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FIXED POINT THEOREMS IN FUZZY METRIC SPACES, FUZZY 2-METRIC SPACES AND FUZZY 3-METRIC SPACES USING SEMI-COMPATIBILITY

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Abstract. The object of this paper is to introduce the notion of semi-compatible maps in fuzzy metric spaces, fuzzy 2-metric spaces and fuzzy 3-metric spaces and to establish three common fixed point theorems for these spaces for four self-maps. These results improve, extend and generalize the results of [16]. As an application , these results have been used to obtain translation and generalization of Grabeic's contraction principle in the new settings. All the result presented in this paper are new.

1. Introduction

Zadeh [20] initiated the concept of fuzzy sets in 1965. Many authors used this concept in Topology and Analysis and developed the theory of fuzzy sets and its application. Kramosil and Michalek [9] introduced the concept of fuzzy metric space. George and Veeramani [4] modified this concept of fuzzy metric space and defined Hausdroff topology on fuzzy metric space. Grabiec [6] obtained the fuzzy version of Banach contraction principle, which is a milestone in developing fixed-point theory in fuzzy metric space. Sessa [15] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weakly commuting maps in metric spaces.

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Jungck [8] soon enlarged this concept to compatible maps. The concepts of R-weakly commuting maps and compatible maps in fuzzy metric space have been introduced by Vasuki [19] and Mishra et al [10] respectively. Cho, Sharma and Sahu [2] introduced the cnocept of semi-compatible maps in a d-topological space. They define a pair of self maps (S, T) to be semi-compatible if conditions (i) $Sy = Ty$ implies $STy = TSy$. (ii) $\{Sx_n\} \to x, \{Tx_n\} \to x$ implies $STx_n \to Tx$, as $n \to \infty$ hold. However, in fuzzy metric space (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So, we define a semi-compatible pair of self-maps in fuzzy metric space by condition (ii) only. Recently Jungck and Rohades [7] termed a pair of self-maps to be coincidentally commuting or equivalently weak-compatible if they commute at their coincidence points. The concept of 2-metric space was initiated by Gahler [4] whose abstract properties were suggested by the area function in Euclidean space. Wenzi [17] initiated the concept of probabilistic 2-metric spaces . Chang and Huang [1] established some fixed point results in probabilistic 2-metric spaces associating it with a partial order relation. Recently, Sharma [16] proved certain common fixed point theorems for three self-maps in fuzzy metric spaces, fuzzy 2-metric spaces and fuzzy 3-metric spaces.

In this paper we prove three fixed point theorems for four self-maps which generalize substantially, the results of Sharma [16] in fuzzy metric spaces, fuzzy 2-metric spaces and fuzzy 3-metric spaces respectively by using the concept of semi-compatibility and weak-compatibility.

2. Preliminaries

DEFINITION 1. :A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c and $d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = min\{a, b\}$.

DEFINITION 2. ([9]) :The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is

a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:
$(FM-1)$ $M(x, y, 0) = 0$;
$(FM-2)$ $M(x, y, t) = 1$, for all $t > 0$ iff $x = y$;
$(FM-3)$ $M(x, y, t) = M(y, x, t)$;
$(FM-4)$ $M(x, y, t) * M(y, z, s) \geq M(x, z, t+s)$;
$(FM-5)$ $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

EXAMPLE 3. : Let (X, d) be a metric space. Define $a*b = \min\{a, b\}$ and for all $x, y \in X$,
$M(x, y, t) = t/(t + d(x, y))$ for all $t > 0$ and $M(x, y, 0) = 0$.
Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

LEMMA 4. *(Grabiec* [6]*) : For all $x, y \in X, M(x, y, .)$ is a non-decreasing function.*

DEFINITION 5. (Grabiec [6]): Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if $Lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $Lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

REMARK 6. : Since * is continuous, it follows from $(FM-4)$ that the limit of a sequence in a fuzzy metric space is unique, if it exists.

In this paper $(X, M, *)$ is considered to be the fuzzy metric space with condition
$(FM-6)$ $Lim_{t\to\infty} M(x, y, t) = 1$.

DEFINITION 7. : A function M is continuous in fuzzy metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then
$Lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for each $t > 0$.

Definition 8. :A binary operation $* : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b*c \leq d*e*f$ whenever $a \leq d$, $b \leq e$ and $c \leq f$, for all a,b,c,d,e and $f \in [0,1]$.

