East Asian Math. J. **23** (2007), No. 2, pp. 257–260

A NOTE ON THE SOLUTION OF A NONLINEAR SINGULAR INTEGRAL EQUATION WITH A SHIFT IN GENERALIZED HÖLDER SPACE

Ioannis K. Argyros

Abstract. Using the center instead of the Lipschitz condition we show how to provide weaker sufficient convergence conditions of the modified Newton Kantorovich method for the solution of nonlinear singular integral equations with Curleman shift (NLSIES). Finer error bounds on the distances involved and a more precise information on the location of the solution are also obtained and under the same computational cost than in [1].

1. Introduction

In this note we are concerned with the problem of approximating a locally unique solution u^* of the following nonlinear singular integral equation with Curleman shift (NLSIES):

$$\begin{aligned}(1)\quad (Q(u)(t) &= a(t)u(t) + b(t)u(a(t)) + \frac{c(t)}{\pi i}\int_L \frac{u(\tau)}{\tau - t}d\tau \\ &+ \frac{d(t)}{\pi i}\int_L \frac{u(\tau)}{\tau - \alpha(t)}d\tau - \frac{1}{\pi i}\int_L \left\{\frac{F(\tau, u(\tau))}{\tau - t} + \frac{G(\tau, u(\tau))}{\tau - \alpha(t)}\right\} d\tau\end{aligned}$$

where L is a simple closed Lyapunov contour which divides the complex plane into the interior a domain D^+ and the exterior domain D^-, $u(t)$ is the uknown function to be determined and $\alpha : L \to L$ is a

Received June 18, 2007.
2000 Mathematics Subject Classification: 65G99, 47H17, 49M15.
Key words and phrases: Hölder space, Modified Newton-Kantorovich method, singular integral equation with a shift.

homeomorphism which is shift preserving orientation, satisfying the Curleman condition:

$$\alpha(\alpha(t)) = \alpha_2(t) = t, \quad t \in L,$$

whose derivative $\alpha'(t)$ satisfied Hölder condition, $\alpha'(t) \neq 0$ for all $t \in L$.

Here we are motivated by optimization considerations and the elegant paper by Amer [1].

We have observed that Lipschitz condition

$$\|Q'(u_1) - Q'(u_2)\| \leq \ell_1 \|u_1 - u_2\| \tag{2}$$

used in [1, Lemma 1.6] is not really needed to show the main result in [1, Th.3.2]. Instead the weaker center Lipschitz condition

$$\left\|Q'(u_0)^{-1}[Q'(u) - Q'(u_0)]\right\| \leq \ell_0 \|u - u_0\| \tag{3}$$

is needed. That is the proofs of all results in [1] using (2) go through with (3) replacing (2). Set

$$\ell = \left\|Q'(u_0)^{-1}\right\| \cdot \ell_1. \tag{4}$$

In general

$$\ell_0 \leq \ell \tag{5}$$

holds and $\frac{\ell}{\ell_0}$ can be arbitrarily large [3], [2]. In contrast to [1] we provide the semilocal convergence result for the modified Newton's method

$$u_{n+1} = u_n - Q'(u_0)^{-1} Q(u_n) \tag{6}$$

in an affine invariant form. We refer the reader to [2] and the references there for the advantages of this approach.

2. Convergence analysis

Theorem 1. *If*

the conditions of Theorem 2.1 and Lemma 3.1 in [1] *with (3) replacing (2)*

$u_0 \in H_{\varphi,m}(L)$ *such that:*

$$\left\|Q'(u_0)^{-1} Q(u_0)\right\| \leq \eta = J(\gamma_1)\gamma_2, \tag{7}$$

(8) $$h_0 = \ell_0 \eta \leq \frac{1}{2}$$

and

(9) $$E_{\varphi,m}(u_0; r) \subseteq E_{\varphi,m}(u_0; r_0),$$

where,

(10) $$r = \frac{1 - \sqrt{1 - 2h_0}}{\ell_0},$$

then sequence $\{u_n\}$ $(n \geq 0)$ *generated by modified Newton method (6) is well defined, remains in the ball* $E_{\varphi,m}(u_0; r)$ *for all* $n \geq 0$ *and converges to a unique solution* u^* *of equation (1) in* $E_{\varphi,m}(u_0, r)$. *Moreover the following estimates hold for all* $n \geq 0$

(11) $$\|u_n - u^*\| \leq \frac{a_0^n}{1 - a_0} \eta,$$

where,

(12) $$a_0 = 1 - \sqrt{1 - 2h_0.}$$

The definition of $J(\gamma_1), \gamma_2, H_{\varphi,m}(L)$ and $E_{\varphi,m}(u_0; r)$ can be found in [1].

REMARK 2. *Set*

(13) $$h = \ell \eta$$

(14) $$r_1 = \frac{1 - \sqrt{1 - 2h}}{\ell}$$

provided that the Newton-Kantorovich hypothesis [4]

(15) $$h \leq \frac{1}{2}$$

holds,

(16) $$a = 1 - \sqrt{1 - 2h.}$$

If equality holds in (5) then 1 reduces to Theorem 2.3 in [1]. *Otherwise it is an improvement. Indeed note that in this case:*

(17) $$h \leq \frac{1}{2} \Longrightarrow h_0 \leq \frac{1}{2}$$

but not vice versa;

$$a_0 < a$$

and

$$r < r_1.$$

The claims made in the introduction are justified.

Note also that in case (15) is violated but weaker condition (3) holds, then Theorem 3.2 in [1] *cannot guarantee the convergence of sequence* $\{u_n\}$ *to* u^* *(or the existence of* u^**). However our Theorem 1 can be used for this reason. Hence the applicability of the modified Newton method for solving equation (1) has been extended. Note also that the computational cost is less since the computation of* ℓ *is more expensive than the computation of* ℓ_0*. Moreover condition (2) cannot even hold at all (in general) where weaker (3) may hold.*

REFERENCES

[1] Amer, S.M. *On solution of nonlinear singular integral equations with shift in generalized Hölder space*, Chaos, Solitons and Fractals, **12**, (2001), 1323-1334.

[2] Argyros, I. K., *On the Newton-Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., **169**, (2004), 315-332.

[3] Argyros, I.K., *Newton Methods*, Nova Science Publ. Corp., New York, 2005.

[4] Kantorovich, L. V. and Akilov, G.P., *Functional Analysis in normed spaces*, Pergamon Press, Oxford, 1982.

Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, U.S.A.
iargyros@cameron.edu