Commun. Korean Math. Soc. **22** (2007), No. 2, pp. 289–295

PRODUCT OF PL FIBRATORS AS CODIMENSION-k FIBRATORS

YOUNG HO IM AND YONGKUK KIM

ABSTRACT. We describe some conditions under which the product of two groups with certain property is a group with the same property, and we describe some conditions under which the product of hopfian manifolds is another hopfian manifold. As applications, we find some PL fibrators among the product of fibrators.

The question of whether the collection of codimension-k PL fibrators is closed under the Cartesian product operator remains unsolved but seems unlikely, in view of the examples described in [6]. As some effort to solve the question in [7], we give some partial answers.

1. Definitions and Notation

A proper map $p : M \to B$ between locally compact ANRs is an *approximate fibration*, a concept introduced and studied by Coram and Duvall [1] [2], if it has the following approximate homotopy lifting property: given an open cover Ω of B, an arbitrary space X, and maps $f : X \to M$ and $F : X \times I \to B$ satisfying $pf = F_0$, there exists a map $F' : X \times I \to M$ such that $F'_0 = f$ and pF' is Ω-close to F. The latter means that each $x \in X$ corresponds $U_x \in \Omega$ with $\{pF'(x), F(x)\} \subset U_x$.

Fibrator properties of manifolds afford quick detection of approximate fibrations. Throughout the rest of this paper all manifolds are oriented PL manifolds. When N is a fixed closed PL n-manifold, M is a PL manifold, B is a polyhedron, and $p : M \to B$ is a PL map, then p is said to be *N-like* if each $p^{-1}(b)$ collapses to an n-complex homotopy equivalent to N. We call N a *codimension-k PL fibrator* if, for every PL $(n+k)$-manifold M and N-like PL map $p : M \to B$, p is an approximate fibration. If N is a codimension-k PL fibrator for all $k > 0$, we simply call N a *PL fibrator*.

Received June 12, 2006; Revised February 2, 2007.

2000 *Mathematics Subject Classification.* Primary 57N15, 55R65; Secondary 57N25, 54B15.

Key words and phrases. approximate fibration, degree of a map, codimension-k PL fibrator, hopfian manifold, normally cohopfian, sparsely abelian.

This work was supported by Korea Research Foundation Grant (KRF-2004-002-C00026).

A group Γ is said to be: *hopfian* if each epimorphism $\Gamma \to \Gamma$ is an isomorphism; *cohopfian* if each monomorphism $\Gamma \to \Gamma$ is an isomorphism; and *normally cohopfian* if each monomorphism $\Gamma \to \Gamma$ with image a normal subgroup of Γ is an isomorphism. Moreover, Γ is *sparsely abelian* if it contains no nontrivial abelian normal subgroup A such that Γ/A is isomorphic to a normal subgroup of Γ. Groups Γ that are both sparsely abelian and normally cohopfian have the useful feature that every homomorphism $\Gamma \to \Gamma$ with, at worst, abelian kernel and normal image necessarily is an automorphism. For brevity a group Γ which is both normally cohopfian and sparsely abelian will be said to have Property NCSA. This Property is useful for detecting rich fibrator properties: a closed, aspherical, orientable n-manifold N is known to be a PL o-fibrator if it is a codimension-2 PL o-fibrator and $\pi_1(N)$ has Property NCSA [5, Theorem 8.1].

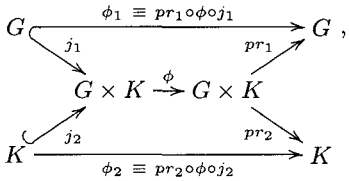
The (absolute) degree of a map is computed with integer coefficients and is understood to be a nonnegative number. Explicitly, a map $f : N \to N'$ between closed, orientable n-manifolds is said to have *degree* d if there are choices of generators $\gamma \in H_n(N;\mathbb{Z}), \gamma' \in H_n(N';\mathbb{Z})$ such that $f_*(\gamma) = d\gamma'$, where $d \geq 0$ is an integer. A closed, orientable manifold N is said to be *hopfian* if every degree 1 map $N \to N$ which induces an isomorphism at the fundamental group level is a homotopy equivalence. As a result, when $\pi_1(N)$ is a hopfian group, N is a hopfian manifold if and only if all degree 1 maps $N \to N$ are homotopy equivalences. According to Hausmann [11, Proposition 1], every closed orientable manifold of dimension at most 4 is hopfian.

