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ON FUZZY β-COMPACT* SPACES AND FUZZY β-FILTERS

M. K. Uma, E. Roja, and G. Balasubramanian

Abstract. In this paper we introduce the concept of fuzzy β-compact* spaces. Besides giving some interesting properties of fuzzy β-compact* spaces we also give a characterization on fuzzy β-compact spaces by making use of newly introduced concept of fuzzy β-filters.

1. Introduction

A. S. Bin Shahna [2] introduced the notion of fuzzy compact spaces, fuzzy Lindelof spaces and investigated some of their properties. The concept of fuzzy filters was introduced and studied in [3]. In this paper we introduce the concept of fuzzy β-compact spaces and study their properties. In this connection we introduce the concept of fuzzy β-filters and making use of this concept we give characterization of fuzzy β-compact spaces.

Let λ be any fuzzy set in the fuzzy topological space (X, T). Then λ is called *fuzzy β-open set* [1] if $\lambda \leq \mathrm{cl}\,\mathrm{Int}\,\mathrm{cl}\,\lambda$. The complement of a fuzzy β-open set is called *fuzzy β-closed*. A fuzzy space X is called *fuzzy G_δ-space* if every fuzzy G_δ-set of X is fuzzy open [2].

Let (X, T) be fuzzy topological space and Y be ordinary subset of X. Then $T_Y = (\lambda/Y : \lambda \in T)$ is a fuzzy topology on Y and is called the *induced or relative topology* [1]. The pair (Y, T_Y) is called a *fuzzy subspace* of (X, T). (Y, T_Y) is called *fuzzy open /fuzzy closed/ fuzzy β-open subspace* if the characteristic function of Y viz χ_Y is

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fuzzy open/fuzzy closed / fuzzy β-open respectively. M. A. DE Prada Vincente and others defined fuzzy filters and fuzzy point in [3]. Rekha Srivastava and others defined the fuzzy product topology in [5].

2. Fuzzy β-compact* spaces

Bin Shahna introduced the concept of fuzzy G_δ space and fuzzy compact space [2]. In this section, we introduce the concept of fuzzy β-compact* space. Some interesting properties are studied.

Throughout this section S will denote fuzzy space consisting of a single point with the fuzzy topology $\{0,1\}$. Friedler [4] shows that if (X,T) is a fuzzy topological space, then $S \times X$ is fuzzy homeomorphic to X.

NOTE. Let (X,T) be a fuzzy topological space. For any fuzzy topological space (Z,R), p_{2x} denotes the projection of $X \times Z$ to Z.

In 1991 A. S. Bin Shahna [2] defined fuzzy compactness as follows. A fuzzy topological space X is said to be *fuzzy compact* if the projection $p_{2X} \colon X \times Z \to Z$ is fuzzy closed for any fuzzy topological space Z. Motivated by this definition we shall define, a fuzzy topological space (X,T) to be *fuzzy β-compact** if the projection $p_{2X} : X \times Z \to Z$ is fuzzy β-closed for any fuzzy topological space (Z,R).

PROPOSITION 1. *An M-fuzzy β-continuous image of a fuzzy β-compact* space is fuzzy β-compact*.*

Proof. Let $f \colon (X,T) \to (Y,S)$ be a M-fuzzy β-continuous mapping from a fuzzy β-compact* space (X,T) onto a fuzzy topological space (Y,S) and I_Z, be the identity mapping on a fuzzy topological space (Z,R). Then $f \times I_Z \colon X \times Z \to Y \times Z$ is M-fuzzy β-continuous. Let μ be a fuzzy β-closed set of $Y \times Z$. Then $(f \times I_Z)^{-1}(\mu)$ is fuzzy β-closed set of $X \times Z$. Since $p_{2X} \colon X \times Z \to Z$ is fuzzy β-closed, $p_{2X}((f \times I_Z)^{-1}(\mu)) = p_{2Y}(\mu)$ is a fuzzy β-closed set of (Z,R) showing that p_{2Y} is a fuzzy β-closed mapping. Hence (Y,S) is fuzzy β-compact*. □

DEFINITION 1. Let (X,T), (Y,S) and (Z,R) be any three fuzzy topological spaces. An M-fuzzy β-continuous map $f \colon (X,T) \to (Y,S)$

is *fuzzy β-perfect* if $f \times I_z : X \times Z \to Y \times Z$ is a fuzzy β-closed map for any fuzzy topological space (Z, R).

