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INTEGRAL INEQUALITIES AND APPLICATIONS FOR BOUNDING THE ČEBYŠEV FUNCTIONAL

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Abstract. Some inequalities related to the Hölder integral inequality and applications for bounding the Čebyšev functional are given.

1. Introduction

The integral Hölder inequality, namely

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \le \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt\right)^{\frac{1}{q}}, \tag{1.1}$$

plays an important role in Mathematical Analysis and its applications. Here the complex-valued functions $f, g : [a,b] \to \mathbb{C}$ are p and $q-$integrable respectively on $[a,b]$, where $p, q > 1$ and $1/p + 1/q = 1$.

In order to provide sharper bounds for the Hölder inequality, Abramovich, Mond and Pečarić considered in [1] the function $\Phi : [a,b] \to \mathbb{R}$ given by

$$\Phi(x) := \left|\int_a^x f(t)\,g(t)\,dt\right| + \left(\int_x^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_x^b |g(t)|^q\,dt\right)^{\frac{1}{q}} \tag{1.2}$$

and proved that $\Phi(\cdot)$ is nondecreasing on $[a,b]$. As a consequence we can observe that

$$\inf_{x\in[a,b]} \Phi(x) = \left|\int_a^b f(t)\,g(t)\,dt\right|$$

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and

$$\sup_{x\in[a,b]} \Phi(x) = \left(\int_a^b |f(t)|^p \, dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \, dt\right)^{\frac{1}{q}}.$$

Using geometrical arguments, G.S. Mahajani [8] obtained the following results for the absolute value of the integral $\int_a^x f(t)\,dt$:

1. *If f has a bounded derivative on $[a,b]$, namely $|f'(t)| \leq M$ $(M>0)$ and if $\int_a^b f(t)\,dt = 0$, then*

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{1}{8} \cdot M \cdot (b-a)^2,$$

for any $x \in [a,b]$.

2. *If, additional to the conditions given above, $f(a) = f(b) = 0$, then*

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{1}{16} \cdot M \cdot (b-a)^2.$$

Analytic proofs of these results were given by P.R. Beesack, [9, p. 474]. For other results related to the Mahajani inequality see Chapter XV of [9].

In this paper some similar results are obtained and applied in obtaining bounds for the quantities $\left|\int_a^x f(t)\,dt\right|$,

$$\int_a^b \left|\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(t)\,dt\right|^r dx, \qquad r\in[1,\infty)$$

and

$$\sup_{x\in[a,b]} \left|\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(t)\,dt\right|,$$

under various assumptions for the function $f:[a,b]\to\mathbb{R}$.

These results are also utilized to provide bounds for the *Čebyšev functional*

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx,$$

where $f, g : [a, b] \to \mathbb{C}$ are Lebesgue integrable functions, in terms of the *shifted integral means*:

$$\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt$$

with $r \in [1, \infty)$. This is possible due to the following representation result obtained in [2]:

$$T(f, g) = -\frac{1}{b-a}\int_a^b \left(\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right) f'(x)\,dx,$$

that holds for g Lebesgue integrable and f absolutely continuous on $[a, b]$.

For recent results on bounding the Čebyšev functional $T(\cdot, \cdot)$ see [2], [3] and [5] where further references are provided.

2. The Results

The following result may be stated:

Theorem 1. *Let $f : [a, b] \to \mathbb{C}$ be a Lebesgue measurable function and $p : [a, b] \to [0, \infty)$ a Lebesgue integrable weight with $\int_a^b p(t)\,dt = 1$. For any $r > 1$ and $x \in [a, b]$, we have the inequality:*

$$\int_a^x p(t)\,|f(t)|^r\,dt + \frac{\left|\int_a^b p(t) f(t)\,dt - \int_a^x p(t) f(t)\,dt\right|^r}{\left[1 - \int_a^x p(t)\,dt\right]^{r-1}} \le \int_a^b p(t)\,|f(t)|^r\,dt. \tag{2.1}$$

In particular,

$$\int_a^x |f(t)|^r\,dt + \frac{\left|\int_a^b f(t)\,dt - \int_a^x f(t)\,dt\right|^r}{(b-x)^{r-1}} \le \int_a^b |f(t)|^r\,dt. \tag{2.2}$$

Proof. Obviously,

$$\int_a^b p(t) f(t)\,dt - \int_a^x p(t) f(t)\,dt = \int_x^b p(t) f(t)\,dt \tag{2.3}$$

for any $x \in [a, b]$.

