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IDEALS AND BRANCHES OF BCC-ALGEBRAS

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Abstract. Basic properties of branches of weak BCC-algebras and ideals of BCC-algebras containing only atoms are described.

1. Introduction

In 1966, Y. Imai and K. Iséki defined two classes of algebras of type (2,0) called BCK-algebras and BCI-algebras [9, 10]. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. A. Wroński [17] solved this problem and proved that BCK-algebras do not form a variety. In connection with this problem Y. Komori introduced in [14] a notion of BCC-algebras which is a generalization of a notion of BCK-algebras and proved that the class of all BCC-algebras is not a variety, but the variety generated by BCC-algebras, that is the smallest variety containing the class of all BCC-algebras, is finitely based [14]. W. A. Dudek [3] redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued in [1, 4, 6, 7, 8]. some open - rather hard - problems are posed in [5].

In this short note we describe basic properties of branches in BCC-algebras and ideals of BCC-algebras containing only atoms.

2. Preliminaries

Definition 2.1. A *weak BCC-algebra X* is an abstract algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms

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(i) $((x * y) * (z * y)) * (x * z) = 0$,
(ii) $x * x = 0$,
(iii) $x * 0 = x$,
(iv) $x * y = y * x = 0 \longrightarrow x = y$.

A weak BCC-algebra satisfying the identity

(v) $0 * x = 0$,

is called a *BCC-algebra.* A BCC-algebra with the condition

(vi) $(x * (x * y)) * y = 0$

is called a *BCK-algebra.*

One can prove (see [2, 3] or [17]) that a BCC-algebra is a BCK-algebra iff it satisfies the identity

(vii) $(x * y) * z = (x * z) * y$.

An algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra.* A BCI-algebra satisfies also (vii) (cf. [11]). A weak BCC-algebra is a BCI-algebra iff it satisfies (vii).

A (weak) BCC-algebra which is not a BCK-algebra (respectively, BCI-algebra) is called *proper.* A proper BCC-algebra has at least four elements. Moreover, for every $n \geq 4$ there exists at least one proper BCC-algebra (cf. [2, 3]). Analogous result are valid for weak BCC-algebras (cf. [4]).

In all these algebras one can defined a natural partial order $\leq$ putting

$$x \leq y \longleftrightarrow x * y = 0.$$

In all BCC/BCK-algebras we have $0 \leq x$ for every $x \in X$. Moreover, from (i) it follows that in any (weak) BCC-algebra

$$x \leq y \longrightarrow x * z \leq y * z, \tag{1}$$

$$x \leq y \longrightarrow z * y \leq z * x \tag{2}$$

for all $x, y, z \in X$.

In BCC-algebras we also have

$$x * y \leq x \tag{3}$$

for all $x, y \in X$ (cf. [3]).

We say that two elements $x, y \in X$ are *comparable* if $x \leq y$ or $y \leq x$. An algebra X is *linearly ordered* if each its two elements are comparable. A linearly ordered weak BCC-algebra (BCI-algebra) is a BCC-algebra (BCK-algebra, respectively).

The set of all elements comparable with 0, i.e., the set

$$B(X) = \{x \in X \mid 0 \leq x\}$$

is called a *BCK-part* of BCI-algebra X.

3. Branches and atoms

DEFINITION 3.1. An element a of a weak BCC-algebra X is called an *atom* if $x \leq a$ implies $x = 0$ or $x = a$. The set of all atoms is denoted by $A(X)$.

LEMMA 3.1. (cf. [6]) *If $a \neq b$ are non-zero atoms of a BCC-algebra X then $a * b = a$.*

Note that in fact this lemma is valid also in the case when $a = 0$ or $b = 0$. Moreover, from this lemma it follows that the set of all atoms of a given BCC-algebras is its subalgebra. For weak BCC-algebras it is not true (see Example 3.1 below).