2. Products of PL fibrators

In this section we discuss fibrator properties of products of hopfian manifolds. Say that a group G is *incommensurable with another group* K if there is no nontrivial homomorphism $G \to K$, and a group G is *hereditary incommensurable with another group* K if there is no nontrivial homomorphism $H \to K$ for any subgroup H of G. For example, perfect groups are incommensurable with all Abelian groups; finite groups, with torsion free groups; and infinite simple groups, with finite groups. In particular, finite groups are hereditary incommensurable with torsion free groups.

Lemma 2.1. *If G and K are normally cohopfian groups such that G is incommensurable with K, then $G \times K$ is normally cohopfian.*

Proof. Let $\phi : G \times K \to G \times K$ be a monomorphism with $\phi(G \times K)$ normal in $G \times K$. Consider the following diagram



where j_1 and j_2 are inclusions, and pr_1 and pr_2 are projections. Write, for any $x \in G$ and $y \in K$,

$$\phi(x,1) = (\phi_1(x), \psi_1(x)),\ \phi(1,y) = (\psi_2(y), \phi_2(y)).$$

Then $\phi_1 : G \to G, \psi_1 : G \to K, \psi_2 : K \to K$ and $\phi_2 : K \to K$ are homomorphisms. From the fact that $pr_2 \circ \phi \circ j_1(G)$ is trivial, we have $pr_2 \circ \phi(G \times K) = pr_2 \circ \phi \circ j_2(K)$. Hence we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{j_1} & G \times K & \xrightarrow{pr_2} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_2} & & \\ 0 & \longrightarrow & G & \xrightarrow{j_1} & G \times K & \xrightarrow{pr_2} & K & \longrightarrow & 0, \end{array}$$

where the horizontal sequences are exact sequences of groups. Note also that for any $(x,y) \in G \times K$,

$$\begin{aligned} \phi(x,y) &= \phi((x,1)(1,y)) = \phi(x,1)\phi(1,y) \\ &= (\phi_1(x),1)(\psi_2(y), \phi_2(y)) = (\phi_1(x)\psi_2(y), \phi_2(y)). \end{aligned}$$

Now we show that ϕ_1 and ϕ_2 are isomorphisms. Then applying the Five-Lemma, we have that ϕ is an isomorphism.

First, we claim that ϕ_2 is an isomorphism. For $y \in \mathrm{Ker}\phi_2$, $pr_2 \circ \phi(1,y) = \phi_2(y) = 1$ and so $\phi(1,y) \in G \times 1$. Since $\phi(1,y) = (\psi_2(y), \phi_2(y)) = (1,1)$ and ϕ is a monomorphism, $y = 1$ and ϕ_2 is also a monomorphism. But since $\phi_2(K) = pr_2 \circ \phi(G \times K) \lhd K$ and K is normally cohopfian, ϕ_2 is an isomorphism.

Next, we show that ϕ_1 is an isomorphism. Now we have known that ϕ_2 is injective. To show that $\phi_1(G)$ is normal in G, for any $x, g \in G$, one wants to have $g\phi_1(x)g^{-1} \in \phi_1(G)$. To have this, one may observe $(g,1)\phi(x,1)(g,1)^{-1} = (g\phi_1(x)g^{-1}, 1)$. Since $\phi(G \times K)$ is a normal subgroup of $G \times K$, it follows that $(g\phi_1(x)g^{-1}, 1) \in \phi(G \times K)$. This means that $(g\phi_1(x)g^{-1}, 1) = \phi(a,b) = (\phi_1(a)\psi_2(b), \phi_2(b))$ for some $(a,b) \in G \times K$ with $b \in \ker(\phi_2)$. Since ϕ_2 is injective, $b = 1$ and so $\psi_2(b) = 1$. Thus $g\phi_1(x)g^{-1} = \phi_1(a) \in \phi_1(G)$, which is required. Since ϕ_1 is a monomorphism, by the normal cohopfianness of G, ϕ_1 is an isomorphism. □