Utilising the Hölder inequality, we have for $r > 1$, $1/r + 1/q = 1$ that

$$
\begin{aligned}
(2.4) \qquad & \left|\int_x^b p(t) f(t)\, dt\right| \\
& \leq \left(\int_x^b p(t)\, dt\right)^{\frac{1}{q}} \left(\int_x^b p(t) |f(t)|^r\, dt\right)^{\frac{1}{r}} \\
& = \left(\int_a^b p(t)\, dt - \int_a^x p(t)\, dt\right)^{\frac{1}{q}} \\
& \qquad \times \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt\right)^{\frac{1}{r}} \\
& = \left(1 - \int_a^x p(t)\, dt\right)^{\frac{1}{q}} \\
& \qquad \times \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt\right)^{\frac{1}{r}}
\end{aligned}
$$

for each $x \in [a, b]$.

Utilising (2.3) and (2.4) and taking the power r, we get

$$
\begin{aligned}
& \left|\int_a^b p(t) f(t)\, dt - \int_a^x p(t) f(t)\, dt\right|^r \\
& \leq \left(1 - \int_a^x p(t)\, dt\right)^{\frac{r}{q}} \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt\right),
\end{aligned}
$$

which gives the desired inequality (2.1). □

COROLLARY 1. *With the assumptions of Theorem 1 and if*

$$\int_a^b p(t) f(t)\, dt = 0,$$

then

$$\frac{1+\left(\int_a^x p(t)\,dt\right)^{1-r}\left[1-\int_a^x p(t)\,dt\right]^{r-1}}{\left[1-\int_a^x p(t)\,dt\right]^{r-1}}\cdot\left|\int_a^x p(t)\,f(t)\,dt\right|^r \leq \int_a^b p(t)\,|f(t)|^r\,dt \tag{2.5}$$

for any $x\in[a,b]$.

Proof. Since $\int_a^b p(t)\,f(t)\,dt=0$, then, by (2.1), we have

$$\int_a^x p(t)\,|f(t)|^r\,dt+\frac{\left|\int_a^x p(t)\,f(t)\,dt\right|^r}{\left[1-\int_a^x p(t)\,dt\right]^{r-1}}\leq\int_a^b p(t)\,|f(t)|^r\,dt \tag{2.6}$$

for any $x\in[a,b]$.

Utilising Hölder's inequality for $r>1$, $1/r+1/q=1$, we have

$$\begin{aligned}\left|\int_a^x p(t)\,f(t)\,dt\right|^r &\leq\left(\int_a^x p(t)\,dt\right)^{\frac{r}{q}}\int_a^x p(t)\,|f(t)|^r\,dt\\ &=\left(\int_a^x p(t)\,dt\right)^{r-1}\int_a^x p(t)\,|f(t)|^r\,dt.\end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7), we get the desired result (2.5). □

REMARK 1. *If* $\int_a^b f(t)\,dt=0$, *then from inequality (2.2) we get the following result as well:*

$$\frac{1+(x-a)^{1-r}(b-x)^{r-1}}{(b-x)^{r-1}}\cdot\left|\int_a^x f(t)\,dt\right|^r\leq\int_a^b |f(t)|^r\,dt, \tag{2.8}$$

or, equivalently

$$\left|\int_a^x f(t)\,dt\right| \leq\left(\frac{(b-x)^{r-1}}{1+(x-a)^{1-r}(b-x)^{r-1}}\right)^{1/r}\cdot\left(\int_a^b |f(t)|^r\,dt\right)^{1/r}, \tag{2.9}$$

which is a Mahajani type result.

The following result is of interest:

COROLLARY 2. *Let $g:[a,b]\to\mathbb{C}$ be a Lebesgue integrable function and $p:[a,b]\to[0,\infty)$ an integrable weight with $\int_a^b p(t)\,dt=1$. Then*

$$\frac{1+\left(\int_a^x p(t)\,dt\right)^{1-r}\left[1-\int_a^x p(t)\,dt\right]^{r-1}}{\left[1-\int_a^x p(t)\,dt\right]^{r-1}} \times\left|\int_a^x p(t)\,g(t)\,dt-\int_a^x p(t)\,dt\cdot\int_x^b p(t)\,g(t)\,dt\right|^r \leq \int_a^b p(t)\left|g(t)-\int_a^b p(s)\,g(s)\,ds\right|^r dt, \tag{2.10}$$

for any $x\in[a,b]$.

