East Asian Math. J. **23** (2007), No. 1, pp. 9–22

ON UPPER AND LOWER SEMI-IRRESOLUTE FUZZY MULTIFUNCTIONS

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Abstract. New classes of multifunctions called fuzzy upper and fuzzy lower semi-irresolute (semi-continuous) multifunctions in fuzzy topological spaces are introduced in this paper. We also obtain some characterizations of this class and some basic interesting properties of such fuzzy multifunctions. We discuss mutual relationship and also relationship with other existing such multifunctions.

1. Introduction

The fundamental concept of the fuzzy sets was first introduced by Zadeh in his classical paper [20] of 1965. The theory of fuzzy topological spaces was introduced and developed by Chang [5] and since then various notions in classical topology have been extended to fuzzy topological spaces. In 1985 Papageorgious [11] introduced the notion of a fuzzy multifunctions and extended the concept of fuzzy continuous functions of the fuzzy multivalued case by the introduction of fuzzy upper and lower semi-continuous multifunctions. In [10], M.N. Mukherjee and S. Malakar have redefined the fuzzy lower inverse and lower semi-continuity of a fuzzy multifunctions in terms of the notions of quasi-coincidence of Pu and Liu [12, 13]. The concepts of the upper and lower preirresolute multifunctions was introduced in [17]. In this paper we introduce and study the fuzzy upper and lower

Received December 1, 2005. Revised March 26, 2007.
2000 Mathematics Subject Classification: 54A40.
Key words and phrases: Fuzzy lower and upper inverse of a fuzzy multifunction; Fuzzy lower and upper semi-irresolute (semi-continuous) fuzzy multifunctions.

semi-irresolute (semi-continuous) multifunctions and the definition we follow is more general than that of [10].

2. Preliminaries

Throughout this paper, by $(X,\ T)$ or simply by X we will mean a topological space in the classical sense, and $(Y,\ T_1)$ or simply Y will stand for a fuzzy topological space (fts. for short) as defined by Chang [5]. Fuzzy sets in Y will be denoted by $\lambda, \gamma, \delta, \rho$, etc., and although subsets of X will be denoted by A, B, U, V, etc. A fuzzy point in Y with support $y \in Y$ and value $\alpha (0 < \alpha \leq 1)$ is denoted by y_α . Let λ be fuzzy set in Y and y_α a fuzzy point in Y. We say that $y_\alpha \in \lambda$ if and only if $y_\alpha \leq \lambda$. The value of a fuzzy set λ at some $y \in Y$ will be denoted by $\lambda(y)$, and the support of a fuzzy set λ in Y, that is, the set of all fuzzy points y of Y at which $\lambda(y) \neq 0$. A fuzzy set λ in Y is said to be *quasi-coincident* (q-coincident) with a fuzzy set μ, denoted by $\lambda q \mu$, if and only if there exists $y \in Y$ such that $\lambda(y) + \mu(y) > 1$ [12]. If they are not q-coincident, we shall write $\lambda \not{q} \mu$. $\lambda \not{q} \mu$ if and only if $\lambda(y) \leq 1 - \mu(y)$. It is known that [12] $\lambda \leq \mu$ if and only if λ and $1 - \mu$ are not q-coincident, denoted by $\lambda \not{q} (1 - \mu)$. μ is called a *quasi-neighbourhood* (q-neighbourhood, q nbds. for short) of λ if and only if there exists a fuzzy open set δ in Y such that $\lambda q \delta \leq \mu$ [12]. A fuzzy set λ of Y is called a *fuzzy semi-neighbourhood* of a fuzzy point y_α in Y if there exists a fuzzy semi-open set μ in Y such that $y_\alpha \in \mu \leq \lambda$. A fuzzy set λ in an fts Y is called *fuzzy semi-open set* (resp. fuzzy α-open set) if $\lambda \leq \text{cl int } \lambda$ (resp. $\lambda \leq \text{int cl int } \lambda$), and complement of such sets is called fuzzy semi-closed set (resp. fuzzy α-closed set). The intersection of all fuzzy semi-closed sets of Y containing λ is called the *fuzzy semi-closure* of λ and is denoted by $\text{fscl}(\lambda)$. The union of all fuzzy semi-open sets contained in λ is called the *fuzzy semi-interior* of λ and is denoted by $\text{fsint}(\lambda)$. The fuzzy sets λ and μ are said to be *disjoint fuzzy sets* if $\mu \wedge \lambda = 0$. A fuzzy set λ in an fts Y is called *fuzzy regularly-open set* if and only if $\lambda = \text{int cl } \lambda$. A subset A of X is called a *semi-neighbourhood* of a point x of X if there exists $U \in SO(X)$ such that $x \in U \subseteq A$. The family of all fuzzy semi-open sets of a space Y is denoted by $FSO(Y)$ and $FSO(Y, y_\alpha)$ denotes the family $\{\lambda \in FSO(Y) \mid \lambda(y_\alpha) \neq 0\}$. The family of all semi-open sets of a space X is denoted by $SO(X)$ and $SO(X,x)$ denotes the family $\{A \in SO(X) \mid x \in A\}$, where x is a point of X. If $U \subset X$, we define $F(U) = \bigvee_{x \in U} F(x)$.

