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ON THE STABILITY OF A GENERALIZED ADDITIVE FUNCTIONAL EQUATION II

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ABSTRACT. For an odd mapping, we study a generalized additive functional equation in Banach spaces and Banach modules over a C^*-algebra. And we obtain generalized solutions of a generalized additive functional equation and so generalize the Cauchy–Rassias stability.

1. INTRODUCTION

In 1940 S. M. Ulam [13] raised the stability problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". In 1941 D. H. Hyers [4] solved this stability problem in real Banach spaces and a number of mathematicians obtained substantial generalizations concerning the stability of functional equations [1, 11, 12]. The following theorem which is called the Cauchy–Rassias stability is a generalized solution to the stability problem:

Theorem A. For two real Banach spaces X, Y, let $f : X \to Y$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \geq 0$ with $p \neq 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique $\mathbb{R}$-linear mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{2\theta}{|2^p - 2|}\|x\|^p$$

for all $x \in X$.

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Actually, the above Cauchy–Rassias stability in real Banach spaces was obtained by D. H. Hyers [4] for the case $p = 0$, by Th. M. Rassias [11] for the case $p \in (0, 1)$, and by Z. Gajda [1] for the case $p > 1$. In particular, Th. M. Rassias and P. Šemrl [12] gave an example to show that it does not occur for the case $p = 1$.

Furthermore, P. Găvruta [2] obtained a generalization of Theroem A :

For an abelian group G and a Banach space Y, let $f : G \to Y$ be a mapping. Assume that there exists a function $\varphi : G \times G \to [0, \infty)$ such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty,$$

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $L : G \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$.

C. Park studied the Cauchy–Rassias stability in Banach modules over a C^*-algebra in [3, 7, 8, 9, 10]. J. Lee and D. Shin [6] also investigated the Cauchy–Rassias stability in Banach spaces and Banach modules over a C^*-algebra and so obtained generalizations of it.

Throughout this paper, we assume that r is a positive rational number and that d, l are integers with $1 < l < \frac{d}{2}$. We let k be a positive integer and m a positive integer with $m \le d$ and we note that ${}_dC_l := \frac{d!}{l!(d-l)!}$.

Let X and Y be Banach spaces and $f : X \to Y$ an odd mapping. The main purpose of this paper is to solve the following functional equation which is called a *generalized additive functional equation* in Banach spaces

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)} x_j}{r}\right)$$

$$= ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1) \sum_{j=1}^{d} f(x_j) \tag{1.1}$$

for all $x_1, \ldots, x_d \in X$. The solution of the functional equation (1.1) is called a *generalized additive mapping*.

In [6] we obtained generalizations of the Cauchy–Rassias stability of (1.1). Especially, for the cases $r > k$ and $p > 1$, or $r < k$ and $0 < p < 1$ we obtained the

Cauchy–Rassias stability of (1.1) in [6]. In this paper, we obtain another generalizations of the Cauchy–Rassias stability of (1.1) and the Cauchy–Rassias stability for another cases $r < k$ and $p > 1$, or $r > k$ and $0 < p < 1$. For our proof, in [6] we assumed the existence of a function $\varphi : X^d \to [0,\infty)$ satisfying

$$\widetilde{\varphi}(x_1,\dots,x_d) := \sum_{j=0}^{\infty} \frac{r^j}{k^j} \varphi\left(\frac{k^j}{r^j}x_1,\dots,\frac{k^j}{r^j}x_d\right) < \infty,$$

but in this paper we assume the existence of a function $\varphi : X^d \to [0,\infty)$ satisfying

$$\widetilde{\varphi}(x_1,\dots,x_d) := \sum_{i=1}^{\infty} \frac{k^i}{r^i} \varphi\left(\frac{r^i}{k^i}x_1,\dots,\frac{r^i}{k^i}x_d\right) < \infty.$$

Furthermore, we consider a non-negative integer s with $s < \frac{k}{2}$ and generalize the Cauchy–Rassias stability of (1.1) :