In particular,

$$\frac{1+(x-a)^{1-r}(b-x)^{r-1}}{(b-x)^{r-1}}\cdot\left|\int_a^x g(t)\,dt-\frac{x-a}{b-a}\int_a^b g(t)\,dt\right|^r \leq \int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt, \tag{2.11}$$

for each $x\in[a,b]$.

The proof is by Corollary 1 applied for $f(t)=g(t)-\int_a^b p(s)\,g(s)\,ds$, $t\in[a,b]$. Then inequality (2.11) follows by (2.8) on choosing $f(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds$.

A similar result concerning the supremum of the weight can be stated as well:

PROPOSITION 1. *Let p,f be as in Theorem 1. Then we have the inequality*

$$\int_a^x |f(t)|\,dt+\frac{\left|\int_a^b p(t)\,f(t)\,dt-\int_a^x p(t)\,f(t)\,dt\right|}{\sup\limits_{t\in[x,b]} p(t)} \leq \int_a^b |f(t)|\,dt \tag{2.12}$$

for any $x\in[a,b]$.

Proof. We have

$$\begin{aligned}
&\left|\int_a^b p(t)\, f(t)\, dt - \int_a^x p(t)\, f(t)\, dt\right| \\
&\qquad = \left|\int_x^b p(t)\, f(t)\, dt\right| \\
&\qquad \leq \sup_{t\in[x,b]} p(t) \int_x^b |f(t)|\, dt \\
&\qquad = \sup_{t\in[x,b]} p(t) \left[\int_a^b |f(t)|\, dt - \int_a^x |f(t)|\, dt\right],
\end{aligned}$$

which easily implies (2.12). □

Corollary 3. *If p and f are as in Corollary 1, then we have*

$$\frac{\sup\limits_{t\in[a,x]} p(t) + \sup\limits_{t\in[x,b]} p(t)}{\sup\limits_{t\in[x,b]} p(t)} \left|\int_a^x p(t)\, f(t)\, dt\right| \leq \int_a^b |f(t)|\, dt, \tag{2.13}$$

for each $x \in [a,b]$.

Proof. Obviously

$$\left|\int_a^x p(t)\, f(t)\, dt\right| \leq \sup_{t\in[a,x]} p(t) \int_a^x |f(t)|\, dt \tag{2.14}$$

for any $x \in [a,b]$ and (2.12) becomes, under the assumption that $\int_a^b p(t)\, f(t)\, dt = 0$,

$$\int_a^x |f(t)|\, dt + \frac{\left|\int_a^x p(t)\, f(t)\, dt\right|}{\sup\limits_{t\in[x,b]} p(t)} \leq \int_a^b |f(t)|\, dt, \tag{2.15}$$

hence by (2.14) and (2.15) we deduce the desired result (2.13). □

Corollary 4. *If p, g are as in Corollary 2, then*

$$\frac{\sup_{t\in[a,x]} p(t) + \sup_{t\in[x,b]} p(t)}{\sup_{t\in[x,b]} p(t)} \times \left| \int_a^x p(t)\,g(t)\,dt - \int_a^x p(t)\,dt \cdot \int_x^b p(t)\,g(t)\,dt \right| \leq \int_a^b \left| g(t) - \int_a^b p(s)\,g(s)\,ds \right| dt, \tag{2.16}$$

for any $x \in [a,b]$.

The following result holds as well.

Proposition 2. *With the above assumptions for f and p we have*

$$\int_a^x p(t)\,|f(t)|\,dt + \left| \int_a^b p(t)\,f(t)\,dt - \int_a^x p(t)\,f(t)\,dt \right| \leq \int_a^b p(t)\,|f(t)|\,dt \tag{2.17}$$

for any $x \in [a,b]$.

