}{8} - \frac{1}{2}x, \frac{15}{16} - \frac{1}{2}x], & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mathscr{D}^{\sim}(x) = \begin{cases} [\frac{3}{8}, \frac{1}{2}], & 0 \le x \le \frac{1}{6}, \\ [\frac{1}{4} + \frac{3}{4}x, \frac{1}{2}], & \frac{1}{6} \le x \le \frac{1}{4}, \\ [\frac{1}{4} + \frac{3}{4}x, \frac{5}{16} + \frac{3}{4}x], & \frac{1}{4} \le x \le \frac{1}{2}, \\ [\frac{7}{8} - \frac{1}{2}x, \frac{15}{16} - \frac{1}{2}x], & \frac{1}{2} \le x \le 1, \end{cases}$$

respectively.

(2) We know that $IVFSSO(I) = \{\tilde{0}, \tilde{1}, \mathscr{C} \mid \mathscr{C} \in IVF(I),\ \mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{B}\}$ and $IVFSSC(I) = \{\tilde{0}, \tilde{1}, \mathscr{C} \mid \mathscr{C} \in IVF(I),\ \mathscr{B}^c \subseteq \mathscr{C} \subseteq \mathscr{A}^c\}$. Then the IVF strong semi-interior $\mathscr{D}_\circ$ and the IVF strong semi-closure $\mathscr{D}_-$ of $\mathscr{D}$ are given by

$$\mathscr{D}_\circ(x) = \begin{cases} [\frac{1}{4} + \frac{3}{4}x, \frac{5}{16} + \frac{3}{4}x], & 0 \le x \le \frac{1}{6}, \\ [\frac{3}{8}, \frac{5}{16} + \frac{3}{4}x], & \frac{1}{6} \le x \le \frac{1}{4}, \\ [\frac{3}{8}, \frac{1}{2}], & \frac{1}{4} \le x \le \frac{1}{2}, \\ [\frac{1}{4}, \frac{3}{8}], & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mathscr{D}_-(x) = \begin{cases} [\frac{1}{2}, \frac{5}{8}], & 0 \le x \le \frac{1}{3}, \\ [\frac{1}{4} + \frac{3}{4}x, \frac{5}{8}], & \frac{1}{3} \le x \le \frac{5}{12}, \\ [\frac{1}{4} + \frac{3}{4}x, \frac{5}{16} + \frac{3}{4}x], & \frac{5}{12} \le x \le \frac{1}{2}, \\ [\frac{5}{8}, \frac{3}{4}], & \frac{1}{2} < x \le 1, \end{cases}$$

respectively.

PROPOSITION 3.16. *Let $\mathscr{A}$ and $\mathscr{B}$ be IVF sets in (X, τ). Then the following assertions are valid:*

(i) $\mathscr{A}^\circ \subseteq \mathscr{A}_\circ \subseteq \mathscr{A}^\ominus \subseteq \mathscr{A} \subseteq \mathscr{A}^\sim \subseteq \mathscr{A}_- \subseteq \mathscr{A}^-$.
(ii) $\mathscr{A} \subseteq \mathscr{B} \Rightarrow \mathscr{A}_\circ \subseteq \mathscr{B}_\circ$, $\mathscr{A}_- \subseteq \mathscr{B}_-$.
(iii) $\tilde{0}_\circ = \tilde{0}$, $\tilde{0}_- = \tilde{0}$, $\tilde{1}_\circ = \tilde{1}$, $\tilde{1}_- = \tilde{1}$.
(iv) $(\mathscr{A}_\circ)_\circ = \mathscr{A}_\circ$, $(\mathscr{A}_-)_- = \mathscr{A}_-$.
(v) $\mathscr{A} \in IVFSSO(X) \Leftrightarrow \mathscr{A}_\circ = \mathscr{A}$.
(vi) $\mathscr{A} \in IVFSSC(X) \Leftrightarrow \mathscr{A}_- = \mathscr{A}$.
(vii) $(\mathscr{A} \cap \mathscr{B})_\circ \subseteq \mathscr{A}_\circ \cap \mathscr{B}_\circ$.
(viii) $(\mathscr{A} \cup \mathscr{B})_- \supseteq \mathscr{A}_- \cup \mathscr{B}_-$.