For an odd mapping $f : X \to Y$, if there exist a function $\varphi : X^d \to [0,\infty)$ and a constant $\alpha \geq 0$ satisfying certain conditions and

$$\left\| rf\left(\frac{k-2s}{r}x\right) - (k-2s)f(x)\right\| \leq \alpha\varphi(\underbrace{x,\dots,x}_{m \text{ times}},\ \underbrace{0,\dots,0}_{d-m \text{ times}})$$

for all $x \in X$, then there exists a unique generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\alpha}{k-2s}\widetilde{\varphi}(\underbrace{x,\dots,x}_{m \text{ times}},\ \underbrace{0,\dots,0}_{d-m \text{ times}})$$

for all $x \in X$.

On the other hand, we investigate the Cauchy–Rassias stability of a generalized additive functional equation in Banach modules over a unital C^*-algebra. Let A be a unital C^*-algebra with unitary group $\mathcal{U}(A)$ and X and Y left Banach modules over A. We consider an odd mapping $f : X \to Y$ and the following generalized additive functional equation in Banach modules over A

$$rf\left(\frac{\sum_{j=1}^{d} ux_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}ux_j}{r}\right)$$

$$= ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} uf(x_j) \tag{1.2}$$

for all $u \in \mathcal{U}(A)$ and all $x_1,\dots,x_d \in X$.

In fact, Park [9, 10] obtained results for several special cases in (1.2) and we obtained generalized solutions of (1.2) in [6]. In this paper, we obtain another generalizations of the Cauchy–Rassias stability of (1.2) which contain C. Park's results.

2. GENERALIZED ADDITIVE MAPPINGS IN BANACH SPACES

In this section, let X and Y be Banach spaces. For a given mapping $f : X \to Y$, we define Df by the following :

$$Df(x_1,\ldots,x_d) := rf\left(\frac{\sum_{j=1}^d x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1\\ \sum_{j=1}^d \iota(j)=l}} rf\left(\frac{\sum_{j=1}^d(-1)^{\iota(j)}x_j}{r}\right)$$
$$-\left({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1\right)\sum_{j=1}^d f(x_j)$$

for all $x_1,\ldots,x_d \in X$.

First of all, we give sufficient conditions for an odd mapping to be additive.

Proposition 2.1. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$. Assume that there exists a function $\varphi : X^d \to [0,\infty)$ such that

$$\widetilde{\varphi}(x_1,\ldots,x_d) := \sum_{i=1}^{\infty}\frac{k^i}{r^i}\varphi\left(\frac{r^i}{k^i}x_1,\ldots,\frac{r^i}{k^i}x_d\right) < \infty, \tag{2.1}$$

$$\|Df(x_1,\ldots,x_d)\| \le \varphi(x_1,\ldots,x_d), \tag{2.2}$$

for all $x_1,\ldots,x_d \in X$. Then f is additive.

Proof. For an odd mapping f satisfying $rf\left(\frac{k}{r}x\right) = kf(x)$, if we replace $\frac{k}{r}x$ by x then we get $kf\left(\frac{r}{k}x\right) = rf(x)$ and so $\frac{k^n}{r^n}f\left(\frac{r^n}{k^n}x\right) = f(x)$ for all positive integer n and all $x \in X$. So by the definition of Df and (2.2), we have

$$\frac{k^n}{r^n}Df\left(\frac{r^n}{k^n}x_1,\ldots,\frac{r^n}{k^n}x_d\right) = Df(x_1,\ldots,x_d)$$

and

$$\frac{k^n}{r^n}\left\|Df\left(\frac{r^n}{k^n}x_1,\ldots,\frac{r^n}{k^n}x_d\right)\right\| \le \frac{k^n}{r^n}\varphi\left(\frac{r^n}{k^n}x_1,\ldots,\frac{r^n}{k^n}x_d\right)$$

for all positive integer n and all $x_1,\ldots,x_d \in X$. From (2.1), we obtain

$$\lim_{n\to\infty}\frac{k^n}{r^n}\varphi\left(\frac{r^n}{k^n}x_1,\ldots,\frac{r^n}{k^n}x_d\right) = 0$$

and so it is straightforward to see that $Df(x_1, \dots, x_d) = 0$ for all $x_1, \cdots, x_d \in X$. Therefore, f is additive by [6, Lemma 2.1]. □

Now we want to change the condition

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$ in Proposition 2.1. In the following theorem, we remark that m and k can be different positive integers.