Definition 9. (Sharma[16]) :The 3-tuple $(X, M, *)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t, t_1, t_2, t_3 > 0$:
$(FM'-1)$ $M(x,y,z,0) = 0$;
$(FM'-2)$ $M(x,y,z,t) = 1$, for all $t > 0$, iff at least two of the three points
$(FM'-3)$ $M(x,y,z,t) = M(y,x,z,t) = M(z,x,y,t)$;(symmetry)
$(FM'-4)$ $M(x,y,z,t_1+t_2+t_3) \geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$;
This corresponds to tetrahedral inequality in 2-metric space.
$(FM'-5)$ $M(x,y,z,.) : [0,\infty) \to [0,1]$ is left continuous.

The function value $M(x,y,z,t)$ may be interpreted as the probability that the area of triangle is less than t.

Definition 10. : Let $(X, M, *)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if

$$Lim_{n\to\infty} M(x_n, x, a, t) = 1,$$

for all $a \in X$ and for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $Lim_{n\to\infty} M(x_n, x_{n+p}, a, t) = 1$ for all $a \in X, t > 0$ and $p > 0$. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 11. : A function M is continuous in fuzzy 2- metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then
$Lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t)$
for all $a \in X$ and for each $t > 0$.

Definition 12. :A binary operation $* : [0,1]^4 \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a * b * c * d \leq e * f * g * h$ whenever $a \leq e$, $b \leq f, c \leq g$ and $d \leq h$, for all a, b, c, d, e, f, g and $h \in [0,1]$.

Definition 13. (Sharma [16]) :The 3-tuple $(X, M, *)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions for all $x, y, z, u, w \in X$ and $t, t_1, t_2, t_3, t_4 > 0$:

$(FM'' - 1)$ $M(x, y, z, w, 0) = 0$;

$(FM'' - 2)$ $M(x, y, z, w, t) = 1$, for all $t > 0$, iff at-least two of the four points are equal

$(FM'' - 3)$ $M(x, y, z, w, t) = M(y, x, z, w, t) = M(w, z, x, y, t) = \cdots$; (symmetry)

$(FM'' - 4)$ $M(x, y, z, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$;

$(FM'' - 5)$ $M(x, y, z, w, .) : [0, \infty) \to [0, 1]$ is left continuous.

Definition 14. : Let $(X, M, *)$ be a fuzzy 3-metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if

$$Lim_{n \to \infty} M(x_n, x, a, b, t) = 1$$

for all $a, b \in X$ and for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $lim_{n \to \infty} M(x_n, x_{n+p}, a, b, t) = 1$ for all $a, b \in X, t > 0$ and $p > 0$. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 15. : A function M is continuous in fuzzy 3- metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then
$Lim_{n \to \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$
for all $a, b \in X$ and for each $t > 0$.

Lemma 16. *(Mishra et al [10]):Let (X, M,*) be a fuzzy metric space. If there exists a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$, $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.*

Lemma 4, 16 and remark 1 hold for a fuzzy 2-metric space and a fuzzy 3-metric space as well.

Compatible Mapping

In this section, we deal with the concept of semi-compatible mappings, compatible mappings and weak-compatible mappings and some properties of it for our main result.

DEFINITION 17. (Mishra et al [10])): Let A and B mappings from a fuzzy metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if $Lim_{n\to\infty}M(ABx_n, BAx_n, t) = 1$ for all $t > 0$,
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty}Ax_n = Lim_{n\to\infty}Bx_n = x \in X$.
Now we extend this concept in fuzzy 2-metric spaces and in fuzzy 3-metric spaces as follows:
Let A and B mappings from a fuzzy 2-metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if
$Lim_{n\to\infty}M(ABx_n, BAx_n, a, t) = 1$ for all $t > 0$ and for all $a \in X$.
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty}Ax_n = Lim_{n\to\infty}Bx_n = x \in X$.
Let A and B mappings from a fuzzy 3-metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if
$Lim_{n\to\infty}M(ABx_n, BAx_n, a, b, t) = 1$ for all $t > 0$ and for all $a, b \in X$.
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty}Ax_n = Lim_{n\to\infty}Bx_n = x \in X$.

DEFINITION 18. :Let A and B mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings are said to be weak-compatible if they commute at their coincidence points i.e. $Ax = Bx$ implies $ABx = BAx$.

REMARK 19. : Let (A, S) be pair of self -mappings of a fuzzy metric space $(X, M, *)$. Then (A, S) is commuting implies (A, S) is compatible. Also (A, S) is compatible implies (A, S) is weak-compatible but the converse is not true.

DEFINITION 20. ([12]): Let A and B mappings from a fuzzy metric space, $(X, M, *)$ into itself. The mappings are said to be semi-compatible if
$Lim_{n\to\infty}M(ABx_n, Bx, t) = 1$ for all $t > 0$,
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty}Ax_n = Lim_{n\to\infty}Bx_n = x \in X$.