DEFINITION 2.1 ([11]). Let (X,T) be a topological space in the classical sense and (Y,T_1) be an fts. $F : X \to Y$ is called a *fuzzy multifunction* if and only if for each $x \in X, F(x)$ is a fuzzy set in Y.

DEFINITION 2.2 ([10]). For a fuzzy multifunction $F : X \to Y$, the *upper inverse* $F^+(\lambda)$ and *lower inverse* $F^-(\lambda)$of a fuzzy set λ in Y are defined as follows:

$$F^+(\lambda) = \{x \in X \mid F(x) \leq \lambda\} \text{ and } F^-(\lambda) = \{x \in X \mid F(x)q\lambda\}.$$

LEMMA 2.1 ([10]). *For a fuzzy multifunction $F : X \to Y$, we have $F^-(1-\lambda) = X - F^+(\lambda)$, for any fuzzy set λ in Y.*

For further details of the pre-requisites which have not been explained here or hereafter, one may refer to [2, 3, 5, 10, 11, 13, 19].

3. Upper and lower semi-irresolute fuzzy multifunctions

DEFINITION 3.1. A fuzzy multifunction $F : X \to Y$ is said to be

(a) *fuzzy upper semi-irresolute (briefly f.u.s.i) at a point $x \in X$* if for each $\lambda \in FSO(Y)$ containing $F(x)$ (therefore, $F(x) \leq \lambda$), there exists $U \in SO(X,x)$ such that $F(U) \leq \lambda$ (therefore $U \subset F^+(\lambda)$).
(b) *fuzzy lower semi-irresolute (briefly f.l.s.i) at a point $x \in X$* if for each $\lambda \in FSO(Y)$ with $F(x)q\lambda$, there exists $U \in SO(X,x)$ such that $U \subseteq F^-(\lambda)$.
(c) *fuzzy upper semi-irresolute (fuzzy lower semi-irresolute)* if it is fuzzy upper semi-irresolute (fuzzy lower semi-irresolute) at each point $x \in X$.

THEOREM 3.1. *The following assertions are equivalent for a fuzzy multifunction $F : X \to Y$:*

1. *F is fuzzy upper semi-irresolute at a point $x \in X$;*
2. *For each $\lambda \in FSO(Y)$ with $F(x) \leq \lambda, x \in cl\ int\ (F^+(\lambda))$;*

3. *For each* $\lambda \in FSO(Y)$ *with* $F(x) \leq \lambda$, *there exists an open set* U *of* X *such that* $x \in U \subseteq$ *cl* $F^+(\lambda) \subseteq F^+(\lambda)$.

Proof. (1) $\Rightarrow$ (2) Let $\lambda \in FSO(Y)$ and $F(x) \leq \lambda$. Then (1), there exists $U \in SO(X, x)$ such that $F(U) \leq \lambda$. Since $U \in SO(X)$, we have $x \in U \subseteq$ cl int U. Since $F(U) \leq \lambda$, $U \subseteq F^+(\lambda)$ and so $x \in U \subseteq$ cl int $U \subseteq$ cl int $F^+(\lambda)$.

(2) $\Rightarrow$ (1) Let $\lambda \in FSO(Y)$ and $F(x) \leq \lambda$. This means $x \in F^+(\lambda)$ and hence $x \in F^+(\lambda) \cap$ cl int $F^+(\lambda) =$ sint $F^+(\lambda)$. Put $U =$ sint $F^+(\lambda)$. Then $U \in SO(X, x)$ and $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$. This proves that F is fuzzy upper semi-irresolute at a point $x \in X$.

(2) $\Rightarrow$ (3) Let $\lambda \in FSO(Y)$ and $F(x) \leq \lambda$. This means $x \in F^+(\lambda)$. Then by (2), $x \in$ cl int $F^+(\lambda)$. We have $F^+(\lambda) \subseteq$ cl int $F^+(\lambda)$. Therefore $F^+(\lambda) \subseteq$ cl(int($F^+(\lambda)$)) $\subseteq$ cl $F^+(\lambda)$. Then U is open set in X and $x \in U \subseteq F^+(\lambda)$. Hence $x \in U \subseteq F^+(\lambda) \subseteq$ cl $F^+(\lambda)$.