The set

$$B(a) = \{x \in X | a \leq x\}$$

where a is an atom of X, is called a *branch* of X. An element a is called *initial* for $B(a)$. In the case when there exists an $b \neq a$ such that $B(a) \subset B(b)$ we say that a branch $B(a)$ is *improper*. So, a branch $B(a)$ is proper if no $b \in X$ such that $b \neq a$ and $b \leq a$. The set of all initial elements of proper branches of X is denoted by $I(X)$. Obviously $I(X) \subset A(X)$.

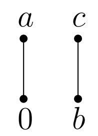
EXAMPLE 3.1. Consider on the set $X = \{0, a, b, c\}$ two operations defined by the following tables:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	b	b
b	b	b	0	0
c	c	c	a	0

Table 1

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	c
b	b	b	0	0
c	c	c	a	0

Table 2

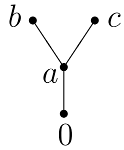


Algebras $(X, *, 0)$ defined by these two tables are proper weak BCC-algebras (cf. [4]). In these algebras we have $A(X) = \{0, a, b\}$, $B(0) = \{0, a\}$, $B(a) = \{a\}$, $B(b) = \{b, c\}$, $I(X) = \{0, b\}$. The branches $B(0)$ and $B(b)$ are proper, the branch $B(a)$ is improper. $I(X)$ is a subalgebra in both these algebras, but $A(X)$ is a subalgebra only in the algebra defined by Table 1.

EXAMPLE 3.2. Consider on the set $X = \{0, a, b, c\}$ two operations defined by the following tables:

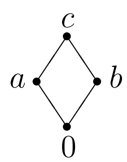
$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	a
c	c	c	a	0

Table 3



$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

Table 4



Algebras $(X, *, 0)$ defined by these tables are proper BCC-algebras (cf. [3]). In these algebras $B(0) = B(X) = X$ and $I(X) = \{0\}$. No other proper branches. The algebra defined by Table 3 has one improper branch $B(a) = \{a, b, c\}$, the algebra defined by Table 4 has two improper branches: $B(a) = \{a, c\}$ and $B(b) = \{b, c\}$.

DEFINITION 3.2. A nonempty subset A of a weak BCC-algebra X is called a *chain* if each its two elements are comparable. A chain initiated by a is denoted by $C(a)$. In the case $B(a) = C(a)$ we say also that $B(a)$ is a *linear branch.* A branch containing only one element is called *trivial.* A branch which has at least two incomparable elements and is the set-theoretic union at least two chains is called *expanded.*

Each BCC-algebra is a linear or expanded branch. A BCC-algebra defined by Table 3 is a set-theoretic union of two chains: $C_1(0) = \{0, a, b\}$ and $C_2(0) = \{0, a, c\}$. A branch $B(a)$ is a union of chains $C_1(a) = \{a, b\}$ and $C_2(a) = \{b, c\}$. A BCC-algebra defined by Table 4 is a union of chains $C_1(0) = \{0, a, c\}$ and $C_2(0) = \{o, b, c\}$. Weak BCC-algebras defined in Example 3.1 have two linear branches: $B(0)$ and $B(b)$.

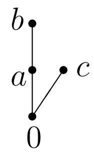
DEFINITION 3.3. A BCC-algebra X is called
implicative if $x * (y * x) = x$,

positive implicative if $(x * y) * y = x * y$,
commutative if $x \wedge y = y \wedge x$
holds for all $x, y \in X$, where $x \wedge y = y * (y * x)$.

EXAMPLE 3.3. Consider on the set $X = \{0, a, b, c\}$ two operations defined by the following tables:

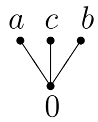
$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	a
c	c	c	c	0

Table 5



$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Table 6



The algebra defined by Table 5 is a positive implicative proper BCC-algebra (Table 12 in [3]) which is not implicative and commutative. The algebra defined by Table 6 is a commutative, positive implicative and implicative BCC-algebra in which all elements are atoms (see our Proposition 4.5 and Corollary 4.3).