Now we extend this concept in fuzzy 2-metric spaces and in fuzzy 3-metric spaces as follows:
Let A and B mappings from a fuzzy 2-metric space, $(X, M, *)$ into itself. The mappings are said to be semi-compatible if
$Lim_{n\to\infty} M(ABx_n, Bx, a, t) = 1$ for all $t > 0$ and for all $a \in X$.
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty} Ax_n = Lim_{n\to\infty} Bx_n = x \in X$.
Let A and B mappings from a fuzzy 3-metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if
$Lim_{n\to\infty} M(ABx_n, Bx, a, b, t) = 1$ for all $t > 0$ and for all $a, b \in X$.
When-ever, $\{x_n\}$ is a sequence in X such that,
$Lim_{n\to\infty} Ax_n = Lim_{n\to\infty} Bx_n = x \in X$.

In [12] it is shown that the semi-compatibility of (A, B) need not imply the semi-compatibility (B, A). Further, an example of a pair of self maps is given which is commuting hence compatible, weak-compatible but it is not semi-compatible. For a detailed discussion of semi-compatibility we refer to [11], [12] and [14] .

Proposition 21. *([12]) : Let A and S be self maps on a fuzzy metric space (X, M, *). If S is continuous then (A, S) is semi-compatible iff (A, S) is compatible.*

It is easy to verify that the above proposition is true for fuzzy 2-metric spaces and for fuzzy 3-metric spaces as well.

3. MAIN RESULTS

Theorem 22. *:Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying :*
(3.11) $A(X) \subseteq T(X), B(X) \subseteq S(X)$.
(3.12) *A or S is continuous.*
(3.13) *Pair (A, S) is semi-compatible and (B, T) is weak-compatible.*
(3.14) *there exists $k \in (0, 1)$ such that $\forall x, y \in X$ and $t > 0$*
$M(Ax, By, kt) \geq Min\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\}$
(3.15) $Lim_{t\to\infty} M(x, y, t) = 1, \forall x, y \in X$ *and* $t > 0$.
Then A, B, S and T have unique common fixed point in X.

Proof. :Let $x_0 \in X$ be any arbitrary point as $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $\exists x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$. Now using (3.14) with $x = x_{2n}, y = x_{2n+1}$ we get,

$$\begin{aligned} M(Ax_{2n},&Bx_{2n+1}, kt) \\ &= M(y_{2n+1}, y_{2n+2}, kt) \\ &\geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t), \\ &\qquad M(Ax_{2n}, Sx_{2n}, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t), \\ &\qquad M(y_{2n}, y_{2n+1}, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)\} \qquad \text{(i)} \end{aligned}$$

Thus,

$$\begin{aligned} M(y_{2n+1}, &y_{2n+2}, t) \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t/k)\} \qquad \text{(ii)} \end{aligned}$$

Putting (ii) in (i) we get,

$$\begin{aligned} M(y_{2n+1},&y_{2n+2}, kt) \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t/k), \\ &\qquad\qquad M(y_{2n}, y_{2n+1}, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, t/k), M(y_{2n}, y_{2n+1}, t)\} \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, t/k^2), M(y_{2n}, y_{2n+1}, t/k^2), \\ &\qquad\qquad M(y_{2n}, y_{2n+1}, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, t/k^2), M(y_{2n}, y_{2n+1}, t)\} \\ &\geq \cdots\cdots\cdots \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, t/k^m), M(y_{2n}, y_{2n+1}, t)\} \end{aligned}$$

Taking limit as $m \to \infty$ we get,

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t), \forall t > 0.$$

Similarly, we can get,

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+2}, y_{2n+1}, t), \forall t > 0.$$

Thus for all n and $t > 0$
$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$
Therefore,

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_{n-1}, y_n, t/k) \\ &\geq M(y_{n-2}, y_{n-1}, t/k^2) \\ &\geq \ldots \ldots \\ &\geq M(y_0, y_1, t/k^n) \end{aligned}$$

Hence, $Lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1, \forall t > 0.$
Now, for any integer p

$$\begin{aligned} &M(y_n, y_{n+p}, t) \\ &\geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p). \end{aligned}$$

Therefore $Lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1 = 1.$
Hence, $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore $\{y_n\}$ converges to $u \in X$. Its subsequences

$$\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$$

also converge to u. i.e.

$$\{Ax_{2n}\} \to u \quad \text{and} \quad \{Bx_{2n+1}\} \to u \tag{1}$$

$$\{Sx_{2n}\} \to u \quad \text{and} \quad \{Tx_{2n+1}\} \to u \tag{2}$$

Case I: A is continuous
In this case
$AAx_{2n} \to Az, ASx_{2n} \to Az$
And semi-compatibility of the pair (A, S) gives, $Lim_{n\to\infty} ASx_{2n} = Sz$.
As limit of a sequence in fuzzy metric space is unique, we have

$$Az = Sz \tag{3}$$

Step I: Putting $x = z, y = x_{2n+1}$ in (3.14) we get,

$$\begin{aligned} &M(Az, Bx_{2n+1}, kt) \\ &\quad \geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sz, Tx_{2n+1}, t), M(Az, Sz, t)\} \end{aligned}$$