DEFINITION 2. Let (X, T) , (Y, S) and (Z, R) be any three fuzzy topological spaces. An M-fuzzy β-continuous mapping $f : (X, T) \to (Y, S)$ of a fuzzy topological space (X, T) into a fuzzy topological space (Y, S) is called a *fuzzy β-quasi perfect mapping* if $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy β-closed for any fuzzy G_δ- space (Z, R).

DEFINITION 3. A fuzzy topological space (X, T) is called *fuzzy β-Lindelöf ** if the projection $p_{2x} : X \times Z \to Z$ is a fuzzy β-closed mapping for any fuzzy G_δ- space (Z, R).

PROPOSITION 2. *If $f : (X, T) \to (Y, S)$ is a fuzzy β-perfect mapping of a fuzzy topological space (X, T) onto a fuzzy β-compact* space (Y, S), then (X, T) is fuzzy β-compact*.*

Proof. Since f is fuzzy β-perfect, $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy β-closed mapping for any fuzzy topological space (Z, R). For (X, T) to be fuzzy β-compact* we show that $p_{2X} : X \times Z \to Z$ is fuzzy β-closed mapping. Noting that p_{2X} is the composition of two fuzzy β-closed mappings $f \times I_Z$ and $p_{2Y} : Y \times Z \to Z$, the result follows. □

PROPOSITION 3. *Let (X, T) be a fuzzy topological space and (Y, S) be a fuzzy β-compact* space. Then the projection $p : X \times Y \to X$ is fuzzy β-perfect mapping.*

Proof. We show that $p \times I_Z : (X \times Y) \times Z \to X \times Z$ is a fuzzy β-closed mapping for any fuzzy topological space (Z, R). Since (Y, S) is fuzzy β-compact*. $p_{2Y} : Y \times (X \times Z) \to X \times Z$ is fuzzy β- closed mapping. Then $p \times I_Z$ is fuzzy β-closed follows by noting that it is the composition $p_{2Y} \circ h$,where $h : (X \times Y) \times Z \to Y \times (X \times Z)$ is a fuzzy β-homeomorphism. □

PROPOSITION 4. *A fuzzy topological space (X, T) is fuzzy β- compact* if and only if the constant mapping $c : X \to S$ is fuzzy β-perfect.*

Proof. Suppose (X, T) is fuzzy β-compact*. We show that $c \times I_Z : X \times Z \to S \times Z$ is fuzzy β-closed for any fuzzy topological space (Z, R). Since (X, T) is fuzzy β-compact*, $p_{2X} : X \times Z \to Z$ is fuzzy β-closed. Note that $S \times Z$ is fuzzy homeomorphic to (Z, R). Now

$c \times I_z = h \circ p_{2X}$, being a composition of a fuzzy β-closed mapping p_{2X} and a fuzzy β-homeomorphism $h\colon Z \to S \times Z$ is fuzzy β-closed. Hence c is fuzzy β-perfect.

Conversely, if $c : X \to S$ is fuzzy β-perfect, then $c \times I_Z\colon X \times Z \to S \times Z$ is fuzzy β-closed for any fuzzy topological space (Z, R). We show that $p_{2X}\colon X \times Z \to Z$ is fuzzy β-closed. Since $p_{2X} = h \circ (c \times I_Z)$, where $h : S \times Z \to Z$ is a fuzzy β- homeomorphism, p_{2X} is fuzzy β-closed and hence (X, T) is fuzzy β-compact*. □

Proposition 5. *The product of two fuzzy β-compact* spaces is fuzzy β-compact*.*

Proof. Let (X, T) and (Y, S) be two fuzzy β-compact* spaces. Then $p_{2X} : X \times (Y \times Z) \to (Y \times Z)$ and $p_{2Y} : Y \times Z \to Z$ are fuzzy β-closed mappings for any fuzzy topological space (Z, R). We show that $p_2\colon (X \times Y) \times Z \to Z$ is fuzzy β-closed. Since $p_2 = p_{2Y} \circ p_{2X}$ $\{X \times (Y \times Z)$ and $(X \times Y) \times Z$ are fuzzy homeomorphic$\}$, being a composition of two fuzzy β-closed mappings is fuzzy β-closed and hence $X \times Y$ is fuzzy β-compact*. □

Note. The finite product of fuzzy β-compact* spaces is fuzzy β-compact*.