Proof. We have

$$\begin{aligned} \left| \int_a^b p(t)\,f(t)\,dt - \int_a^x p(t)\,f(t)\,dt \right| &= \left| \int_x^b p(t)\,f(t)\,dt \right| \\ &\leq \int_x^b p(t)\,|f(t)|\,dt \\ &= \int_a^b p(t)\,|f(t)|\,dt - \int_a^x p(t)\,|f(t)|\,dt, \end{aligned}$$

which is clearly equivalent to (2.17). □

Corollary 5. *If p and f are as in Corollary 1, then*

$$2\left| \int_a^x p(t)\,f(t)\,dt \right| \leq \int_a^b p(t)\,|f(t)|\,dt \tag{2.18}$$

for any $x \in [a,b]$.

Corollary 6. *If* p, g *are as in Corollary 2, then*

$$\left|\int_a^x p(t)\,g(t)\,dt - \int_a^x p(t)\,dt \cdot \int_x^b p(t)\,g(t)\,dt\right| \le \frac{1}{2}\int_a^b p(t)\left|g(t) - \int_a^b p(s)\,g(s)\,ds\right| dt. \tag{2.19}$$

Remark 2. *If in Corollary 6 we choose the uniform weight* $p(t) = \frac{1}{b-a}$, $t \in [a,b]$, *then (2.19) becomes:*

$$\left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{2}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt, \tag{2.20}$$

for each $x \in [a,b]$.

The inequality (2.20) can be seen as the limiting case of (2.11) where $r \to 1$, $r > 1$.

Remark 3. *We observe that (2.20) produces the following Mahajani type inequality, which is, in a sense, the limiting case of (2.9) for* $r \to 1, r > 1$:

$$\left|\int_a^x f(t)\,dt\right| \le \frac{1}{2}\int_a^b |f(t)|\,dt, \tag{2.21}$$

provided $\int_a^b f(t)\,dt = 0$.

3. Applications for Grüss Type Inequalities

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$ consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.1}$$

In 1934, G. Grüss [6] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{3.2}$$

provided m, M, n, N are real numbers with the property

$$\begin{aligned} &-\infty < m \leq f \leq M < \infty, \\ &-\infty < n \leq g \leq N < \infty \quad \text{ a.e. on } \quad [a,b]. \end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible in (3.2) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $T(f,g)$ was derived in 1882 by Čebyšev [4] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|T(f,g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \tag{3.4}$$

where $\|f'\|_{\infty} := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general.

Čebyšev's inequality (3.4) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_{\infty}[a,b]$.

In 1970, A.M. Ostrowski [11] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{3.5}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (3.3) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [7] (see also [10, p. 210]) obtained the following result as well:

$$|T(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{3.6}$$

provided f, g are absolutely continuous and $f', g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|T(f,g)| \leq \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|\bar{f}(t)\right|^p dt\right)^{\frac{1}{p}} \\ \qquad \text{where } p>1,\ 1/p+1/q=1, \end{cases} \tag{3.7}$$

where

$$\bar{f}(t) := f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds, \qquad t\in[a,b].$$

For $\gamma = 0$, we get from the first inequality

$$|T(f,g)| \leq \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt \tag{3.8}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x\in[a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (3.7) we can deduce the following result obtained by Cheng and Sun [5]

$$|T(f,g)| \leq \frac{1}{2}(M-m)\cdot\frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt. \tag{3.9}$$

The constant $\frac{1}{2}$ is best in (3.9) as shown by Cerone and Dragomir in [3].

For $r>1$, we define

$$I(r) := \int_a^b \frac{[(b-x)(x-a)]^{r-1}}{(b-x)^{r-1}+(x-a)^{r-1}}dx.$$

For $r=2$, we have

$$I(2) = \frac{1}{b-a}\int_a^b (b-x)(x-a)\,dx = \frac{(b-a)^2}{6}. \tag{3.10}$$

For $r>2$, since the inequality

$$\begin{aligned}\frac{(b-x)^{r-1}+(x-a)^{r-1}}{2} &\geq \left[\frac{(b-x)+(x-a)}{2}\right]^{r-1}\\ &= \frac{1}{2^{r-1}}(b-a)^{r-1}\end{aligned}$$

holds, then

$$(b-x)^{r-1}+(x-a)^{r-1} \geq 2^{2-r}(b-a)^{r-1}, \qquad x\in[a,b],$$

and so

$$\begin{aligned}I(r) &\leq \frac{2^{r-2}}{(b-a)^{r-1}}\int_a^b [(b-x)(x-a)]^{r-1}\,dx \\ &\leq \frac{2^{r-2}}{(b-a)^{r-1}}\int_0^1 (b-(1-t)a-tb)^{r-1} \\ &\qquad\times((1-t)a+tb-a)^{r-1}(b-a)\,dt \\ &= \frac{2^{r-2}}{(b-a)^{r-1}}\int_0^1 (b-a)^{r-1}(1-t)^{r-1} \\ &\qquad\times(b-a)^{r-1}t^{r-1}(b-a)\,dt \\ &= 2^{r-2}(b-a)^r B(r,r),\ r\geq 2,\end{aligned} \tag{3.11}$$

where $B(\cdot,\cdot)$ is the well known Euler beta function.