Theorem 2.2. *Let $f : X \to Y$ be an odd mapping. Assume that there exist a function $\varphi : X^d \to [0, \infty)$ satisfying (2.1) and (2.2) and a constant $\alpha \ge 0$ such that*

$$\left\| rf\left(\frac{k}{r}x\right) - kf(x) \right\| \le \alpha\varphi(\underbrace{x, \dots, x}_{m\ times}, \underbrace{0, \dots, 0}_{d-m\ times}) \tag{2.3}$$

for all $x \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{\alpha}{k}\widetilde{\varphi}(\underbrace{x, \dots, x}_{m\ times}, \underbrace{0, \dots, 0}_{d-m\ times}) \tag{2.4}$$

for all $x \in X$.

Proof. If we replace $\frac{k}{r}x$ by x and divide by r in (2.3), then we get

$$\left\| f(x) - \frac{k}{r}f\left(\frac{r}{k}x\right) \right\| \le \frac{\alpha}{r}\varphi(\underbrace{\frac{r}{k}x, \dots, \frac{r}{k}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}})$$

for all $x \in X$. If we replace x by $\frac{r^n}{k^n}x$ in the above inequality, then we get

$$\left\| f\left(\frac{r^n}{k^n}x\right) - \frac{k}{r}f\left(\frac{r^{n+1}}{k^{n+1}}x\right) \right\| \le \frac{\alpha}{r}\varphi\left(\underbrace{\frac{r^{n+1}}{k^{n+1}}x, \dots, \frac{r^{n+1}}{k^{n+1}}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}\right)$$

If we multiply by $\frac{k^n}{r^n}$ in the above inequality, then we get

$$\left\| \frac{k^n}{r^n}f\left(\frac{r^n}{k^n}x\right) - \frac{k^{n+1}}{r^{n+1}}f\left(\frac{r^{n+1}}{k^{n+1}}x\right) \right\| \le \frac{\alpha}{r}\frac{k^n}{r^n}\varphi\left(\underbrace{\frac{r^{n+1}}{k^{n+1}}x, \dots, \frac{r^{n+1}}{k^{n+1}}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}\right)$$

for all $x \in X$ and all positive integers n. From the above inequality, we have

$$\left\| \frac{k^q}{r^q} f\left(\frac{r^q}{k^q}x\right) - \frac{k^n}{r^n} f\left(\frac{r^n}{k^n}x\right) \right\| \le \frac{\alpha}{k} \sum_{i=q+1}^{n} \frac{k^i}{r^i} \varphi \Bigg(\underbrace{\frac{r^i}{k^i}x, \ldots, \frac{r^i}{k^i}x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}} \Bigg)$$

for all $x \in X$ and all non negative integers q, n with $q < n$. This shows that the sequence $\{\frac{k^n}{r^n} f(\frac{r^n}{k^n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{k^n}{r^n} f(\frac{r^n}{k^n}x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{k^n}{r^n} f\left(\frac{r^n}{k^n}x\right)$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $L(-x) = -L(x)$ for all $x \in X$, which means that L is an odd mapping. On the other hand, we have

$$\begin{aligned} \|DL(x_1, \ldots, x_d)\| &= \lim_{n\to\infty} \frac{k^n}{r^n} \left\| Df\left(\frac{r^n}{k^n}x_1, \ldots, \frac{r^n}{k^n}x_d\right) \right\| \\ &\le \lim_{n\to\infty} \frac{k^n}{r^n} \varphi\left(\frac{r^n}{k^n}x_1, \ldots, \frac{r^n}{k^n}x_d\right) = 0 \end{aligned}$$

for all $x_1, \ldots, x_d \in X$ and so L is additive by [6, Lemma 2.1].