(3) $\Rightarrow$ (2) Let $\lambda \in FSO(Y)$ and $F(x) \leq \lambda$. This means $x \in F^+(\lambda)$. Then by (3), there exists an open set U of X such that $x \in U \subseteq$ cl $F^+(\lambda)$. Since $U \in SO(X)$ and $U \subseteq F^+(\lambda)$, we have $x \in U \subseteq$ cl int $(U) \subseteq$ cl int $(F^+(\lambda))$. This means that $x \in$ cl int $(F^+(\lambda))$. □

THEOREM 3.2. *The following assertions are equivalent for a fuzzy multifunction* $F : X \to Y$*:*

1. F *is fuzzy lower semi-irresolute at a point* $x \in X$*;*
2. *For each* $\lambda \in FSO(Y)$ *with* $F(x)q\lambda, x \in$ *cl int* $(F^-(\lambda))$*;*
3. *For each* $\lambda \in FSO(Y)$ *with* $F(x)q\lambda$, *there exists an open set* U*of* X *such that* $x \in U \subseteq$ *cl* $F^-(\lambda)$.

Proof. The proof is similar to that of Theorem 3.1. □

THEOREM 3.3. *The following assertions are equivalent for a fuzzy multifunction* $F : X \to Y$*:*

1. F *is fuzzy upper semi-irresolute;*
2. *For each point* x *of* X *and each fuzzy semi-neighbourhood* λ *of* $F(x), F^+(\lambda)$ *is a semi-neighbourhood of* x;
3. *For each point* x *of* X *and each fuzzy semi-neighbourhood* λ *of* $F(x)$, *there exists a semi-neighbourhood* U *of* x *such that* $F(U) \leq \lambda$;
4. $F^+(\lambda) \in SO(X)$ *for each* $\lambda \in FSO(Y)$;

5. *$F^{-}(\delta)$ is a semi-closed set in X for each fuzzy semi-closed set δ of Y;*
6. *scl $(F^{-}(\mu) \subseteq F^{-}($fscl $(\mu))$ for each fuzzy set μ of Y.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and μ be a fuzzy semi-neighbourhood of $F(x)$. Then there exists $\lambda \in FSO(Y)$ such that $F(x) \leq \lambda \leq \mu$. By (1), there exists $U \in SO(X, x)$ such that $F(U) \leq \lambda$. Therefore $x \in U \subseteq F^{+}(\mu)$ and hence $F^{+}(\mu)$ is a semi-neighbourhood of x.

(2) $\Rightarrow$ (3) Let $x \in X$ and λ be a fuzzy semi-neighbourhood of $F(x)$. Put $U = F^{+}(\lambda)$. Then by (2), U is semi-neighbourhood of x and $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$.

(3)$\Rightarrow$ (4) Let $\lambda \in FSO(Y)$, we want to show that $F^{+}(\lambda) \in SO(X)$. So let $x \in F^{+}(\lambda)$. Then there exists a semi-neighbourhood G of x such that $F(G) \leq \lambda$. Therefore for some $U \in SO(X, x), U \subseteq G$ and $F(U) \leq \lambda$. Therefore we get $x \in U \subseteq F^{+}(\lambda)$and hence $F^{+}(\lambda) \in SO(X)$.

(4)$\Rightarrow$ (5) Let δ be a fuzzy semi-closed set in Y. So, we have $X - F^{-}(\delta) = F^{+}(1 - \delta) \in SO(X)$ and hence $F^{-}(\delta)$ is semi-closed set in X.

(5) $\Rightarrow$ (6) Let μ be any fuzzy set in Y. Since fscl(μ) is fuzzy semi-closed set in Y, $F^{-}($fscl$(\mu))$ is semi-closed set in X and $F^{-}(\mu) \subseteq F^{-}$ (fscl(μ)). Therefore, we obtain scl$(F^{-}(\mu)) \subseteq F^{-}($fscl$(\mu))$.

(6)$\Rightarrow$ (1) Let $x \in X$ and $\lambda \in FSO(Y)$ with $F(x) \leq \lambda$. Now $F^{-}(1-\lambda) = \{x \in X \mid F(x) q (1-\lambda)\}$. So, for x not belongs to $F^{-}(1-\lambda)$. Then, we must have $F(x) \not{q} (1-\lambda)$ and this is implies $F(x) \leq 1-(1-\lambda) = \lambda$ which is true. Therefore $x \notin F^{-}(1-\lambda)$ by (6), $x \notin$ scl$(F^{-}(1-\lambda))$ and there exists $U \in SO(X, x)$ such that $U \cap F^{-}(1-\lambda) = \phi$. Therefore, we obtain $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$. This proves F is fuzzy upper semi-irresolute. □

THEOREM 3.4. *The following assertions are equivalent for a fuzzy multifunction $F : X \to Y$:*

1. *F is fuzzy lower semi-irresolute;*
2. *For each $\lambda \in FSO(Y)$ and each $x \in F^{-}(\lambda)$, there exists $U \in SO(X, x)$ such that $U \subseteq F^{-}(\lambda)$;*
3. *$F^{-}(\lambda) \in SO(X)$ for every $\lambda \in FSO(Y)$;*

4. $F^+(\delta)$ *is a semi-closed set in* X *for every fuzzy semi-closed set* δ *of* Y;
5. *int cl* $F^+(\mu) \subseteq F^+$*(fscl(*μ*))* *for every fuzzy set* μ *of* Y;
6. F*(int cl (*A*))* $\leq$ *fscl(*$F(A)$*)* *for every subset* A *of* X;
7. F*(sclA)* $\leq$ *fscl(*$F(A)$*)* *for every subset* A *of* X;
8. *scl(*$F^+(\mu)$*)* $\subseteq F^+$*(fscl(*μ*))* *for every fuzzy set* μ *of* Y.