Proposition 6. *Let (X_1, T_1), (X_2, T_2) and (X_3, T_3) be any three fuzzy topological spaces. The composition $g \circ f\colon (X_1, T_1) \to (X_3, T_3)$ of fuzzy β-quasi perfect mappings $f : (X_1, T_1) \to (X_2, T_2)$ and $g : (X_2, T_2) \to (X_3, T_3)$ is fuzzy β-quasi perfect.*

Proof. Let (Z, R) be a fuzzy G_δ space. Then $(g \circ f) \times I_z : X_1 \times Z \to X_3 \times Z$ is fuzzy β-closed follows from the identify $(g \circ f) \times I_z = (g \times I_z) \circ (f \times I_Z)$ by noting that f and g are fuzzy β-quasi perfect mappings. □

Proposition 7. *Let (X_1, T_1), (X_2, T_2) and (X_3, T_3) any three fuzzy topological spaces. Let $f : (X_1, T_1) \to (X_2, T_2)$ and $g : (X_2, T_2) \to (X_3, T_3)$ be M-fuzzy β-continuous mappings. Then*

(a) *if $g \circ f$ is fuzzy β-quasi perfect and f is surjective, then g is fuzzy β-quasi perfect.*
(b) *if $g \circ f$ is fuzzy β-quasi perfect and g is injective, then f is fuzzy β-quasi perfect.*

Proof. (a) We show that $g \times I_z \colon X_2 \times Z \to X_3 \times Z$ is fuzzy β-closed mapping for any fuzzy G_δ space (Z, R). Let μ be a fuzzy β-closed set of $X_2 \times Z$. Then $(f \times I_Z)^{-1}(\mu)$ is a fuzzy β-closed set of $X_1 \times Z$. Because $(g \circ f) \times I_Z$ is fuzzy β-closed and

$$[(g \circ f) \times I_Z)(f \times I_Z)^{-1}](\mu) = (g \times I_Z)(\mu),$$

it follows that $(g \times I_Z)(\mu)$ is fuzzy β-closed set of $X_3 \times Z$.

(b) We show that $f \times I_Z \colon X_1 \times Z \to X_2 \times Z$ is fuzzy β-closed mapping for any fuzzy G_δ space (Z, R). Let μ be a fuzzy β-closed set of $X_1 \times Z$. Then $((g \circ f) \times I_Z)(\mu)$ is a fuzzy β-closed set of $X_3 \times Z$. Because $g \times I_z$ is M-fuzzy β-continuous and $(g \times I_Z)^{-1}(((g \circ f) \times I_Z)(\mu)) = (f \times I_Z)(\mu)$, it follows that $(f \times I_Z)(\mu)$ is fuzzy β-closed set of $X_2 \times Z$. □

Proposition 8. *An M-fuzzy β-continuous image of a fuzzy β-Lindelöf* space is fuzzy β-Lindelöf*.*

Proof. Let f be a M-fuzzy β-continuous mapping from a fuzzy β-Lindelöf* sapce (X, T) onto a fuzzy β-Lindelöf* space (Y, S) and I_Z be the identity mapping on a fuzzy G_δ space (Z, R). Then $f \times I_z :$ $X \times Z \to Y \times Z$ is M-fuzzy β-continuous. Let μ be a fuzzy β-closed set of $Y \times Z$. Then $(f \times I_Z)^{-1}(\mu)$ is fuzzy β-closed set of $X \times Z$. Since $p_{2X} :$ $X \times Z \to Z$ is fuzzy β-closed mapping, $p_{2X}((f \times I_Z)^{-1}(\mu)) = p_{2Y}(\mu)$ is a fuzzy β-closed set of (Z, R) showing that p_{2Y} is fuzzy β-closed mapping. Hence (Y, S) is fuzzy β- Lindelöf*. □

Proposition 9. *If $f : (X, T) \to (Y, S)$ is a fuzzy β-quasi perfect mapping of a fuzzy topological space (X, T) onto a fuzzy β- Lindeöf* space (Y, S), then (X, T) is fuzzy β-Lindelöf*.*