In order to prove that L satisfies (2.4), if we put $q = 0$ and let $n \to \infty$ in the last inequality then we obtain

$$\begin{aligned} \|f(x) - L(x)\| &\le \frac{\alpha}{k} \sum_{i=1}^{\infty} \frac{k^i}{r^i} \varphi \Bigg(\underbrace{\frac{r^i}{k^i}x, \ldots, \frac{r^i}{k^i}x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}} \Bigg) \\ &= \frac{\alpha}{k} \widetilde{\varphi}(\underbrace{x, \ldots, x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}}) \end{aligned}$$

for all $x \in X$.

Now to prove the uniqueness of L, let $L' : X \to Y$ be another additive mapping satisfying (2.4). Since L and L' are additive, we have

$$\begin{aligned} \|L(x) - L'(x)\| &= \frac{k^n}{r^n} \left\| L\left(\frac{r^n}{k^n}x\right) - L'\left(\frac{r^n}{k^n}x\right) \right\| \\ &\le \frac{k^n}{r^n} \left(\left\| L\left(\frac{r^n}{k^n}x\right) - f\left(\frac{r^n}{k^n}x\right) \right\| + \left\| L'\left(\frac{r^n}{k^n}x\right) - f\left(\frac{r^n}{k^n}x\right) \right\| \right) \end{aligned}$$

$$\le 2\frac{\alpha}{k}\frac{k^n}{r^n}\widetilde{\varphi}\Bigg(\underbrace{\frac{r^n}{k^n}x,\ldots,\frac{r^n}{k^n}x}_{m\text{ times}},\ \underbrace{0,\ldots,0}_{d-m\text{ times}}\Bigg)$$

$$= 2\frac{\alpha}{k}\sum_{i=1}^{\infty}\frac{k^{n+i}}{r^{n+i}}\varphi\Bigg(\underbrace{\frac{r^{n+i}}{k^{n+i}}x,\ldots,\frac{r^{n+i}}{k^{n+i}}x}_{m\text{ times}},\ \underbrace{0,\ldots,0}_{d-m\text{ times}}\Bigg)$$

which goes to zero as $n \to \infty$ for all $x \in X$ by (2.1). Consequently, L is a unique additive mapping satisfying (2.4), as desired. □

If $\alpha = 0$ in the above theorem, then Theorem 2.2 becomes Proposition 2.1. On the other hand, we obtained the Cauchy–Rassias stability for the cases $r > k$ and $p > 1$, or $r < k$ and $0 < p < 1$ in [6, Corollary 2.4]. Here we prove the following corollary which is the Cauchy–Rassias stability for different cases from [6, Corollary 2.4].

Corollary 2.3. *Let $f : X \to Y$ be an odd mapping. When $r < k$ and $p > 1$, or $r > k$ and $0 < p < 1$, assume that there exist constants $\theta \ge 0$ and $\alpha \ge 0$ such that*

$$\|Df(x_1,\ldots,x_d)\| \le \theta\sum_{j=1}^{d}\|x_j\|^p,$$

$$\left\|rf\left(\frac{k}{r}x\right) - kf(x)\right\| \le \alpha\theta m\|x\|^p$$

for all $x_1,\ldots,x_d \in X$ and all $x \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{\alpha\theta m}{k}\frac{r^{p-1}}{k^{p-1} - r^{p-1}}\|x\|^p$$

for all $x \in X$.