Proof. This is immediate from Definition 3.14. □

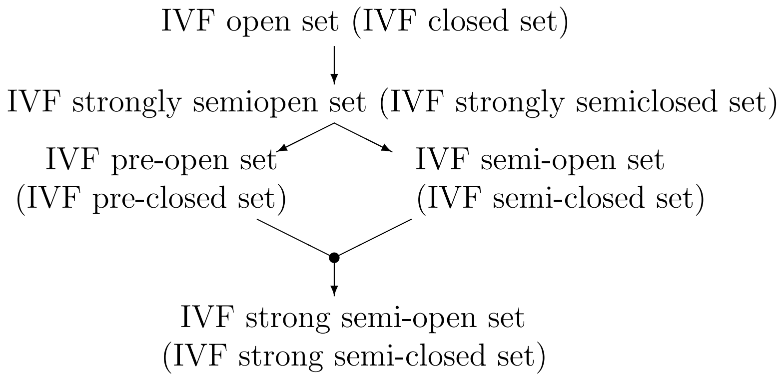
THEOREM 3.17. *For an IVF set $\mathscr{A}$ in (X, τ), we have*

(i) $(\mathscr{A}_\circ)^c = (\mathscr{A}^c)_-$.
(ii) $(\mathscr{A}_-)^c = (\mathscr{A}^c)_\circ$.

Proof. (i) Note that $\mathscr{A}_\circ \subseteq \mathscr{A}$ by Proposition 3.16(i). Thus $\mathscr{A}^c \subseteq (\mathscr{A}_\circ)^c$. Since $\mathscr{A}_\circ$ is an IVF strongly semiopen set, $(\mathscr{A}_\circ)^c$ is an IVF strongly semiclosed set in X. Hence $(\mathscr{A}^c)_- \subseteq (\mathscr{A}_\circ)^c$. Now $\mathscr{A}^c \subseteq (\mathscr{A}^c)_-$ by Proposition 3.16(i), and $(\mathscr{A}^c)_-$ is an IVF strongly semiclosed set in X. It follows that $((\mathscr{A}^c)_-)^c \subseteq \mathscr{A}$ and $((\mathscr{A}^c)_-)^c$ is an IVF strongly semiopen set in X so that $((\mathscr{A}^c)_-)^c \subseteq \mathscr{A}_\circ$; whence $(\mathscr{A}_\circ)^c \subseteq (\mathscr{A}^c)_-$. Therefore (i) is valid.

(ii) Notice that $\mathscr{A} \subseteq \mathscr{A}_-$ by Proposition 3.16(i), which implies that $(\mathscr{A}_-)^c \subseteq \mathscr{A}^c$. Since $\mathscr{A}_-$ is an IVF strongly semiclosed set in X, $(\mathscr{A}_-)^c$ is an IVF strongly semiopen set in X. Thus $(\mathscr{A}_-)^c \subseteq (\mathscr{A}^c)_\circ$. On the other hand, $(\mathscr{A}^c)_\circ \subseteq \mathscr{A}^c$ by Proposition 3.16(i), and hence $\mathscr{A} \subseteq ((\mathscr{A}^c)_\circ)^c$. Since $(\mathscr{A}^c)_\circ$ is an IVF strongly semiopen set, $((\mathscr{A}^c)_\circ)^c$ is an IVF strongly semiclosed set in X. Since $\mathscr{A}_-$ is the smallest IVF strongly semiclosed set containing $\mathscr{A}$, it follows that $\mathscr{A}_- \subseteq ((\mathscr{A}^c)_\circ)^c$ so that $(\mathscr{A}^c)_\circ \subseteq (\mathscr{A}_-)^c$. Thus (ii) is valid. □

We have the following diagram described relations between IVF open (IVF closed) set, IVF strongly semiopen (IVF strongly semiclosed) set, IVF semi-open (IVF semi-closed) set, IVF pre-open (IVF pre-closed) set, and IVF strong semi-open (IVF strong semi-closed) set which reverse implications do not valid.



4. Conclusions

As a generalization of fuzzy sets, the notion of interval-valued fuzzy sets was introduced by Gorzałczany [5], and recently there has been progress in the study of such sets by several researchers (see [4], [7], [8], [9], [10], [11], [13]). Using the concept of interval-valued fuzzy (IVF) sets, Jun et al. [6] introduced the notion of IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets, and investigated relationships between IVF semiopen (semiclosed) sets, IVF preopen (preclosed) sets and IVF α-open (α-closed) sets. The aim of this paper is to introduce the notions of IVF strongly semiopen (semiclosed) sets and IVF (strong) semi-interior (IVF (strong) semi-closure), and to investigate several properties. Characterizations of an IVF strongly semiopen set are provided.

Future work will focus on studying the interval-valued fuzzy strong semi-open mappings and the interval-valued fuzzy strong semi-continuity.

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