A different possibility to bound $I(r)$ is by utilising the inequality between the harmonic and geometric means, namely

$$\frac{2\alpha\beta}{\alpha+\beta}\leq\sqrt{\alpha\beta}, \qquad \alpha,\beta>0.$$

Therefore

$$\frac{(b-x)^{r-1}(x-a)^{r-1}}{(b-x)^{r-1}+(x-a)^{r-1}}\leq\frac{1}{2}\sqrt{(b-x)^{r-1}(x-a)^{r-1}}, \qquad r>1$$

for $x \in [a,b]$, which gives by integration

$$I(r) \leq \frac{1}{2}\int_a^b (b-x)^{\frac{r-1}{2}} (x-a)^{\frac{r-1}{2}}\,dx = \frac{1}{2}(b-a)^r \int_0^1 t^{\frac{r-1}{2}}(1-t)^{\frac{r-1}{2}}\,dt = \frac{1}{2}(b-a)^r B\left(\frac{r+1}{2},\frac{r+1}{2}\right) \tag{3.12}$$

for $r > 1$.

REMARK 4. *If we compare* $2^{r-2}B(r,r)$ *with* $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ *for* $r \geq 2$, *using the Maple computer package, we observe that the bound provided by (3.11) for* $I(r)$ *is better than the one provided for (3.12). However, the second one is also valid for* $r \in (0,1)$. *The plot concerning the variations of* $2^{r-2}B(r,r)$ *and* $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ *on* $[2,\infty)$ *is depicted in Figure 1. However, we do not have an analytic proof to show that*

$$2^{r-2}B(r,r) \leq \frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$$

for any $r \in [2,\infty)$.

The following lemma may be stated.

LEMMA 1. *For* $r > 1$ *we have the inequality*

$$\int_a^b \left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right|^r dt \leq I(r)\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt. \tag{3.13}$$

In particular:

$$\int_a^b \left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right|^2 dt \leq \frac{(b-a)^2}{6}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt. \tag{3.14}$$

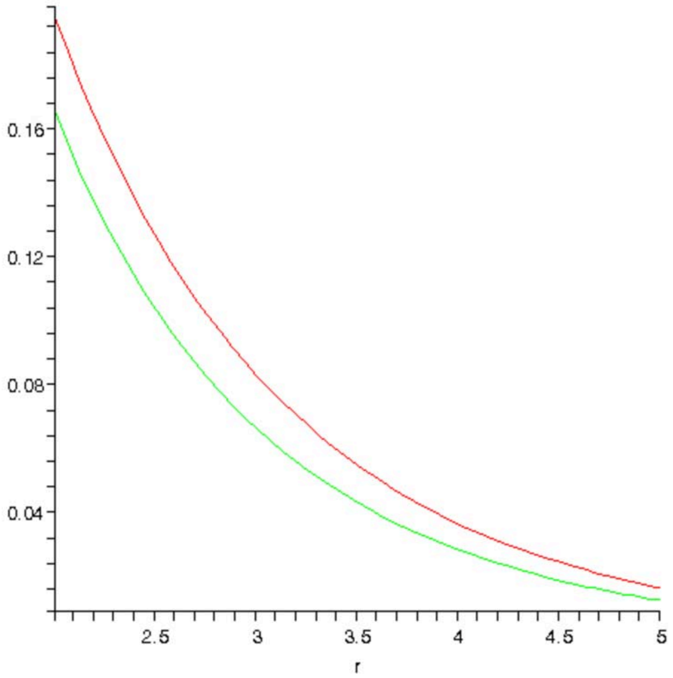

FIGURE 1. The plot of $2^{r-2}B\left(r,r\right)$ and $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ for $r\geq 2$.