Proof. Let $\varphi : X^d \to [0,\infty)$ be $\varphi(x_1,\ldots,x_d) = \theta\sum_{j=1}^{d}\|x_j\|^p$. When $r < k$ and $p > 1$, or $r > k$ and $0 < p < 1$, since $0 < \left(\frac{r}{k}\right)^{p-1} < 1$ we get

$$\widetilde{\varphi}(x_1,\ldots,x_d) := \sum_{i=1}^{\infty}\frac{k^i}{r^i}\varphi\left(\frac{r^i}{k^i}x_1,\ldots,\frac{r^i}{k^i}x_d\right)$$

$$= \frac{\theta r^{p-1}}{k^{p-1} - r^{p-1}}\sum_{j=1}^{d}\|x_j\|^p$$

By applying Theorem 2.2, there exists a unique additive mapping $L : X \to Y$ such that

$$\begin{aligned}\|f(x) - L(x)\| &\leq \frac{\alpha}{k}\widetilde{\varphi}(\underbrace{x, \dots, x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}) \\ &\leq \frac{\alpha\theta m}{k}\frac{r^{p-1}}{k^{p-1} - r^{p-1}}||x||^p\end{aligned}$$

for all $x \in X$. □

From now on, we denote the greatest integer less than or equal to a real number β by $[\beta]$. We now introduce the following equation in [6, Lemma 2.5]. For an odd mapping $f : X \to Y$, under the definition of Df we have

$$Df(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}}) = ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x)\right) \tag{2.5}$$

$$+ \sum_{t=1}^{[\frac{k-1}{2}]} {}_kC_t({}_{d-k}C_{l-t} - {}_{d-k}C_{l+t-k})\left(rf\left(\frac{k-2t}{r}x\right) - (k-2t)f(x)\right)$$

for all $x \in X$.

The following corollary is an application of Theorem 2.2. Since the proof of the following is clear by Theorem 2.2 and (2.5), we omit it.

Corollary 2.4. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \dots, [\frac{k-1}{2}]\}$ and all $x \in X$. Assume that there exists a function $\varphi : X^d \to [0, \infty)$ satisfying (2.1) and (2.2). Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{k({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)}\widetilde{\varphi}(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Now we are ready to generalize Theorem 2.2. Here we consider a given integer $s \in \{0, 1, 2, \dots, [\frac{k-1}{2}]\}$.

Theorem 2.5. *Let $f : X \to Y$ be an odd mapping. Assume that there exist a function $\varphi : X^d \to [0,\infty)$ satisfying (2.2) and a constant $\alpha \geq 0$ such that*

$$\widetilde{\varphi}(x_1,\ldots,x_d) := \sum_{i=1}^{\infty} \frac{(k-2s)^i}{r^i} \varphi\left(\frac{r^i}{(k-2s)^i}x_1,\ldots,\frac{r^i}{(k-2s)^i}x_d\right) < \infty, \tag{2.6}$$

$$\left\| rf\left(\frac{k-2s}{r}x\right) - (k-2s)f(x)\right\| \leq \alpha\varphi(\underbrace{x,\ldots,x}_{m\ times}, \underbrace{0,\ldots,0}_{d-m\ times}) \tag{2.7}$$

for all $x_1,\ldots,x_d \in X$ and all $x \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\alpha}{k-2s}\widetilde{\varphi}(\underbrace{x,\ldots,x}_{m\ times}, \underbrace{0,\ldots,0}_{d-m\ times}) \tag{2.8}$$

for all $x \in X$.

Proof. Since our proof is similar to that of Theorem 2.2 when we replace k by $k-2s$, we omit the details. □

We remark that when $s = 0$, Theorem 2.5 is the same as Theorem 2.2. Finally, we complete our generalizations of the Cauchy–Rassias stability of a generalized additive functional equation by giving the following result which is complementary cases of Corollary 2.4.

Corollary 2.6. *Let $s \in \{1,2,\ldots,[\frac{k-1}{2}]\}$ and let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0,1,2,\ldots,[\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$. Assume that there exists a function $\varphi : X^d \to [0,\infty)$ satisfying (2.6) and (2.2), then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(k-2s)_kC_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})}\widetilde{\varphi}(\underbrace{x,\ldots,x}_{k\ times}, \underbrace{0,\ldots,0}_{d-k\ times})$$

for all $x \in X$.