Proof. Since f is fuzzy β-quasi perfect mapping, $f \times I_Z \colon X \times Z \to$ $Y \times Z$ is fuzzy β-closed for any fuzzy G_δ space (Z, R). For (X, T) to be fuzzy β-Lindelöf* we show that $p_{2X} \colon X \times Z \to Z$ is fuzzy β-closed. Noting that p_{2X} is the composition of two fuzzy β-closed mappings $f \times I_Z$ and $p_{2Y} \colon Y \times Z \to Z$ the result follows. □

Proposition 10. *Let (X, T) be a fuzzy G_δ space and (Y, S) be a fuzzy β-Lindelöf* space. Then the projection $p : X \times Y \to X$ is a fuzzy β-quasi perfect mapping.*

Proof. We show that $p \times I_Z \colon (X \times Y) \times Z \to X \times Z$ is a fuzzy β-closed mapping for any fuzzy G_δ space (Z, R). Since (Y, S) is fuzzy β-Lindelöf*, $p_{2Y} \colon Y \times (X \times Z) \to X \times Z$ is fuzzy β-closed. That $p \times I_Z$ is fuzzy β-closed follows by noting that it is the composition $p_{2Y} \circ h$, where $h \colon (X \times Y) \times Z \to Y \times (X \times Z)$ is a fuzzy β- homeomorphism. □

Proposition 11. *A fuzzy space (X, T) is fuzzy β-Lindelöf* iff the constant mapping $c \colon X \to S$ is fuzzy β-quasi perfect.*

Proof. Suppose (X, T) is fuzzy β-Lindelöf*. We show that $c \times I_Z \colon X \times Z \to S \times Z$ is fuzzy β-closed mapping for any fuzzy G_δ space (Z, R). Since (X, T) is fuzzy β-Lindelöf* $p_{2X} : X \times Z \to Z$ is fuzzy β-cloesd. Note that $S \times Z$ is fuzzy β-homeomorphic to (Z, R). Now $c \times I_Z = h \circ p_{2X}$, being the composition of fuzzy β-closed mapping p_{2X} and a β-homeomorphism $h \colon Z \to S \times Z$ is fuzzy β- closed. Hence c is fuzzy β-quasi perfect.

Conversely, if $c : X \to S$ is fuzzy β-quasi perfect, then $c \times I_Z : X \times Z \to S \times Z$ is fuzzy β-closed for any fuzzy G_δ space (Z, R). We show that $p_{2X} \colon X \times Z \to Z$ is fuzzy β-closed. Since $p_{2X} = h \circ (c \times I_Z)$, where $h : S \times Z \to Z$ is a fuzzy β- homeomorphism, p_{2X} is fuzzy β-closed and hence (X, T) is fuzzy β- Lindelöf*. □

Proposition 12. *A product of two fuzzy β-Lindelöf* spaces is a fuzzy β-Lindelöf* space.*

Proof. Let (X, T) and (Y, S) be two fuzzy β-Lindelöf* spaces. Since (X, T) and (Y, S) are fuzzy β-Lindelöf* spaces, $p_{2X} \colon X \times (Y \times Z) \to (Y \times Z)$ and $p_{2Y} : Y \times Z \to Z$ are fuzzy β-closed mappings for any fuzzy G_δ space (Z, R). We show that $p_2 \colon (X \times Y) \times Z \to Z$ is fuzzy β-closed.

Clearly, $p_2 = p_{2Y} \circ p_{2X}$ ($X \times (Y \times Z)$ and $(X \times Y) \times Z$ are fuzzy homeomorphic), being composition of two fuzzy β-closed mappings, is fuzzy β-closed and hence $X \times Y$ is fuzzy β-Lindelöf*. □

Proposition 13. *Let (X, T) be a fuzzy β-compact* space. If (Y, S) is fuzzy β-Lindelöf* space, then $X \times Y$ is a fuzzy β- Lindelöf*.*

3. Fuzzy filter characterization on fuzzy β-compact spaces

In this section fuzzy β-filters are introduced and filter characterization on fuzzy β-comapct space is discussed. For fuzzy filter characterization on fuzzy β-compact spaces we use the definition of fuzzy β-compact spaces defined by Balasumbramanian [1].