The proof follows by the inequality (2.11) which is equivalent with

$$\begin{aligned}(3.15)\quad &\int_a^b\left|\int_a^x g\left(t\right)dt-\frac{x-a}{b-a}\int_a^b g\left(t\right)dt\right|^r dx\\ &\leq\frac{\left(b-x\right)^{r-1}\left(x-a\right)^{r-1}}{\left(b-x\right)^{r-1}+\left(x-a\right)^{r-1}}\int_a^b\left|g\left(t\right)-\frac{1}{b-a}\int_a^b g\left(s\right)ds\right|^r dt\end{aligned}$$

for any $x\in\left[a,b\right]$.

Also, if we take the supremum over $x \in [a,b]$ in (3.15) for $r = 2$, then we get

$$\sup_{x\in[a,b]} \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right|^2 \leq \frac{b-a}{4}\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^2 dt. \tag{3.16}$$

Therefore the following lemma may be stated:

LEMMA 2. *With the above assumptions, we have*

$$\sup_{x\in[a,b]} \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{(b-a)^{\frac{1}{2}}}{2}\left(\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^2 dt \right)^{\frac{1}{2}}. \tag{3.17}$$

Also, on utilising the inequality (2.20), we get the following result as well:

LEMMA 3. *With the above assumptions, we have*

$$\int_a^b \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right| dx \leq \frac{1}{2}(b-a)\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt. \tag{3.18}$$

We can now state the following result that provides upper bounds for the absolute value of the Čebyšev functional $T(f,g)$.

THEOREM 2. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then:*

$$(3.19) \quad |T(f,g)| \leq \begin{cases} \frac{1}{2}\int_a^b |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \\ \frac{1}{2}\cdot \sup_{x\in[a,b]} |f'(x)| \int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \\ [I(r)]^{\frac{1}{r}} \left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt\right)^{\frac{1}{r}} \\ \quad \times \left(\frac{1}{b-a}\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ \quad \text{where } r > 1,\ 1/r + 1/q = 1. \end{cases}$$

In particular, for $r = 2$, we have

$$(3.20) \quad |T(f,g)| \leq \frac{\sqrt{6}}{6}(b-a)\left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \times \left(\frac{1}{b-a}\int_a^b |f'(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Proof. Utilising the identity (2.13) from [2], namely,

$$T(f,g) = -\frac{1}{b-a}\int_a^b \bar{G}(x)\,f'(x)\,dx,$$

where

$$\bar{G}(x) = \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt, \qquad x \in [a,b],$$

we have

$$|T(f,g)| \leq \frac{1}{b-a}\int_a^b \left|\bar{G}(x)\right| |f'(x)|\,dx =: T.$$

Using Lemma 2, we have

$$\begin{aligned} T &\leq \sup_{x\in[a,b]} \left|\bar{G}(x)\right| \cdot \frac{1}{b-a}\int_a^b |f'(x)|\,dx \\ &\leq \frac{(b-a)^{\frac{1}{2}}}{2} \cdot |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \\ &= \frac{1}{2} \cdot \int_a^b |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}}, \end{aligned}$$

and the first inequality in (3.19) is proved.

Utilising Lemma 3, we have

$$\begin{aligned} T &\leq \sup_{x\in[a,b]} |f'(x)| \cdot \frac{1}{b-a}\int_a^b \left|\bar{G}(x)\right| dx \\ &\leq \frac{1}{2}\sup_{x\in[a,b]} |f'(x)| \int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt, \end{aligned}$$

which proves the second inequality in (3.19).

Now, from Hölder's inequality and Lemma 1, we also have

$$\begin{aligned} T &\leq \frac{1}{b-a}\left(\int_a^b \left|\bar{G}(x)\right|^r dx\right)^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ &\leq \frac{1}{b-a}\left[I(r) \cdot \int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt\right]^{\frac{1}{r}} \\ &\qquad \times \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ &= \frac{1}{b-a}\left[I(r)\right]^{\frac{1}{r}} \left[\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt\right]^{\frac{1}{r}} \\ &\qquad \times \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \end{aligned}$$

and the last part of (3.19) is also proved. □

REMARK 5. *It is an open question whether or not the constants* $\frac{1}{2}$ *in (3.19) and* $\frac{\sqrt{6}}{6}$ *in (3.20) are best possible.*

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