Proof. For a given $s \in \{1,2,\ldots,[\frac{k-1}{2}]\}$, since an odd mapping $f : X \to Y$ satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0,1,2,\dots,[\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$, equation (2.5) becomes the following equation

$$\begin{aligned} &Df(\underbrace{x,\dots,x}_{k \text{ times}}, \underbrace{0,\dots,0}_{d-k \text{ times}}) \\ &= {}_kC_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k}) \left(rf\left(\frac{k-2s}{r}x\right) - (k-2s)f(x) \right) \end{aligned}$$

for all $x \in X$. So if there exists a function $\varphi : X^d \to [0,\infty)$ satisfying (2.6) and (2.2), then we have $\alpha^{-1} = {}_kC_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})$ and $m = k$ in (2.7). Thus, Theorem 2.5 gives the proof. □

3. Generalized Additive Mappings in Banach Modules over a C^*-Algebra

In this section, we solve a generalized additive functional equation (1.2) in Banach modules over a unital C^*-algebra. Throughout this section, let X and Y be left Banach modules over a unital C^*-algebra A with unitary group $\mathcal{U}(A)$. For a given mapping $f : X \to Y$ and $u \in \mathcal{U}(A)$, we define D_uf by the following :

$$\begin{aligned} D_uf(x_1,\dots,x_d) := rf\left(\frac{\sum_{j=1}^d ux_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} rf\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)}ux_j}{r}\right) \\ - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^d uf(x_j) \end{aligned}$$

for all $x_1,\dots,x_d \in X$.

In the following proposition, we give sufficient conditions for an odd mapping to be A-linear additive.

Proposition 3.1. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$. Assume that there exists a function $\varphi : X^d \to [0,\infty)$ such that

$$\widetilde{\varphi}(x_1,\dots,x_d) := \sum_{i=1}^{\infty} \frac{k^i}{r^i}\varphi\left(\frac{r^i}{k^i}x_1,\dots,\frac{r^i}{k^i}x_d\right) < \infty, \tag{3.1}$$

$$\|D_uf(x_1,\dots,x_d)\| \le \varphi(x_1,\dots,x_d) \tag{3.2}$$

for all $u \in \mathcal{U}(A)$ and all $x_1,\dots,x_d \in X$, then f is A-linear additive.

Proof. For an odd mapping f satisfying $rf\left(\frac{k}{r}x\right) = kf(x)$, if we replace $\frac{k}{r}x$ by x then we get $kf\left(\frac{r}{k}x\right) = rf(x)$ and so $\frac{k^n}{r^n}f\left(\frac{r^n}{k^n}x\right) = f(x)$ for all positive integer n and all $x \in X$. So by the definition of $D_u f$ and (3.2), we have

$$\frac{k^n}{r^n}D_u f\left(\frac{r^n}{k^n}x_1, \dots, \frac{r^n}{k^n}x_d\right) = D_u f(x_1, \dots, x_d)$$

and

$$\frac{k^n}{r^n}\left\|D_u f\left(\frac{r^n}{k^n}x_1, \dots, \frac{r^n}{k^n}x_d\right)\right\| \le \frac{k^n}{r^n}\varphi\left(\frac{r^n}{k^n}x_1, \dots, \frac{r^n}{k^n}x_d\right)$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \dots, x_d \in X$. On the other hand, from (3.1) we know that

$$\lim_{n\to\infty}\frac{k^n}{r^n}\varphi\left(\frac{r^n}{k^n}x_1, \dots, \frac{r^n}{k^n}x_d\right) = 0$$

for all $x_1, \dots, x_d \in X$. Thus, we conclude that $D_u f(x_1, \dots, x_d) = 0$ for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$ and so f is A-linear additive by [6, Lemma 3.1]. □

In order to obtain generalizations of the Cauchy–Rassias stability in Banach modules over a unital C^*-algebra, we change the condition

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$ in Proposition 3.1.

