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DOMAINS OF HYPERHOLOMORPHY AND HYPER STEIN DOMAINS ON CLIFFORD ANALYSIS

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ABSTRACT. We give definitions of hyperholomorphic functions of quaternionic functions of two quaternionic variables. We investigate properties of hyperholomorphic functions on quaternion analysis, and obtain equivalence relations for domains of hyperholomorphy and hyper Stein domains in a domain of $\mathbf{C}^2 \times \mathbf{C}^2$.

1. INTRODUCTION

In quaternion analysis, F. Brackx, R. Delanghe and F. Sommen [2], M. Naser [10] represented one quaternion variable $z = x_1 + ix_2 + jx_3 + kx_4$ by a pair $z = z_1 + z_2 j$ of two complex variables $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, and established the identification $\mathcal{V} \cong \mathbf{C}^2$ between the quaternion field $\mathcal{V}$ and the space $\mathbf{C}^2$, and introduced the notion of hyperholomorphy of quaternion valued functions $f = f_1 + f_2 j$ of a quaternion variables $z = z_1 + z_2 j$ in $\mathbf{C}^2$ and proved that any complex valued harmonic function f_1 in a domain of holomorphy D in $\mathbf{C}^2$ has a hyperconjugate harmonic function f_2 such that the quaternion valued function $f = f_1 + f_2 j$ is hyperholomorphic in D in the sense of M. Naser [10]. R. Futer has give a definition of regular quaternionic function in $\mathbf{R}^4$ and developed a theory of quaternionic functions in 1935. Also, C. A. Deavours [3] and A. Sudbery [14] have developed a theory of regular functions defined by R. Futer [4]. F. Brackx [1] has given several results for (k)-monogenic functions which are an extension of regular quternionic functions. Recently, the theory of quaternion analysis has been applied for physics in F. Gürsey and H. C. Tze [6]. K. Nôno [12] showed the validity of the converse proving that, if any complex valued harmonic function f_1 in a domain D in $\mathbf{C}^2$ has a hyperconjugate

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harmonic function f_2, then D is a domain of holomorphy in $\mathbf{C}^2$.

2. Notations on Quaternion Analysis

The field $\mathcal{V}$ of quaternions

$$z = x_1 + ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathbf{R} \tag{2.1}$$

is a four dimensional non-commutative $\mathbf{R}$-field generated by four base elements 1, i, j and k with the following non-commutative multiplication rule:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

We associate two complex numbers

$$z_1 := x_1 + ix_2, \quad z_2 := x_3 + ix_4 \in \mathbf{C}$$

to (2.1), regarding as

$$z = z_1 + z_2 j \in \mathcal{V}.$$

Thus, we identify $\mathcal{V}$ with $\mathbf{C}^2 \cong \mathbf{R}^4$. Identifying the element i with the imaginary unit $\sqrt{-1}$, we have a canonical inclusion:

$$\mathbf{C} := \{\text{complex numbers}\} \subset \mathcal{V} := \{\text{quaternions}\}.$$

We define the non-commutative multiplication of two quaternions $z = z_1 + z_2 j, w = w_1 + w_2 j \in \mathcal{V}$ by

$$zw := (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1}) j \in \mathcal{V}, \qquad \overline{z_1} := x_1 - ix_2.$$

The quaternion conjugate z^* of $z = z_1 + z_2 j \in \mathcal{V}$ is defined by

$$z^* := \overline{z_1} - z_2 j.$$

The absolute value

$$|z| := \sqrt{|z_1|^2 + |z_2|^2}$$

coincides with the usual norm of $z \in \mathbf{C}^2$.

We use the following quaternion differential operators:

$$\frac{\partial}{\partial z} := \frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}}, \qquad \frac{\partial}{\partial z^*} := \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}},$$

where $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z_1}}, \frac{\partial}{\partial z_2}$ and $\frac{\partial}{\partial \overline{z_2}}$ are usual differential operators :

$$\frac{\partial}{\partial z_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \overline{z_1}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

So, we have

$$\frac{\partial}{\partial z_1}j = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)j = \frac{1}{2}\left(j\frac{\partial}{\partial x_1} - ij\frac{\partial}{\partial x_2}\right) = \frac{1}{2}\left(j\frac{\partial}{\partial x_1} + ji\frac{\partial}{\partial x_2}\right) = j\frac{\partial}{\partial \overline{z_1}}$$

and

$$\frac{\partial}{\partial \overline{z_1}}j = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)j = \frac{1}{2}\left(j\frac{\partial}{\partial x_1} + ij\frac{\partial}{\partial x_2}\right) = \frac{1}{2}\left(j\frac{\partial}{\partial x_1} - ji\frac{\partial}{\partial x_2}\right) = j\frac{\partial}{\partial z_1}.$$