Taking limit as $n \to \infty$, using (1), (2) and (3) we get,

$$\begin{aligned} M(Az, z, kt) &\geq Min\{M(z, z, t), M(Az, z, t), M(Az, Az, t)\} \\ &= M(Az, z, t), \forall t > 0. \end{aligned}$$

By lemma 16, $Az = z$. Thus $Az = Sz = z$.

Step 2: As $A(X) \subseteq T(X), \exists u \in X$ such that $z = Az = Tu$.
Putting $x = x_{2n}, y = u$ in (3.14) we get,

$$\begin{aligned} &M(Ax_{2n}, Bu, kt) \\ &\qquad \geq Min\{M(Bu, Tu, t), M(Sx_{2n}, Tu, t), M(Ax_{2n}, Sx_{2n}, t)\} \end{aligned}$$

Taking limit as $n \to \infty$ we get,

$$\begin{aligned} M(z, Bu, kt) &\geq Min\{M(Bu, z, t), M(z, z, t), M(z, z, t)\} \\ &= M(Bu, z, t), \forall t > 0. \end{aligned}$$

By lemma 16, we get, $z = Bu = Tu$ and the weak compatibility of (B, T) gives $TBu = BTu$ i.e. $Tz = Bz$.

Step 3: Putting $x = z, y = z$ in (3.14) we have,
$M(Az, Bz, kt) \geq Min\{M(Bz, Tz, t), M(Sz, Tz, t), M(Az, Sz, t)\}$.
Using the results from above steps we get,

$$\begin{aligned} M(z, Bz, kt) \geq &\ Min\{M(Bz, Bz, t), M(z, Bz, t), M(z, z, t)\} \\ &= M(z, Bz, t), \forall t > 0. \end{aligned}$$

which gives $Bz = z$.
Combining the results from the above steps we get, $z = Az = Bz = Sz = Tz$. i.e. z is common fixed point of A, B, S and T in this case.
Case II: S is continuous
In this case
$SAx_{2n} \to Sz, S^2x_{2n} \to Sz$
And semi-compatibility of the pair (A, S) gives $Lim_{n\to\infty} ASx_{2n} = Sz$.

Step 4: Putting $x = Sx_{2n}, y = x_{2n+1}$ in (3.14) we get,

$$\begin{aligned} M(ASx_{2n}, Bx_{2n+1}, kt) \geq Min\{&M(Bx_{2n+1}, Tx_{2n+1}, t), \\ &M(SSx_{2n}, Tx_{2n+1}, t), M(ASx_{2n}, SSx_{2n}, t)\} \end{aligned}$$

Taking limit as $n \to \infty$ we get,

$$\begin{aligned} M(Sz, z, kt) &\geq Min\{M(z, z, t), M(Sz, z, t), M(Sz, Sz, t)\} \\ &= M(Sz, z, t), \forall t > 0. \end{aligned}$$

By lemma 16, we get, $Sz = z$.

Step 5: Putting $x = z, y = x_{2n+1}$ in (3.14) we get,

$$M(Az, Bx_{2n+1}, kt) \geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, t), \\ M(Sz, Tx_{2n+1}, t), M(Az, Sz, t)\}.$$

Taking limit as $n \to \infty$ we get,

$$M(Az, z, kt) \geq Min\{M(z, z, t), M(z, z, t), M(Az, z, t)\} \\ = M(Az, z, t), \forall t > 0.$$

By lemma 16, we get, $Az = z$. Therefore, $Az = Sz = z$.
Now, apply step 2 and 3 of case 1 to get $Tz = Bz = z$.
Thus, $z = Az = Bz = Sz = Tz$. i.e. z is common fixed point of A, B, S and T in this case also.

Uniqueness: Let u be another common fixed point of A, B, S and T.
Then $u = Au = Bu = Su = Tu$.
Putting $x = z$ and $y = u$ in (3.14) we get,
$M(Az, Bu, kt) \geq Min\{M(Bu, Tu, t), M(Sz, Tu, t), M(Az, Sz, t)\}$
i.e. $M(z, u, kt) \geq M(z, u, t)$
Which yields $z = u$ and therefore z is the unique common fixed point of the four self maps A, B, S and T.
The following example validates theorem 3.1.