Theorem 3.2. *Let $f : X \to Y$ be an odd mapping. Assume that there exist a function $\varphi : X^d \to [0, \infty)$ satisfying (3.1) and (3.2) and a constant $\alpha \ge 0$ such that*

$$\left\|rf\left(\frac{k}{r}x\right) - kf(x)\right\| \le \alpha\varphi(\underbrace{x, \dots, x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}) \tag{3.3}$$

all $x \in X$. Then there exists a unique A-linear additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{\alpha}{k}\widetilde{\varphi}(\underbrace{x, \dots, x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}) \tag{3.4}$$

for all $x \in X$.

Proof. By the same reasoning as the proof of Theorem 2.2, there exists a unique additive mapping $L : X \to Y$ defined by

$$L(x) := \lim_{n\to\infty}\frac{k^n}{r^n}f\left(\frac{r^n}{k^n}x\right)$$

for all $x \in X$.

From (3.1) and (3.2), we have

$$\begin{aligned}\|D_uL(x_1,\dots,x_d)\| &= \lim_{n\to\infty}\frac{k^n}{r^n}\left\|D_uf\left(\frac{r^n}{k^n}x_1,\dots,\frac{r^n}{k^n}x_d\right)\right\| \\ &\le \lim_{n\to\infty}\frac{k^n}{r^n}\varphi\left(\frac{r^n}{k^n}x_1,\dots,\frac{r^n}{k^n}x_d\right)=0\end{aligned}$$

and so $D_uL(x_1,\dots,x_d)=0$ for all $u\in\mathcal{U}(A)$ and all $x_1,\cdots,x_d\in X$. Therefore, L is A-linear generalized additive by [6, Lemma 3.1]. □

The following corollary is the Cauchy–Rassias stability for the cases $r<k$ and $p>1$, or $r>k$ and $0<p<1$ which contains several results in [9,10]. By applying Theorem 3.2, since the proof of the following is similar to that of Corollary 2.3, we omit the details.

Corollary 3.3. *Let $f:X\to Y$ be an odd mapping. When $r<k$ and $p>1$, or $r>k$ and $0<p<1$, assume that there exist constants $\theta\ge 0$ and $\alpha\ge 0$ satisfying*

$$\|D_uf(x_1,\dots,x_d)\|\le\theta\sum_{j=1}^{d}\|x_j\|^p,$$

$$\left\|rf\left(\frac{k}{r}x\right)-kf(x)\right\|\le\alpha\theta m\|x\|^p$$

for all $u\in\mathcal{U}(A)$, all $x_1,\dots,x_d\in X$, and all $x\in X$. Then there exists a unique A-linear additive mapping $L:X\to Y$ such that

$$\|f(x)-L(x)\|\le\frac{\alpha\theta m}{k}\frac{r^{p-1}}{k^{p-1}-r^{p-1}}\|x\|^p$$

for all $x\in X$.

On the other hand, we are ready to generalize the Cauchy–Rassias stability of a generalized additive functional equation in Banach modules over a unital C^*-algebra . At first, we introduce the following equation in [6, Lemma 3.5] to apply and generalize Theorem 3.2. For an odd mapping $f:X\to Y$, under the definition of D_uf we have

$$\begin{aligned}(3.5)\quad D_uf(\underbrace{x,\dots,x}_{k\text{ times}},\underbrace{0,\dots,0}_{d-k\text{ times}}) &= ({}_{d-k}C_l-{}_{d-k}C_{l-k}+1)\left(rf\left(\frac{kux}{r}\right)-kuf(x)\right)\\ &+\sum_{t=1}^{[\frac{k-1}{2}]}{}_kC_t({}_{d-k}C_{l-t}-{}_{d-k}C_{l+t-k})\left(rf\left(\frac{k-2t}{r}ux\right)-(k-2t)uf(x)\right)\end{aligned}$$

for all $u\in\mathcal{U}(A)$ and all $x\in X$.

Here we give a generalization of the Cauchy–Rassias stability of (1.2).