Proof. (1) $\Rightarrow$ (2) Let $\lambda \in FSO(Y)$ and each $x \in F^-(\lambda)$ with $F(x)q\lambda$. Then by (1), there exists $U \in SO(X,x)$ such that $U \subseteq F^-(\lambda)$.

(2) $\Rightarrow$ (3) Let $\lambda \in FSO(Y)$ and $x \in F^-(\lambda)$. Then by (2), there exists $U \in SO(X,x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq$ cl int $(U) \subseteq$ cl int $(F^-(\lambda))$ and hence $F^-(\lambda) \in SO(X)$.

(3) $\Rightarrow$ (4) Let δ be a fuzzy semi-closed set in Y. So, we have $X - F^+(\delta) = F^-(1-\delta) \in SO(X)$ and hence $F^+(\delta)$ is semi-closed set in X.

(4) $\Rightarrow$ (5) Let μ be any fuzzy set in Y. Since fscl(μ) is fuzzy semi-closed set in Y, then by (4), we have F^+(fscl(μ)) is semi-closed set in X and $F^+(\mu) \subseteq F^+$ (fsclμ). Therefore, we obtain scl $(F^+(\mu)) \subseteq F^+$(fsclμ) and hence int cl $(F^+(\mu)) \subseteq$ int clF^+(fscl(μ))$\subseteq F^+$(fscl(μ)).

(5) $\Rightarrow$ (6) Let A be any subset of X. By (5), we have int cl $(A) \subseteq$ int cl $F^+(F(A)) \subseteq F^+$(fscl $(F(A))$). Therefore we obtain int cl $(A) \subseteq F^+$(fscl $F(A)$). This implies that F(int cl (A)) $\leq$ fscl $F(A)$.

(6) $\Rightarrow$ (7) Let A be any subset of X. Then scl$(A) = A\cup$ int cl (A), we have F(scl (A)) $= F(A\cup$ int cl $(A)) \leq$ fscl $F(A)$.

(7) $\Rightarrow$ (8) Let μ be any fuzzy set in Y. Then by (7), we have F(scl$F^+(\mu)$) $\leq$ fscl($F(F^+(\mu))$) and hence

$$\text{scl}(F^+(\mu)) \subseteq F^+(\text{ fscl }(F(F^+(\mu)))) \subseteq F^+(\text{fscl }\mu).$$

Therefore scl$(F^+(\mu)) \subseteq F^+$(fscl μ).

(8) $\Rightarrow$ (1) Let $x \in X$ and $\lambda \in FSO(Y)$ with $F(x)q\lambda$. Now, $F^+(1-\lambda) = \{x \in X \mid F(x) \leq 1-\lambda\}$. So, for x not belongs to $F^+(1-\lambda)$, then, we have $F(x) \nleq 1-\lambda$ and this implies that $F(x)q\lambda$. Therefore $x \notin F^+(1-\lambda)$. Since $1-\lambda$ is fuzzy semi-closed set in Y, by (8), $x \notin$ scl $(F^+(1-\lambda))$ and there exists $U \in SO(X,x)$ such that $\phi = U \cap F^+(1-\lambda) = U \cap (X - F^-(\lambda))$. Therefore, we obtain $U \subseteq F^-(\lambda)$. This proves F is fuzzy lower semi-irresolute. □

DEFINITION 3.2. For a given fuzzy multifunction $F : X \to Y$, a fuzzy multifunction $\text{scl}(F) : X \to Y$ is defined as follows:

$$(\text{scl}F)(x) = \text{scl}F(x) \text{ for each } x \in X.$$

We use scl$F$ and the following Lemma to obtain a characterization of lower semi-irresolute fuzzy multifunction.