The operator

$$\begin{aligned}\frac{\partial^2}{\partial z \partial z^*} &= \left(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial \overline{z_2}}\right)\left(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}\right) \\ &= \frac{\partial}{\partial z_1}\frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}\frac{\partial}{\partial \overline{z_1}} + \frac{\partial}{\partial z_1}j\frac{\partial}{\partial \overline{z_2}} - j\frac{\partial}{\partial \overline{z_2}}j\frac{\partial}{\partial \overline{z_2}} \\ &= \frac{\partial}{\partial z_1}\frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_1}}\frac{\partial}{\partial \overline{z_2}} - jj\frac{\partial}{\partial z_2}\frac{\partial}{\partial \overline{z_2}} \\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right)\end{aligned}$$

is the usual complex Laplacian Δ_z.

In the space $\mathcal{V}^2 \cong \mathbf{C}^2 \times \mathbf{C}^2$ of two quaternion variables $z = z_1 + z_2 j, w = w_1 + w_2 j$ for $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, w_1 = y_1 + iy_2$ and $w_2 = y_3 + iy_4$, we use the quaternion differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial z^*}$ and

$$\frac{\partial}{\partial w} := \frac{\partial}{\partial w_1} - j\frac{\partial}{\partial \overline{w_2}}, \qquad \frac{\partial}{\partial w^*} := \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial \overline{w_2}}.$$

Let D be an open subset of $\mathbf{C}^2$ and $f = f_1 + f_2 j$ be a function defined in D with valued in $\mathcal{V}$, where $z = (z_1, z_2)$:

$$f = f_1 + f_2 j; z = (z_1, z_2) \in D \longrightarrow f(z) = f_1(z_1, z_2) + f_2(z_1, z_2)j \in \mathcal{V},$$

where f_1 and f_2 are complex valued functions.

Definition 2.1. Let D be an open set in $\mathbf{C}^2$. A function $f = f_1 + f_2 j$ is said to be *$L(R)$-hyperholomorphic* in D, if

(a) f_1 and f_2 are continuously differential functions in D,

(b)

$$\frac{\partial}{\partial z^*}f = 0 \ \left(f\frac{\partial}{\partial z^*} = 0\right) \text{ in } D. \tag{2.2}$$

We consider that f is (L-)hyperholomorphic in a subset D of $\mathbf{C}^2$.

The equation (2.2) operate to the following system of equations

$$\frac{\partial}{\partial z^*}f = \left(\frac{\partial}{\partial \overline{z}_1} + j\frac{\partial}{\partial \overline{z}_2}\right)(f_1 + f_2 j) = \left(\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2}\right) + \left(\frac{\partial f_2}{\partial \overline{z}_1} + \frac{\partial \overline{f}_1}{\partial z_2}\right)j$$

$$f\frac{\partial}{\partial z^*} = (f_1 + f_2 j)\left(\frac{\partial}{\partial \overline{z}_1} + j\frac{\partial}{\partial \overline{z}_2}\right) = \left(\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial f_2}{\partial \overline{z}_2}\right) + \left(\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right)j.$$

The above equations of the condition (2.2) are equivalent to the following systems of equations:

$$\frac{\partial f_1}{\partial \overline{z}_1} = \frac{\partial \overline{f}_2}{\partial z_2},\ \frac{\partial f_2}{\partial \overline{z}_1} = -\frac{\partial \overline{f}_1}{\partial z_2},\ \left(\frac{\partial f_1}{\partial \overline{z}_1} = \frac{\partial f_2}{\partial \overline{z}_2},\ \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}\right). \tag{2.3}$$

Let Ω be an open set in $\mathbf{C}^4$ and $f(z,w) = f_1(z,w) + f_2(z,w)j$ be a function defined in Ω with values in $\mathcal{V}$, where $(z,w) = (z_1, z_2, w_1, w_2) \in \Omega$.

Definition 2.2. Let Ω be an open set in $\mathbf{C}^4$, a function $f = f_1 + f_2 j$ is said to be *hyperholomorphic* in Ω, if

(a) $f \in \mathbf{C}^1(\Omega)$ (this means that f_1 and f_2 are continuously differential functions in Ω),

(b)

$$\frac{\partial}{\partial z^*}f = 0,\ f\frac{\partial}{\partial w^*} = 0 \text{ in } \Omega. \tag{2.4}$$

We say that f_2 is a *hyper conjugate harmonic function* of f_1 in Ω. We denote the set of all hyperholomorphic functions in Ω by $\mathcal{V}(\Omega)$.