PROPOSITION 3.1. *If a is an atom of a BCC-algebra X then $a*x = a$ for every $x \notin B(a)$.*

Proof. Indeed, by (3) for every $x \in X$ we have $a * x \le a$ whence we conclude $a * x = 0$ or $a * x = a$. The first is impossible because $x \notin B(a)$. So, $a * x = a$. □

COROLLARY 3.1. [6] *If $a \neq b$ are atoms of a BCC-algebra, then $a * b = a$.*

PROPOSITION 3.2. *An element a of a weak BCC-algebra is its initial element if and only if $0 * (0 * a) = a$.*

Proof. Indeed, for any element $a \in X$ we have $(0 * (0 * a)) * a = ((a * a) * (0 * a)) * (a * 0) = 0$, i.e., $0 * (0 * a) \le a$. Hence, if a is initial, then $0 * (0 * a) = a$.

Conversely, let $0 * (0 * a) = a$ for some $a \in X$. If $z \le a$, then $z * a = 0$ and $a * z = (0 * (0 * a)) * z = ((z * a) * (0 * a)) * (z * 0) = 0$. Thus $a * z = z * a = 0$. Therefore $z = a$. So, $a \in I(X)$. □

PROPOSITION 3.3. *Let X be a weak BCC-algebra. $B(a) \cap B(b) = \phi$ for distinct $a, b \in I(X)$.*

Proof. If $B(a) \cap B(b) \neq \phi$, then there exists at leas one $x \in B(a) \cap B(b)$. Let $x_0 = 0*(0*x)$. Obviously $x_0 = 0*(0*x) = (x*x)*(0*x) \leq (x*0) = x$, i.e., $x_0 \leq x$. If $z \leq x_0$ for some $z \in X$, then $z*x_0 = 0$ and $z*x = 0$. So, $x_0 * z = (0*(0*x))*z = ((z*x)*(0*x))*(z*0) = 0$. Therefore $z = x_0$. This means that x_0 is an initial element of X. Since $b \leq x$, we have $x_0 = 0*(0*x) = (b*x)*(0*x) \leq b*0 = b$, which implies $x_0 = b$ because b is initial. Similarly $x_0 = a$. Whence $a = b$, which is impossible. □

COROLLARY 3.2. *Comparable elements are contained in the same branch.*

Proof. Let$x, y \in X$ be comparable. Without loss of generality we can assume that $x \leq y$. Then $x \in B(a)$ for some $a \in I(X)$. Thus $a \leq x \leq y$ which implies $y \in B(a)$. □

PROPOSITION 3.4. *$x*y, y*x \in B(X)$ for any two comparable elements x and y of a weak BCC-algebra X.*

Proof. Let x and y be comparable. Then $x \leq y$ or $y \leq x$. Suppose $x \leq y$. Then $x*y = 0 \in B(X)$ and $x*x \leq y*x$ by (1). Thus $0 \leq y*x$, i.e., $y*x \in B(X)$. □

PROPOSITION 3.5. *A BCK algebra X is positive implicative iff $x*y = z \Rightarrow z*y = z$.*

Proof. If X is positive implicative then $(x*y)*y = x*y$ let $x*y = z$ then $(x*y)*y = x*y \Rightarrow z*y = z$.
Conversely, suppose that $x*y = z \Rightarrow z*y = z$. Then $(x*y)*y = z*y = z = x*y$. Hence X is positive implicative. □

4. Ideals

DEFINITION 4.1. Let X be a BCC-algebra. A subset $A \subset X$ containing 0 is called
a *BCK-ideal* if $y, x*y \in A$ imply $x \in A$,
a *BCC-ideal* if $y, (x*y)*z \in A$ imply $x*z \in A$,
a *strong BCC-ideal* if $y, (x*y)*z \in A$ imply $x \in A$.

Every BCC-ideal is a BCK-ideal. The converse is not true [7].