Example 1: Let (X, d) be the metric space with $X = [0, 1]$ and $d(x, y) = |x - y|$. Let $(X, M, *)$ be the induced fuzzy metric space,with $a * b = Min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+|x-y|}$. Define self mappings A, B, S and T as follows

$$A(X) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2 \\ 1/3, & \text{otherwise} \end{cases}, \quad Sx = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2 \\ 1, & \text{otherwise} \end{cases}$$

$$B(X) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4 \\ 1/3, & \text{otherwise} \end{cases} \quad and \quad Tx = \begin{cases} 1/3, & \text{if } 0 < x \leq 1/4 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 1/4 < x < 1 \end{cases}$$

Then S is continuous , the pair (A, S) is semi-compatible and (B, T) is weak-compatible. Also $A(X) = B(X) = \{0, 1/3\}, S(X) = [0, 1]$ and $T(X) = \{0, 1/3, 1\}$ satisfy the containment conditions of theorem 3.1. For $k = 1/2$ the condition (3.14) is satisfied and we obtain that 0 is the unique common fixed point of four mappings. □

COROLLARY 23. *:Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying (3.11), (3.14), (3.15) and*
(3.21) S is continuous.
(3.22) Pair (A, S) is compatible and (B, T) is weak-compatible.
Then A, B, S and T have a fixed point in X.

Proof As S is continuous, the proof follows from theorem 22 and proposition 21. If we take $B = A$ in Corollary 23 then we get the following result for three self maps as follows:

COROLLARY 24. *: Let A, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying (3.15) and*
(a) $A(X) \subset S(X) \bigcap T(X)$.
(b) S is continuous.
(c) Pair (A, S) is compatible and (A, T) is weak-compatible.
(d) $M(Ax, Ay, kt) \geq Min\{M(Ay, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\}$
$\forall x, y \in X, t > 0$ and $0 < k < 1$.
Then S, T and A have a unique common fixed point in X.

Remark: This corollary generalizes and improves theorem 1 of [16] in the sense that commutativity of the maps (A, S) and (A, T) has been replaced by their compatibility and weak-compatibility respectively. Further, the required continuities have been reduced from two to one only.
If we take $S = T = I$, the identity map on X in theorem 22, then the conditions (3.11), (3.12) and (3.13) are trivially satisfied. Now, taking only one factor in RHS of the contraction condition (3.14), we get the fuzzy version of Banach contraction principle as given by Grabiec [6] as follows:

COROLLARY 25. *: Let A be a self map on a complete fuzzy metric space $(X, M, *)$ with the condition (FM-6) such that*
$M(Ax, Ay, kt) \geq M(x, y, t), \forall x, y \in X, k \in (0, 1), t > 0$.
Then A has a unique common fixed point in X.

As an application of theorem 3.1, we have the following generalization of the Banach contraction principle.

THEOREM 26. *Let A be a self map on a complete fuzzy metric space $(X, M, *)$ with the condition (FM-6) such that*
(3.51) for some $k \in (0,1)$ and for some $p, q, r \in [0,1]$ with $p+q+r = 1$, $M(Ax, Ay, kt) \geq pM(x, y, t) + qM(Ax, x, t) + rM(Ay, y, t), \forall x, y \in X, t > 0$.
Then A has a unique common fixed point in X.

Proof Let

$$m = min\{M(x, y, t), M(Ax, x, t), M(Ay, y, t)\}$$

Then from (3.51), we have

$$M(Ax, Ay, kt) \geq pm + qm + rm = (p + q + r)m = m$$

and the result follows from theorem 3.1 by taking $A = B, S = T = I$, the identity mapping on X. In the following we extend theorem 22 to fuzzy 2-metric space as follows:

THEOREM 27. *:Let A, B, S and T be self mappings of a complete fuzzy 2-metric space $(X, M, *)$ satisfying (3.11), (3.12),(3.13) and*
(3.61) *there exists such that $\forall x, y, a \in X$ and $t > 0$*

$$M(Ax, By, a, kt) \geq Min\{M(By, Ty, a, t), M(Sx, Ty, a, t), M(Ax, Sx, a, t)\}$$

(3.62) $Lim_{t\to\infty} M(x, y, a, t) = 1, \forall x, y, a \in X$ *and* $t > 0$.
Then A, B, S and T have unique common fixed point in X.