LEMMA 3.1. *If $F : X \to Y$ is a fuzzy multifunction, then $(\text{scl}F)^-(\lambda) = F^-(\lambda)$ for each $\lambda \in FSO(Y)$.*

Proof. Let $\lambda \in FSO(Y)$ and $x \in (\text{scl}F)^-(\lambda)$. This means that $(\text{scl}F)(x)q\lambda$. Since $\lambda \in FSO(Y)$, we have $F(x)q\lambda$ and hence $x \in F^-(\lambda)$. Therefore

$$(\text{scl}F)^-(\lambda) \subseteq F^-(\lambda). \quad (1)$$

Conversely, let $x \in F^-(\lambda)$ since $\lambda \in FSO(Y)$ then $F(x)q\lambda \subseteq (\text{scl}F)(x)q\lambda$ and hence $x \in (\text{scl}F)^-(\lambda)$. Therefore

$$F^-(\lambda) \subseteq (\text{scl}F)^-(\lambda). \quad (2)$$

From (1) and (2), we get $(\text{scl}F)^-(\lambda) = F^-(\lambda)$. □

THEOREM 3.5. *A fuzzy multifunction $F : X \to Y$ is fuzzy lower semi-irresolute if and only if sclF $: X \to Y$ is fuzzy lower semi-irresolute.*

Proof. Suppose F is fuzzy lower semi-irresolute. Let $\lambda \in FSO(Y)$ and $F(x)q\lambda$. This means that $x \in F^-(\lambda)$. Then there exists $U \in SO(X, x)$ such that $U \subseteq F^-(\lambda)$. Therefore we have $x \in U \subseteq \text{cl int}(U) \subseteq \text{cl int}\, F^-(\lambda)$ and hence $F^-(\lambda) \in SO(X)$. Then, by Lemma 3.1, we have $U \subseteq F^-(\lambda) = (\text{scl}F)^-(\lambda)$ and $(\text{scl}F)^-(\lambda) \in SO(X)$,and hence $(\text{scl}F)(x)q\lambda$. Therefore (scl$F$) is fuzzy lower semi-irresolute.

Conversely, suppose (scl$F$) is fuzzy lower semi-irresolute. If for each $\lambda \in FSO(Y)$ with $(\text{scl}F)(x)q\lambda$ and $x \in (\text{scl}F)^-(\lambda)$ then there exists $U \in SO(X, x)$ such that $U \subseteq (\text{scl}F)^-(\lambda)$. By Lemma 3.1 and Theorem 3.4 (3), we have $U \subseteq (\text{scl}F^-(\lambda)) = F^-(\lambda)$ and $F^-(\lambda) \in SO(X)$. Therefore F is fuzzy lower semi-irresolute. □

4. Preservation and some other properties of upper and lower semi-irresolute fuzzy multifunctions

Definition 4.1. A fuzzy set λ of a fts Y is said to be *fuzzy semi-compact relative to* Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by fuzzy semi-open sets of Y has a finite subcover $\{\lambda_i\}_{i=1}^n$ of λ.

Definition 4.2. A fuzzy set λ of a fts Y is said to be *fuzzy semi-Lindelöf relative to* Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by fuzzy semi-open sets of Y has a countable subcover $\{\lambda_n\}_{n\in N}$ of λ.

Definition 4.3. A fts Y is said to be *fuzzy semi-compact* if χ_Y (characteristic function of Y) is fuzzy semi compact relative to Y.

Definition 4.4. A fts Y is said to be *fuzzy semi-Lindelöf* if χ_Y (characteristic function of Y) is fuzzy semi-Lindelöf relative to Y.

Definition 4.5. A fuzzy multifunction $F : X \to Y$ is said to be *punctually fuzzy semi-compact* (resp. *punctually fuzzy semi-Lindelöf*) if, for each $x \in X, F(x)$ is fuzzy semi-compact (resp. fuzzy semi-Lindelöf).

Theorem 4.1. *Let the fuzzy multifunction $F : X \to Y$ be a fuzzy upper semi-irresolute and F is punctually fuzzy semi-compact. If A is a semi-compact relative to X, then $F(A)$ is fuzzy semi-compact relative to Y.*

Proof. Let $\{\lambda_\alpha \mid \alpha \in \Delta\}$ be any cover of $F(A)$ by fuzzy semi-open sets of Y. We claim that $F(A)$ is fuzzy semi-compact relative to Y. For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that $F(x) \leq \cup\{\lambda_\alpha \mid \alpha \in \Delta(x)\}$. Put $\lambda(x) = \cup\{\lambda_\alpha \mid \alpha \in \Delta(x)\}$. Then $F(x) \leq \lambda(x) \in FSO(Y)$ and there exists $U(x) \in SO(X, x)$ such that $F(U(x)) \leq \lambda(x)$. Since $\{U(x) \mid x \in A\}$ is a semi-open cover of A, there exists a finite number of points of A, say, $x_1, x_2, \cdots, x_n$ such that $A \subseteq \cup\{U(x_i) \mid i = 1, 2, \cdots, n\}$. Therefore we obtain $F(A) \leq F\left(\bigcup_{i=1}^{n} U(x_i)\right) \leq \bigcup_{i=1}^{n} F(U(x_i)) \leq \bigcup_{i=1}^{n} \lambda(x_i) \leq \bigcup_{i=1}^{n}\left(\bigcup_{\alpha\in\Delta(x_i)} \lambda_\alpha\right)$. This shows that $F(A)$ is fuzzy semi-compact relative to Y. □