PROPOSITION 4.1. *If A is a strong BCC-ideal of a BCC-algebra X, then $y \in A$ and $x \leq y$ imply $x \in A$.*

Proof. Indeed, $x \leq y$ and $y \in A$ imply $x * y = 0 \in A$. So, $(x * y) * 0, y * 0 \in A$ by (iii). This, according to the definition of a strong BCC-ideal, gives $x \in A$. □

PROPOSITION 4.2. *Every strong BCC-ideal is a BCC-ideal.*

Proof. Let $y, (x * y) * z \in A$, where A is a strong BCC-ideal. Then $x \in A$, whence by (3) and Proposition 4.1, we obtain $x * z \in A$. So, A is a BCC-ideal. □

PROPOSITION 4.3. *If every element of a BCC-algebra X is an atom then every subset of X containing 0 is a subalgebra and a BCK-ideal of X. In this case X is a BCK-algebra.*

Proof. Let A be a subset of X containing 0. If $0, x \in A$, where $x \neq 0$, then $0 * x, x * 0 \in A$ by (iii) and (v). If $x, y \in A$ are non-zero atoms, then $x * y = x$, by Lemma 3.1. Hence $x * y \in A$ and so A is a subalgebra of X. Now, if $x, y * x \in A$ for some $x, y \in X$, then $y * x = 0$ or $y = y * x \in A$ because $y * x \leq y$ (by 3) and y is an atom. In the case $y * x = 0$ we have $y \leq x$ which implies $y = 0$ or $y = x$. So, in any case $y \in A$. Thus A is a BCK-ideal of X. The rest is a consequence of Corollary 8 from [6]. □

PROPOSITION 4.4. *In a BCC-algebra containing only atoms each BCC-ideal is strong.*

Proof. Let A be a BCC-ideal of a BCC-algebra X and let $y, (x * y) * z \in A$ for some $x, y, z \in A$. Then $x * z \in A$, whence, by Lemma 3.1, we conclude $x \in A$. □

As a consequence of the above results we obtain

COROLLARY 4.1. *In a BCC-algebra containing only atoms the following conditions are equivalent:*

a) $0 \in A$,
b) *A is a subalgebra,*
c) *A is a BCC-ideal,*
d) *A is a BCK-ideal,*
e) *a is a strong BCC-ideal.*

Comparing this corollary with Theorem 3 from [6] we have

COROLLARY 4.2. *A BCC-algebra contains only atoms iff each of its subalgebra is a BCC-ideal.*

PROPOSITION 4.5. *A BCC-algebra containing only atoms is a positive implicative and commutative BCK-algebra.*

Proof. Since each element of X is an atom of X, so by Lemma 3.1 for all $x, y \in X$ we have $x * y = x$. Hence $(x * y) * y = x * y$, i.e., X is positive implicative. Moreover $x * (x * y) = x * x = 0$ and $y * (y * x) = y * y = 0$. Thus X is commutative. The rest is a consequence of Corollary 8 from [6]. □

As a consequence of the above result and Theorem 6.2 from [16] we obtain

COROLLARY 4.3. *A BCC-algebra containing only atoms is an implicative BCK-algebra.*

PROPOSITION 4.6. *If every element of a BCC-algebra X is an atom and A is an ideal of X then every element of the quotient algebra X/A is also atom.*

Proof. Let X a BCC-algebra contains only atoms and let A be an ideal of X. Then the quotient algebra $X/A = \{C_x : x \in X\}$, where $C_x = \{y \in X : y*x, x*y \in A\}$, is a BCC-algebra (cf. [7]) with respect to the operation $C_x * C_y = C_{x*y}$. We show that each element in X/A is an atom. Suppose that for $C_x \in X/A$ there exists $C_y \in X/A$ such that $C_y \leq C_x$. Then $C_y * C_x = C_0$, i.e., $C_{y*x} = C_0$ which implies $y * x = 0$. Thus $y \leq x$, whence $y = x$ or $y = 0$ because x is an atom. Hence $C_y = C_x$ or $C_y = C_0$. So C_x is an atom of X/A. □

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