Proof. :As in proof of theorem 22 define sequences $\{y_n\}$ and $\{x_n\}$ in X as follows $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, \ldots$. Now, we show that $\{y_n\}$ is a Cauchy sequence in X. Using (3.61) with $x = x_{2n}, y = x_{2n+1}$ we get,

$$\begin{aligned} M(Ax_{2n}, &Bx_{2n+1}, a, kt) \\ &= M(y_{2n+1}, y_{2n+2}, a, kt) \\ &\geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Sx_{2n}, Tx_{2n+1}, a, t), \\ &\qquad M(Ax_{2n}, Sx_{2n}, a, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, a, t), M(y_{2n}, y_{2n+1}, a, t), \\ &\qquad M(y_{2n}, y_{2n+1}, a, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, a, t), M(y_{2n}, y_{2n+1}, a, t)\} \qquad \text{(iii)} \end{aligned}$$

Thus,

$$M(y_{2n+1}, y_{2n+2}, a, t) \geq Min\{M(y_{2n+1}, y_{2n+2}, a, t/k), \\ M(y_{2n}, y_{2n+1}, a, t/k)\} \qquad \text{(iv)}$$

Putting (iii) in (iv) we get,

$$\begin{aligned} M(y_{2n+1}, \quad & y_{2n+2}, a, kt) \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, a, t/k), M(y_{2n}, y_{2n+1}, a, t/k), \\ &\qquad M(y_{2n}, y_{2n+1}, a, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, a, t/k), M(y_{2n}, y_{2n+1}, a, t)\} \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, a, t/k^2), M(y_{2n}, y_{2n+1}, a, t/k^2), \\ &\qquad M(y_{2n}, y_{2n+1}, a, t)\} \\ &= Min\{M(y_{2n+1}, y_{2n+2}, a, t/k^2), M(y_{2n}, y_{2n+1}, a, t)\} \\ &\geq \cdots \quad \cdots \quad \cdots \\ &\geq Min\{M(y_{2n+1}, y_{2n+2}, a, t/k^m), M(y_{2n}, y_{2n+1}, a, t)\} \end{aligned}$$

Taking limit as $m \to \infty$ we get,

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t), \forall t > 0.$$

Similarly, we can get,

$$M(y_{2n+2}, y_{2n+3}, a, kt) \geq M(y_{2n+2}, y_{2n+1}, a, t), \forall t > 0.$$

Thus for all n and $t > 0$

$M(y_n, y_{n+1}, a, kt) \geq M(y_{n-1}, y_n, a, t)$

Therefore,

$$\begin{aligned} M(y_n, y_{n+1}, a, t) \quad &\geq M(y_{n-1}, y_n, a, t/k) \\ &\geq M(y_{n-2}, y_{n-1}, a, t/k^2) \\ &\geq \ldots ... \\ &\geq M(y_0, y_1, a, t/k^n) \end{aligned}$$

Hence, $Lim_{n\to\infty}M(y_n, y_{n+1}, a, t) = 1, \forall t > 0$.
Now, we prove by induction on p
$Lim_{n\to\infty}M(y_n, y_{n+p}, a, t) = 1, \forall t > 0$. (*)
Clearly (*) is true for $p = 1$. Suppose that (*) is true for $p = m$.
i.e $Lim_{n\to\infty}M(y_n, y_{n+m}, a, t) = 1, \forall t > 0$. (Induction hypothesis)
Now using $(FM' - 4)$

$$\begin{aligned} M(y_n, y_{n+m+1}, a, t) = \ & M(y_n, y_{n+m}, a, t/3) * M(y_n, y_{n+m}, y_{n+m+1}, t/3) \\ & * M(y_{n+m}, y_{n+m+1}, a, t/3) \end{aligned}$$

Therefore,

$$Lim_{n\to\infty}M(y_n, y_{n+m+1}, a, t) = 1 * 1 * 1 = 1.$$

Hence (*) is true for $p = m + 1$. Thus, (*) holds for all p and we get $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore, $\{y_n\}$ converges to $z \in X$ and equations (1) and (2) of theorem 22 hold.

Case I: A is continuous

In this case
$AAx_{2n} \to Az, ASx_{2n} \to Az$
And semi-compatibility of the pair (A, S) gives, $Lim_{n\to\infty}ASx_{2n} = Sz$.
As limit of a sequence in fuzzy metric space is unique, we have

$$Az = Sz \tag{4}$$

Step I: Putting $x = z, y = x_{2n+1}$ in (3.61) we get,

$$\begin{aligned} M(Az, Bx_{2n+1}, a, kt) \geq Min\{ & M(Bx_{2n+1}, Tx_{2n+1}, a, t), \\ & M(Sz, Tx_{2n+1}, a, t), M(Az, Sz, a, t)\} \end{aligned}$$

Taking limit as $n \to \infty$, using (1), (2) and (4) we get,

$$\begin{aligned} M(Az, z, a, kt) & \geq Min\{M(z, z, a, t), M(Az, z, a, t), M(Az, Az, a, t)\} \\ & = M(Az, z, a, t), \forall t > 0. \end{aligned}$$

By lemma 16, $Az = z$. Thus $Az = Sz = z$.