THEOREM 4.2. *Let the fuzzy multifunction $F : X \to Y$ be a fuzzy upper semi-irresolute and F is punctually fuzzy semi-Lindelöf. If A is semi-Lindelöf relative to X, then $F(A)$ is fuzzy semi-Lindelöf relative to Y.*

Proof. The proof is similar to that of Theorem 4.1. □

THEOREM 4.3. *Let $F_\alpha : X_\alpha \to Y_\alpha$ be a fuzzy multifunction for each $\alpha \in \Delta$ and $F : \prod_{\alpha\in\Delta} X_\alpha \to \prod_{\alpha\in\Delta} Y_\alpha$ be a fuzzy multifunction defined by $F(x) = \prod_{\alpha\in\Delta} F_\alpha(x_\alpha)$ for each $x = (x) \in \prod_{\alpha\in\Delta} X_\alpha$. If F is fuzzy upper semi-irresolute (resp. fuzzy lower semi-irresolute), then F_α is fuzzy upper semi-irresolute (resp. fuzzy lower semi-irresolute) for each $\alpha \in \Delta$.*

Proof. Let F be fuzzy upper semi-irresolute and $\lambda_\alpha \in FSO(Y_\alpha)$. Then $\lambda = \lambda_\alpha \times \prod_{\alpha\neq\beta} Y_\beta$ is fuzzy semi-open set in $\prod_{\alpha\in\Delta} Y_\alpha$ by Theorem 3.5 in [18]. Since F is fuzzy upper semi-irresolute by Theorem 3.3, $F^+(\lambda) = F^+(\lambda_\alpha) \times \prod_{\alpha\neq\beta} X$ is semi-open in $\prod_{\alpha\in\Delta} X_\alpha$. Since $P_\alpha : \prod_{\alpha\in\Delta} X_\alpha \to X_\alpha$ is the natural projection mapping, $P_\alpha(F^+(\lambda)) = F_\alpha^+(\lambda_\alpha) \in SO(X_\alpha)$. Thus F_α is fuzzy upper semi-irresolute for each $\alpha \in \Delta$. The proof for fuzzy lower semi-irresolute is similar. □

THEOREM 4.4. *Let $F_\alpha : X \to Y_\alpha$ be a fuzzy multifunction for each $\alpha \in \Delta$ and let $F : X \to \prod_{\alpha\in\Delta} Y_\alpha$ be fuzzy multifunction defined by $F(x) = \prod_{\alpha\in\Delta} F_\alpha(x)$ for each point $x \in X$. If F is fuzzy upper semi-irresolute (resp. fuzzy lower semi-irresolute), then F_α is fuzzy upper semi-irresolute (resp. fuzzy lower semi-irresolute) for each $\alpha \in \Delta$.*

Proof. (i) Suppose that F is fuzzy upper semi-irresolute. Let $x \in X, \alpha \in \Delta$ and $\lambda_\alpha \in FSO(Y_\alpha)$ containing $F_\alpha(x)$. Then we have $F(x) \leq P_\alpha^{-1}(\lambda_\alpha) = \lambda_\alpha \times \prod_{\alpha\neq\beta} Y_\beta \in FSO\left(\prod_{\alpha\in\Delta} Y_\alpha\right)$ by Theorem 3.5 in [18]. Since F is fuzzy upper semi-irresolute, there exists $U \in SO(X, x)$ such that $F(U) \leq P_\alpha^{-1}(\lambda_\alpha)$. Therefore, we obtain $F_\alpha(U) = P_\alpha(F(U)) \leq P_\alpha(P_\alpha^{-1}(\lambda_\alpha)) = \lambda_\alpha$. This shows that F_α is fuzzy upper semi-irresolute.

(ii) Suppose that F is fuzzy lower semi-irresolute. Let $x \in X, \alpha \in \Delta$ and $\lambda_\alpha \in FSO(Y_\alpha)$ with $F_\alpha(x) q \lambda_\alpha$. Then $P_\alpha^{-1}(\lambda_\alpha) = \lambda_\alpha \times \prod_{\alpha \neq \beta} Y_\beta \in FSO\left(\prod_{\alpha \in \Delta} Y_\alpha\right)$. Since F is fuzzy lower semi-irresolute, there exists $U \in SO(X, x)$ such that $U \subseteq F^-(P_\alpha^{-1}(\lambda_\alpha))$. Therefore, we obtain $F(z) q P_\alpha^{-1}(\lambda_\alpha)$ and hence $(F_\alpha(z) q \lambda_\alpha) \times \prod_{\alpha \neq \beta} F(z)$ for each $z \in U$. Consequently, $F_\alpha(z) q \lambda_\alpha$ for each $z \in U$ and hence $U \subseteq F_\alpha^-(\lambda_\alpha)$. Therefore F_α is fuzzy lower semi-irresolute for each $\alpha \in \Delta$. □

DEFINITION 4.6. [3]: Let (Y, T_1) be a fts and let λ be any fuzzy set in Y. λ is called *fuzzy pre-open set* if $\lambda \leq$ int cl λ. The complement of fuzzy pre-open set is called *fuzzy pre-closed set.* The intersection of a fuzzy pre-closed sets containing λ is called the *fuzzy pre-closure* of λ and is denoted by fpcl(λ).