The above equations of the condition (2.4) are equivalent to the following systems of equations:

$$\frac{\partial f_1}{\partial \overline{z}_1} = \frac{\partial \overline{f}_2}{\partial z_2},\ \frac{\partial f_2}{\partial \overline{z}_1} = -\frac{\partial \overline{f}_1}{\partial z_2},\ \frac{\partial f_1}{\partial \overline{w}_1} = \frac{\partial f_2}{\partial \overline{w}_2},\ \frac{\partial f_2}{\partial w_1} = -\frac{\partial f_1}{\partial w_2}. \tag{2.5}$$

The following lemma 2.3 is given in M. Naser[10] and K. Nôno[11].

Lemma 2.3. *If a function $f(z) = f_1(z_1, z_2) + f_2(z_1, z_2)j$ is hyperholomorphic in an open set D in $\mathbf{C}^2$, then function f_1 and f_2 are of class $\mathbf{C}^\infty$ in D.*

By Lemma 2.3 and the definition of hypermholomorphic functions, we have the following lemma:

Lemma 2.4. *If a function $f(z) = f_1(z_1, z_2) + f_2(z_1, z_2)j$ is hyperholomorphic in an open set D in $\mathbf{C}^2$, then the functions f_1 and f_2 are harmonic in D.*

Proof. We have

$$\begin{aligned}\frac{1}{4}\Delta_z f_1 &= \frac{\partial^2 f_1}{\partial z \partial z^*} = \left(\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2}{\partial z_2 \partial \overline{z}_2}\right) f_1 \\ &= \frac{\partial}{\partial z_1}\left(\frac{\partial f_1}{\partial \overline{z}_1}\right) + \frac{\partial}{\partial z_2}\left(\frac{\partial f_1}{\partial \overline{z}_2}\right) \\ &= \frac{\partial}{\partial z_1}\left(\frac{\partial \overline{f}_2}{\partial z_2}\right) + \frac{\partial}{\partial z_2}\left(-\frac{\partial \overline{f}_2}{\partial z_1}\right) \\ &= \frac{\partial^2 \overline{f}_2}{\partial z_1 \partial z_2} - \frac{\partial^2 \overline{f}_2}{\partial z_1 \partial z_2} = 0,\end{aligned}$$

and the function f_2 may be proved by the similar method as in the proof of the case f_1. □

3. Domains of Hyperholomorpy

In this section, according to the regularity of quaternionic functions of two quaternionic variables, we construct a theory of hyperholomorphic functions in $\mathbf{C}^2 \times \mathbf{C}^2 = \mathbf{C}^4 \cong \mathcal{V}^2$. Also, we give definitions of domains of hyperholomorphy of two quaternionic variables, and investigate the equivalence conditions for domains of hyperholomorphy. We can define domains of hyperholomorphy, as the domain of holomorphy in the sense of [6, 7, 9].

Definition 3.1. Let Ω be a domain in $\mathbf{C}^4$ and let $\alpha \in bdry(\Omega)$, $f \in \mathcal{V}(\Omega)$. We say that α is a *regular point* of f if there exists a $z_0 \in \Omega$ such that the power series expansion of f about z_0 converges in a neighborhood of α. We say that α is a *singular point* of f if it's not regular.

Definition 3.2. A domain in D in $\mathbf{C}^2$ is said to be a *domain of hyperholomorphy* if it is impossible to find two domains D_1 and D_2 in $\mathbf{C}^2$ with the following properties:

(a) $D \cap D_1 \supset D_2 \neq \emptyset$, $D_1 \not\subset D$.
(b) For every $(L-)$hyperholomorphic function f defined in D, there exists a $(L-)$hyperholomorphic function g in D_1 such that $f = g$ on D_2.

Definition 3.3. A domain in Ω in $\mathbf{C}^4$ is said to be a *domain of hyperholomorphy* if it is impossible to find two domains Ω_1 and Ω_2 in $\mathbf{C}^4$ with the following properties:

(a) $\Omega \cap \Omega_1 \supset \Omega_2 \neq \emptyset$, $\Omega_1 \not\subset \Omega$.
(b) For every hyperholomorphic function f defined in Ω, there exists a hyperholomorphic function g in Ω_1 such that $f = g$ on Ω_2.

Definition 3.4. If Ω is a domain in $\mathbf{C}^4$, a point $\alpha \in bdry(\Omega)$ is called an *essential point* if and only if α is a singular point for at least one function in $\mathcal{V}(\Omega)$.

Definition 3.5. A domain Ω in $\mathbf{C}^4$ is said to be *hyperholomorphically convex* if for every compact subset K of Ω, the set

$$\hat{K}_\Omega = \left\{(z,w) \in \Omega; |f(z,w)| \leq \sup_{(z',w')\in K} |f(z',w')| \ \text{ for all } f \in \mathcal{V}(\Omega)\right\}$$

is relatively compact in Ω.