Step 2: As $A(X) \subseteq T(X), \exists u \in X$ such that $z = Az = Tu$.
Putting $x = x_{2n}, y = u$ in (3.61) we get,

$$\begin{aligned} M(Ax_{2n}, Bu, a, kt) \geq Min\{ & M(Bu, Tu, a, t), M(Sx_{2n}, Tu, a, t), \\ & M(Ax_{2n}, Sx_{2n}, a, t)\} \end{aligned}$$

Taking limit as $n \to \infty$ we get,
$$\begin{aligned} M(z, Bu, a, kt) &\geq Min\{M(Bu, a, z, t), M(z, z, a, t), M(z, z, a, t)\} \\ &= M(Bu, z, a, t), \forall t > 0. \end{aligned}$$
by lemma 16, we get, $z = Bu = Tu$ and the weak compatibility of (B, T) gives $TBu = BTu$ i.e. $Tz = Bz$.

Step 3: Putting $x = z, y = z$ in (3.61) we have,
$$M(Az, Bz, a, kt) \geq Min\{M(Bz, Tz, a, t), M(Sz, Tz, a, t), M(Az, Sz, a, t)\}.$$
Using the results from above steps we get,
$$\begin{aligned} M(z, Bz, a, kt) &\geq Min\{M(Bz, Bz, a, t), M(z, Bz, a, t), M(z, z, a, t)\} \\ &= M(z, Bz, a, t), \forall t > 0. \end{aligned}$$
which gives $Bz = z$.

Combining the results from the above steps we get, $z = Az = Bz = Sz = Tz$. i.e. z is common fixed point of A, B, S and T in this case.

Case II: S is continuous

In this case

$SAx_{2n} \to Sz, S^2x_{2n} \to Sz$

And semi-compatibility of the pair (A, S) gives $Lim_{n\to\infty} ASx_{2n} = Sz$.

Step 4: Putting $x = Sx_{2n}, y = x_{2n+1}$ in (3.61) we get,
$$M(ASx_{2n}, Bx_{2n+1}, a, kt) \geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(SSx_{2n}, Tx_{2n+1}, a, t), M(ASx_{2n}, SSx_{2n}, a, t)\}$$
Taking limit as $n \to \infty$ we get,
$$\begin{aligned} M(Sz, z, a, kt) &\geq Min\{M(z, z, a, t), M(Sz, z, a, t), M(Sz, Sz, a, t)\} \\ &= M(Sz, z, a, t), \forall t > 0. \end{aligned}$$
by lemma 16, we get, $Sz = z$.

Step 5: Putting $x = z, y = x_{2n+1}$ in (3.61) we get,
$$M(Az, Bx_{2n+1}, a, kt) \geq Min\{M(Bx_{2n+1}, Tx_{2n+1}, a, t), M(Sz, Tx_{2n+1}, a, t), M(Az, Sz, a, t)\}.$$
Taking limit as $n \to \infty$ we get,
$$\begin{aligned} M(Az, z, a, kt) &\geq Min\{M(z, z, a, t), M(z, z, a, t), M(Az, z, a, t)\} \\ &= M(Az, z, a, t), \forall t > 0. \end{aligned}$$
by lemma 16, we get, $Az = z$. Therefore, $Az = Sz = z$.

Now, apply step 2 and 3 of case I to get $Tz = Bz = z$.

Thus, $z = Az = Bz = Sz = Tz$. i.e. z is common fixed point of

A, B, S and T in this case also.
Uniqueness: Let u be another common fixed point of A, B, S and T. Then $u = Au = Bu = Su = Tu$.
Putting $x = z$ and $y = u$ in (3.61) we get,

$$M(Az, Bu, a, kt) \geq Min\{M(Bu, Tu, a, t), \\ M(Sz, Tu, a, t), M(Az, Sz, a, t)\}$$

i.e. $M(z, u, a, kt) \geq M(z, u, a, t)$, which yields $z = u$ and therefore z is the unique common fixed point of the four self maps A, B, S and T. □

Corollary 28. *:Let A, B, S and T be self mappings of a complete fuzzy 2-metric space $(X, M, *)$ satisfying (3.11), (3.61), (3.52) and*
(3.71) S is continuous.
(3.72) *Pair (A, S) is compatible and (B, T) is weak-compatible.*
Then A, B, S and T have a fixed point in X.