Definition 4. A *fuzzy $\beta(\beta^*)$-filter* F on a fuzzy topological space (X, T) is a non empty collection of subset of I^X with the following properties.

(i) $\lambda \in F$ is a fuzzy β-open (β-closed) set in (X, T).
(ii) $0 \notin F$, $1 \in F$.
(iii) $\mu_1, \mu_2 \in F$ then $\mu_1 \wedge \mu_2 \in F$.
(iv) If $\mu \in F$ and γ is a fuzzy β-open (β-closed) set in (X, T). with $\mu \leq \gamma$, then $\gamma \in F$.

Definition 5. A fuzzy $\beta(\beta^*)$-filter F on X is a *fuzzy $\beta(\beta^*)$-ultra filter* if there is no other fuzzy $\beta(\beta^*)$-filter finer than F.

Definition 6. A fuzzy β^*-filter F in a fuzzy topological space (X, T) is called a *fuzzy β^*-prime filter*, if γ and μ are any two fuzzy β- closed sets such that $\gamma \vee \mu \in F$, then $\gamma \in F$ or $\mu \in F$.

Proposition 14. *The following are equivalent for a fuzzy topological space* (X, T).

(a) (X, T) *is fuzzy β-compact.*
(b) *Every fuzzy β^*-filter F satisfies* $\wedge_{\mu \in F} \mu \neq 0$.
(c) *Every fuzzy β^*-prime filter F satisfies* $\wedge_{\mu \in F} \mu \neq 0$.
(d) *Every fuzzy β^*-ultra filter U satisfies* $\wedge_{\mu \in U} \mu \neq 0$.

Proof. (a) $\Rightarrow$ (b). Suppose $\wedge_{\mu \in F} \mu = 0$. Then $\vee_{\mu \in F}(1 - \mu) = 1$. Since $(1 - \mu)$ is fuzzy β-open set and (X, T) is fuzzy β-compact, there must exist a finite sub collection $\{1 - \mu_1, 1 - \mu_2, \cdots, 1 - \mu_n\}$ such that $1 = (1 - \mu_1) \vee (1 - \mu_2) \vee \cdots \vee (1 - \mu_n)$, that is $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n = 0$. Contradiction.

(b) $\Rightarrow$ (c). Follows from the fact that every fuzzy β^*-prime filter is fuzzzy β^*-filter.

(c) $\Rightarrow$ (d). Suppose U is a fuzzy β^*-ultra filter. Let μ, γ be fuzzy β- closed sets such that $\mu \wedge \nu \in U$. Suppose $\mu, \gamma \notin U$ then there exist

μ^*, $\gamma^* \in U$ with $\mu^* \wedge \mu = 0$ and $\gamma * \wedge \gamma = 0$. Since $\mu \vee \gamma$, μ^* and $\gamma^* \in U$, we have $(\mu \vee \gamma) \wedge \mu^* \wedge \gamma^* \in U$. But $(\mu \vee \gamma) \wedge \mu^* \wedge \gamma^* = 0$. This is a contradiction. Therefore U is a fuzzy β*-prime filter. Hence (d) follows from (c).

(d) $\Rightarrow$ (a). Suppose H is a family of fuzzy β-closed sets with finite intersection property. For each $\gamma \in H$ consider a family in the $G_\gamma = \{\mu/\mu$ is a fuzzy β-closed set, $\mu \geq \gamma\}$. Clearly $\gamma \in G_\gamma$. Let $G = \bigcup_{\gamma \in H}\{G_\gamma\}$. Since H has the finite intersection property, G also has the property. Thus there exists a fuzzy β*-ultra filter U such that $H \subset G \subset U$. Hence $\wedge_{\mu \in H}\mu \geq \wedge_{\mu \in G}\mu \geq \wedge_{\mu \in U}\mu$. By hypothesis, $\wedge_{\mu \in U}\mu \neq 0$ and therefore $\wedge_{\mu \in H}\mu \neq 0$. This proves that (X, T) is fuzzy β-compact. $\square$

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M. K. Uma and E. Roja
Department of Mathematics
Sri Sarada College for Women
Salem 636 016
Tamil Nadu, India.

G. Balasubramanian
Department of Mathematics
Periyar University, Salem 636 011
Tamil Nadu, India.