DEFINITION 4.7. A fuzzy multifunction $F : X \to Y$ is said to be *fuzzy pre-open* if for every open set U of $X, F(U)$ is fuzzy pre-open in Y.

LEMMA 4.1. *A fuzzy multifunction $F : X \to Y$ is fuzzy pre-open if and only if for every fuzzy set λ of Y, F^-(fpcl $(\lambda)) \subseteq$ cl $(F^-(\lambda))$.*

Proof. Necessity : Suppose that F is fuzzy pre-open. Let λ be any fuzzy set of Y. Put $U = X-$cl $F^-(\lambda)$, then U is open set in X and $F(U)$ is fuzzy pre-open. Since $U = X-$ cl $(F^-(\lambda)) \subseteq X - F^-(\lambda) = F^+(1-\lambda)$, we have $F(U) \leq 1-\lambda$ and $F(U) \wedge \lambda = \phi$. Since $F(U)$ is fuzzy pre-open, $F(U)\wedge$ fpcl$(\lambda) = \phi$ and hence $F(U) \leq 1-$ fpcl(λ). Therefore we obtain $X-$ cl$(F^-(\lambda)) = U \subseteq F^+(1-$ fpcl$(\lambda)) = X - F^-($fpcl$(\lambda))$ and hence $F^-($fpcl $(\lambda)) \subseteq$ cl $F^-(\lambda)$.

Sufficiency : Let U be any open set in X and put $\lambda = 1 - F(U)$. Then $F(U) \wedge \lambda = \phi$ and $U \cap F^-(\lambda) = \phi$. Since U is open set in X, we have $U \cap F^-($fpcl$(\lambda)) \subseteq U\cap$ cl $(F^-(\lambda)) = \phi$. Therefore, we obtain $F(U)\wedge$ fpcl$(\lambda) = \phi$ and hence fpcl $(\lambda) \leq 1 - F(U) = \lambda$. This shows that λ is fuzzy pre-closed in Y. Thus, $F(U)$ is fuzzy pre-open in Y and hence F is fuzzy pre-open. □

DEFINITION 4.8. A fuzzy multifunction $F : X \to Y$ is said to be

(a) *fuzzy upper semi-continuous* if for each $x \in X$ and each fuzzy open set λ of Y containing $F(x)$, there exists $U \in SO(X, x)$ such that $F(U) \leq \lambda$.

(b) *fuzzy lower semi-continuous* if for each $x \in X$ and each fuzzy open set λ of Y such that $F(x)q\lambda$, there exists $U \in SO(X, x)$ such that $U \subset F^{-}(\lambda)$.

REMARK 4.1. Fuzzy upper semi-continuous and fuzzy lower semi-continuous given in [10] is less in general than the above two concepts. Since the concepts in [10] demand the existence of open neighbourhood whereas our definition demand the existence of semi-open neighbourhood.

LEMMA 4.2. *A fuzzy multifunction $F : X \to Y$ is fuzzy upper semi-continuous (resp. fuzzy lower semi-continuous) if and only if $F^{+}(\lambda) \in SO(X)$ (resp. $F^{-}(\lambda) \in SO(X)$) for every fuzzy open set λ of Y.*

Proof. We prove this lemma only for the case of fuzzy upper semi-continuity of F, the proof of the other case being similar. Let λ be fuzzy open set in Y and $F(x) \leq \lambda$. This means $x \in F^{+}(\lambda)$. By hypothesis, we have $x \in \text{cl int } F^{+}(\lambda)$. Therefore, we obtain $x \in U \subseteq F^{+}(\lambda) \subseteq \text{cl int } F^{+}(\lambda)$ and hence $F^{+}(\lambda) \in SO(X)$.

Conversely, $x \in X$ and λ be fuzzy set in Y such that $F(x) \leq \lambda$. By hypothesis, we have $F^{+}(\lambda) \in SO(X)$. Let $U = F^{+}(\lambda)$, then $U \in SO(X, x)$ and $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$. This shows that F is upper semi-continuous. □

THEOREM 4.5. *If a fuzzy multifunction $F : X \to Y$ is fuzzy lower semi-continuous and fuzzy pre-open, then F is fuzzy lower semi-irresolute.*

Proof. Let $\lambda \in FSO(Y)$. Since F is fuzzy pre-open, by lemma 4.1, we have $F^{-}(\lambda) \subseteq F^{-}(\text{fpcl}\lambda) \subseteq \text{cl } F^{-}(\lambda)$. Since $\lambda \in FSO(Y)$, we have fpcl (λ) = cl int (λ) and hence by lemma 4.2, $F^{-}(\text{fpcl}\lambda) \in \text{SO}(X)$. Hence, we obtain $F^{-}(\lambda) \subseteq F^{-}(\text{fpcl}\lambda) \subseteq \text{cl int } (F^{-}(\text{fpcl}(\lambda)) \subseteq \text{cl int } (F^{-}(\lambda))$. This shows that $F^{-}(\lambda) \in \text{SO}(X)$. Thus F is fuzzy lower semi-irresolute. □