Proof: As S is continuous, the proof follows from theorem 27 and proposition 21. If we take $B = A$ in Corollary 28 then we get the following result for three self maps as follows:

Corollary 29. *: Let A, S and T be self mappings of a complete fuzzy 2-metric space $(X, M, *)$ satisfying (3.62) and*
(a) $A(X) \subset S(X) \bigcap T(X)$.
(b) S is continuous.
(c) Pair (A, S) is compatible and (A, T) is weak-compatible.
(d) $\forall x, y, a \in X, t > 0$ and $0 < k < 1$,

$$M(Ax, Ay, a, kt) \geq Min\{M(Ay, Ty, a, t), \\ M(Sx, Ty, a, t), M(Ax, Sx, a, t)\}.$$

Then S, T and A have a unique common fixed point in X.

Remark: This corollary generalizes and improves theorem 2 of [16] in the sense that commutativity of the maps (A, S) and (A, T) has been replaced by their compatibility and weak-compatibility respectively. Further, the required continuities have been reduced from two to one only.

If we take $S = T = I$, the identity map on X in theorem 27, then the conditions (3.11), (3.12) and (3.13) are trivially satisfied. Now, taking only one factor in RHS of the contraction condition (3.61), we get the Banach contraction principle in fuzzy 2-metric space as follows:

COROLLARY 30. *: Let A be a self map on a complete fuzzy 2-metric space $(X, M, *)$ with the condition (6.12) such that*
$M(Ax, Ay, a, kt) \geq M(x, y, a, t), \forall x, y, a \in X, k \in (0, 1), t > 0.$
Then A has a unique common fixed point in X.

As an application of theorem 3.6, we have the following generalization of the above Banach contraction principle.

THEOREM 31. *Let A be a self map on a complete fuzzy metric space $(X, M, *)$ with the condition (3.62) such that for some $k \in (0, 1)$ and for some $p, q, r \in [0, 1]$ with $p + q + r = 1$,*

$$M(Ax, Ay, a, kt) \geq aM(x, y, a, t) + bM(Ax, x, a, t) + cM(Ay, y, a, t),$$

$\forall x, y, a \in X, t > 0.$
Then A has a unique common fixed point in X.

Proof: The proof can be given on the lines of proof of theorem 3.5. On the same lines we extend theorem 22 and its corollaries to fuzzy 3-metric space as follows:

THEOREM 32. *:Let A, B, S and T be self mappings of a complete fuzzy 3-metric space $(X, M, *)$ satisfying (3.11), (3.12),(3.13) and*
(3.111) *there exists $k \in (0, 1)$ such that $\forall x, y, a, b \in X$ and $t > 0$*

$$M(Ax, By, a, b, kt) \geq Min\{M(By, Ty, a, b, t), M(Sx, Ty, a, b, t), M(Ax, Sx, a, b, t)\}.$$

(3.112) $Lim_{t\to\infty} M(x, y, z, w, t) = 1, \forall x, y, z, w \in X$ *and* $t > 0.$
Then A, B, S and T have unique common fixed point in X.

COROLLARY 33. *:Let A, B, S and T be self mappings of a complete fuzzy 3-metric space $(X, M, *)$ satisfying (3.11), (3.111), (3.112) and*
(3.121) S is continuous.
(3.122) *Pair (A, S) is compatible and (B, T) is weak-compatible.*
Then A, B, S and T have a fixed point in X.

Proof: As S is continuous, the proof follows from theorem 32 and proposition 21. If we take $B = A$ in Corollary 33 then we get the following result for three self maps as follows:

Corollary 34. *: Let A, S and T be self mappings of a complete fuzzy 3-metric space $(X, M, *)$ satisfying (3.92) and*
(a) $A(X) \subset S(X) \bigcap T(X)$.
(b) S is continuous.
(c) Pair (A, S) is compatible and (A, T) is weak-compatible.
(d) $\forall x, y, a, b \in X, t > 0$ and $0 < k < 1$,

$$M(Ax, Ay, a, b, kt) \geq Min\{M(Ay, Ty, a, b, t), \\ M(Sx, Ty, a, b, t), M(Ax, Sx, a, b, t)\}$$

Then S, T and A have a unique common fixed point in X.

Remark : This corollary generalizes and improves theorem 3 of [16] in the sense that commutativity of the maps (A, S) and (A, T) has been replaced by their compatibility and weak-compatibility respectively. Further, the required continuities have been reduced from two to one only. If we take $S = T = I$, the identity map on X in theorem 32, then the conditions (3.11), (3.12) and (3.13) are trivially satisfied. Now, taking only one factor in RHS of the contraction condition (3.121), we get the Banach contraction principle in fuzzy 3-metric space as follows:

Corollary 35. *: Let A be a self map on a complete fuzzy 3-metric space $(X, M, *)$ with the condition (3.112) such that*
$M(Ax, Ay, a, b, kt) \geq M(x, y, a, b, t), \forall x, y, a, b \in X, k \in (0, 1), t > 0$.
Then A has a unique common fixed point in X.

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