Theorem 3.6 ([13]). *Let Ω be a domain of hyperholomorphy. Then, for each compact subset K of Ω,*

$$\left(1 - \frac{\sqrt{2}}{2}\right)\delta(K,\Omega^c) \leq \delta(\hat{K}_\Omega,\Omega^c) \leq \delta(K,\Omega^c). \tag{3.1}$$

Lemma 3.7 ([13]). *If K is a compact subset of a domain Ω in $\mathbf{C}^4$, then $\hat{K}_\Omega$ is a closed set in Ω and is a bounded set in $\mathbf{C}^4$.*

Theorem 3.8. *Let Ω be a domain of hyperholomorphy in $\mathbf{C}^4$. Then Ω is a hyperholomorphically convex set.*

Proof. Let K be any compact subset of Ω and $\{(z_n, w_n)\}$ be a sequence on $\hat{K}_\Omega$ such that $\{(z_n, w_n)\}$ converges to a point $(z^0, w^0) \in \bar{\Omega}$. Assume that $(z^0, w^0) \in \partial\Omega$. From Theorem 3.6 and the definition of $\hat{K}_\Omega$, we have that

$$\left(1 - \frac{\sqrt{2}}{2}\right)\delta(K,\Omega^c) \leq \delta(\hat{K}_\Omega,\Omega^c) \leq \delta((z_n,w_n),\Omega^c). \tag{3.2}$$

for every n. Since the sequence $\{(z_n, w_n)\}$ converges to a point (z^0, w^0), from (3.2), it follows that $\delta(K,\Omega^c)$=0. This is a contradiction. From Lemma 3.7, we have the desired result.

Definition 3.9. A complex manifold X is called *hyperholomorphically separable* if for any $x, y \in X$ with $x \neq y$ there exists a hyperholomorphic function f on X with $f(x) \neq f(y)$.

The following definition is defined by Definition 3.5 and Definition 3.9.

Definition 3.10. A hyper Stein manifold is a *connected complex manifold* that is hyperholomorphically separable and hyperholomorphically convex.

We have the following lemma 3.11, from the result of Y. Katznelson [8].

Lemma 3.11. *A domain Ω in $\mathbf{C}^4$ is a domain of hyperholomorphy if and only if each point $\alpha \in bdry(\Omega)$ is an essential point.*

Lemma 3.12 ([8]). *Let Ω be a domain and $\alpha \in bdry(\Omega)$. Assume that for any compact $K \subset \Omega$ we can find a neighborhood V_K of α with the property that, for any neighborhood V of α with $V \subset V_K$ and any component U of $V \cap \Omega$, there exists a corresponding function $\varphi \in \mathcal{V}(\Omega)$ such that*

$$\sup\{|\varphi(z)| : z \in K\} < \sup\{|\varphi(z)| : z \in U\}.$$

Then α is an essential point.

Definition 3.13. Let Ω be a domain in $\mathbf{C}^4$ and define the distance
(a) $\Delta_\Omega(z) = \inf\{\|z-\alpha\| : \alpha \in bdry(\Omega)\}$ for $z \in \Omega$ and
(b) $\Delta_\Omega(K) = \inf\{\Delta_\Omega(z) : z \in K\}$ for any set $K \subseteq \Omega$.

Theorem 3.14. *For a domain Ω in $\mathbf{C}^4$ the following properties are equivalent:*

(a) *Ω be a domain of hyperholomorphy.*
(b) *Ω is a hyperholomorpically convex set.*
(c) *Ω is a hyper Stein manifold.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Theorem 3.8.

We prove that $(b) \Rightarrow (a)$. Let $\alpha \in bdry(\Omega)$ and let K be compact in Ω. Then $\Delta_\Omega(K)>0$ and $\hat{K}$ is compact, so $\Delta_\Omega(\hat{K})>0$ from (b) and we have $\inf\{\|\alpha - z\prime\| : z\prime \in \hat{K}\} > 0$. Therefore, we can find a $z^0 \in \Omega\backslash\hat{K}$ in any prescribed neighborhood of α. Since $z^0 \notin \hat{K}$, there exists a $\varphi \in \mathcal{V}(\Omega)$ such that compact $\sup\{|\varphi(z)| : z \in K\} < |\varphi(z^0)|$. Thus, for any K relatively compact in Ω, and any neighborhood U of the boundary point α, there is a $\varphi \in \mathcal{V}(\Omega)$ such that $\sup\{|\varphi(z)| : z \in K\} < \sup\{|\varphi(z)| : z \in U\}$. We can now apply Lemma 3.12 to conclude that α is an essential point, and since all boundary points are essential it follows from Lemma 3.11 that Ω is a domain of hyperholomorphy.

We next prove that $(b) \Rightarrow (c)$. Since every domain Ω in $\mathbf{C}^4$ is hyperholomorphically separable, from (b), Ω is hyper Stein manifold. The implication $(c) \Rightarrow (b)$ follows from the definition of hyper Stein manifold. □

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