5. Mutual relationships and relations with other existing fuzzy multifunction

We now show by means of the following examples that

1. fuzzy upper semi-irresolute (fuzzy upper semi-continuous) $\not\Rightarrow$ fuzzy lower semi-irresolute (fuzzy lower semi-continuous).
2. fuzzy lower semi-irresolute (fuzzy lower semi-continuous) $\not\Rightarrow$ fuzzy upper semi-irresolute (fuzzy upper semi-continuous).

Example 5.1. Let $X = \{a, b\}, Y = [0, 1]$. Let T and T_1 respectively the topology on X and fuzzy topology on Y given by $T = \{X, \phi, \{b\}\}, T_1 = \{C_0, C_1, C_{\frac{1}{3}}, C_{\frac{5}{6}}\}$. We use the notation $C_\alpha(0 \leq \alpha \leq 1)$ to denote the constant fuzzy set such that $C_\alpha(y) = \alpha$, for all $y \in Y$. Consider the fuzzy multifunction $F : (X, T) \to (Y, T_1)$ given by $F(a) = C_{\frac{5}{6}}$ and $F(b) = C_{\frac{1}{2}}$. This example is given in [10]. It is shown in [10] that F *is fuzzy lower weakly continuous (f.l.w.c)* and F *is not fuzzy lower almost continuous (f.l.a.c).* Since $\{b\}$ is open set in (X, T) and therefore $\{b\}$ is semi-open set in (X, T). It can be easily verified that F *is fuzzy upper semi-irresolute* and F *is fuzzy upper semi-continuous.* Now $F^-(C_{\frac{1}{3}}) = \{a\}$, which is not semi-open set in (X, T) . This proves that F *is not fuzzy lower semi-irresolute* and also F *is not fuzzy lower semi-continuous.*

Example 5.2. Let $X = \{a, b, c\}$ and $Y = [0, 1]$. Let T and T_1 be respectively the topology on X and fuzzy topology on Y given by $T = \{X, \phi, \{a, c\}\}, T_1 = \{C_0, C_1, C_{\frac{1}{2}}, C_{\frac{1}{3}}\}$. We use the notation $C_\alpha(0 \leq \alpha \leq 1)$ to denote the constant fuzzy set such that $C_\alpha(y) = \alpha$, for all $y \in Y$. Consider the fuzzy multifunction $F : (X, T) \to (Y, T_1)$ given by $F(a) = C_{\frac{5}{6}}, F(b) = C_{\frac{1}{2}}$ and $F(c) = C_{\frac{3}{4}}$. Now $F^+(C_{\frac{1}{2}}) = \{b\}$, which is not semi-open set in (X, T). This shows that F *is not fuzzy upper semi-irresolute* and also F *is not fuzzy upper semi-continuous.* It can be easily verified that F *is fuzzy lower semi-irresolute* and also F *is fuzzy lower semi-continuous.* Now, the fuzzy regularly open sets of (Y, T_1) are C_0, C_1 and $C_{\frac{1}{2}}$ and $F^-\left(C_{\frac{1}{2}}\right) = \{a, c\} \in T$. Thus F *is fuzzy lower almost continuous.*

EXAMPLE 5.3. Let $X = \{a, b, c\}$ and $Y = [0, 1]$. Let T and T_1 be respectively the topology on X and fuzzy topology on Y given by $T = \{X, \phi, \{a, b\}\}, T_1 = \{C_0, C_1, C_{\frac{1}{3}}, C_{\frac{5}{6}}\}$. We use the notation $C_\alpha(0 \leq \alpha \leq 1)$ to denote the constant fuzzy set such that $C_\alpha(y) = \alpha$, for all $y \in Y$. Consider the fuzzy multifunction $F : (X, T) \to (Y, T_1)$ given by $F(a) = C_{\frac{1}{4}}, F(b) = C_{\frac{1}{2}}$ and $F(c) = C_{\frac{11}{12}}$. This example is given in [10]. It is shown in [10] that F *is fuzzy upper weakly continuous (f.u.w.c)* and F *is not fuzzy upper almost continuous (f.u.a.c)*. Now $F^+(C_{\frac{1}{3}}) = \{a\}$, which is not semi-open set in (X, T). This proves that F *is not fuzzy upper semi-irresolute* and also F *is not fuzzy upper semi-continuous*. It can be easily verified that F *is fuzzy lower semi-irresolute* and also F *is fuzzy lower semi-continuous.*

Acknowledgement: The authors wish to thank the referee for his suggestions and corrections which helped to improve this paper.

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