Corollary 3.4. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \dots, [\frac{k-1}{2}]\}$ and all $x \in X$. Assume that there is a function $\varphi : X^d \to [0, \infty)$ satisfying (3.1) *and* (3.2). *Then there exists a unique A-linear additive mapping $L : X \to Y$ satisfying*

$$\|f(x) - L(x)\| \le \frac{1}{k({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)}\widetilde{\varphi}(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Proof. Since an odd mapping $f : X \to Y$ satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \dots, [\frac{k-1}{2}]\}$ and all $x \in X$, by (3.5) we get

$$D_1 f(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}}) = ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x)\right)$$

for all $x \in X$. So, if there is a function $\varphi : X^d \to [0, \infty)$ satisfying (3.1) and (3.2), then Theorem 3.2 gives the proof. □

Remark 3.5. If we let $k = 2$, then the condition in Corollary 3.4 is satisfied automatically and so there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{2({}_{d-2}C_l - {}_{d-2}C_{l-2} + 1)}\widetilde{\varphi}(x, x, \underbrace{0, \cdots, 0}_{d-2 \text{ times}})$$

for all $x \in X$. Therefore, Corollary 3.4 is a generalization of [10, Theorem 3.3]. On the other hand, if $r = 1$ and $k = 3$ in Corollary 3.4 then we obtain [9, Theorem 3.3].

Now we consider an integer $s \in \{0, 1, 2, \dots, [\frac{k-1}{2}]\}$. We obtain the following result which is a generalization of Theorem 3.3.

Theorem 3.6. *Let $f : X \to Y$ be an odd mapping. Assume that there exist a function $\varphi : X^d \to [0, \infty)$ satisfying* (3.2) *and a constant $\alpha \ge 0$ such that*

$$\widetilde{\varphi}(x_1, \dots, x_d) := \sum_{i=1}^{\infty} \frac{(k-2s)^i}{r^i}\varphi\left(\frac{r^i}{(k-2s)^i}x_1, \dots, \frac{r^i}{(k-2s)^i}x_d\right) < \infty, \tag{3.6}$$

$$\left\| rf\left(\frac{k-2s}{r}x\right) - (k-2s)f(x) \right\| \leq \alpha\varphi(\underbrace{x,\ldots,x}_{m \text{ times}}, \underbrace{0,\ldots,0}_{d-m \text{ times}}) \tag{3.7}$$

for all $x_1, \ldots, x_d \in X$ *and all* $x \in X$. *Then there exists a unique* A*-linear additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\alpha}{k-2s}\widetilde{\varphi}(\underbrace{x,\ldots,x}_{m \text{ times}}, \underbrace{0,\ldots,0}_{d-m \text{ times}}) \tag{3.8}$$

for all $x \in X$.

Proof. If we replace k by $k-2s$ in Theorem 3.2, then the proof is the same as that of Theorem 3.2 and so we omit the details. □

When $s \neq 0$ in Theorem 3.6, we obtain the following corollary. The proof of the following, being similar to that of Corollary 2.6, is omitted.

Corollary 3.7. *Let* $s \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$ *and let* $f : X \to Y$ *be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ *with* $t \neq s$ *and all* $x \in X$. *Assume that there exists a function* $\varphi : X^d \to [0, \infty)$ *satisfying* (3.6) *and* (3.2). *Then there exists a unique* A*-linear additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{1}{(k-2s)_kC_s({}_{d-k}C_{l-s} -{}_{d-k} C_{l+s-k})}\widetilde{\varphi}(\underbrace{x,\ldots,x}_{k \text{ times}}, \underbrace{0,\ldots,0}_{d-k \text{ times}})$$

for all $x \in X$.

Remark 3.8. If $r = 3$ and $k = 3$ in Corollary 3.7, then $s = 1$ and $t = 0$ and so there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{3({}_{d-3}C_{l-1} -{}_{d-3} C_{l-2})}\widetilde{\varphi}(x, x, x, \underbrace{0,\ldots,0}_{d-3 \text{ times}})$$

for all $x \in X$.

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