Lemma 2.2. *Suppose G is a sparsely abelian group and K is a group with no non-trivial abelian normal subgroup such that G is hereditary incommensurable with K, then $G \times K$ is sparsely abelian.*

Proof. Suppose $G \times K$ is not sparsely abelian. Then, there is a non-trivial abelian normal subgroup N in $G \times K$ such that $(G \times K)/N$ is isomorphic to a normal subgroup of $G \times K$, i.e., there is a homomorphism $\phi : G \times K \to G \times K$ such that $ker\ \phi = N$, and $im\ \phi$ is a normal subgroup of $G \times K$. Consider $\phi_1 = pr_1 \circ \phi \circ j_1 : G \to G$, $\phi_2 = pr_2 \circ \phi \circ j_2 : K \to K$, and $N_i = ker\ \phi_i$ for $i = 1, 2$. The fact that K has no non-trivial abelian normal subgroup implies $pr_2(N) = 1$, and so $N \subset G \times 1$.

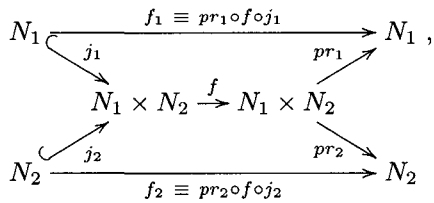
Now we show that $N_1 \times 1 = N$. For $(x,1) \in N_1 \times 1$, $\phi_1(x) = pr_1 \circ \phi(x,1) = 1$. Then $\phi(x,1) \in 1 \times K$. By the incommensuribility of G with K, we have $\phi(x,1) \in G \times 1$. Hence, we have $\phi(x,1) = (1,1)$ and $(x,1) \in N$. Conversely, since $N \subset G \times 1$, for $(x,1) \in N$, $\phi(x,1) = (1,1)$ and $(x,1) \in N_1 \times 1$.

Next we show that $\phi(G \times 1) = \phi(G \times K) \cap G \times 1$. By the incommensurability of G with K, $\phi(G \times 1) \subset G \times 1$, and then $\phi(G \times 1) \subset \phi(G \times K) \cap G \times 1$. For $(g,1) = \phi(x,y) \in \phi(G \times K) \cap G \times 1$, $\phi(x,y) = \phi(x,1)\phi(1,y)$. Then $\phi(1,y) \in G \times 1$. Since $N = N_1 \times 1$ and $\psi : (G/N_1) \times K \cong \phi(G \times K)$, we have $\psi| : (G/N_1) \times 1 \cong \phi(G \times 1)$ and $\psi| : 1 \times K \cong \phi(1 \times K)$. If $\phi(1,y) \neq (1,1)$, we consider the cyclic subgroup K' of K generated by y. Then we have the nontrivial well-defined isomorphism $\psi^{-1} : \phi(1 \times K') \to 1 \times K' \subset 1 \times K$, where $\phi(1 \times K')$ is a subgroup of $G \times 1$. Since G is hereditary incommensurable with K, ψ^{-1} is trivial and so $\phi(1,y) = (1,1)$. As a result, $\phi(x,y) = \phi(x,1) \in \phi(G \times 1)$ and $\phi(G \times 1)$ is a normal subgroup of $G \times 1$.

From the fact that G is a sparsely abelian group, $N_1 \times 1 = N$ must be trivial. □

Remark. For a closed aspherical manifold N with $\chi(N) \neq 0$, $\pi_1(N)$ has no nontrivial abelian normal subgroup by work of Rosset [16].

Now we consider the conditions under which the product of two hopfian manifolds is hopfian. For a given two hopfian manifolds N_1 and N_2, consider the following diagram



where j_1 and j_2 are inclusions, and pr_1 and pr_2 are projections.

Lemma 2.3. *Suppose that dim $N_1 = n$ and dim $N_2 = m$ with $m < n$. If deg $f = 1$ and $\beta_i(N_1) = 0$ for $1 \leq i \leq \min\{m, \frac{n}{2}\}$, then deg $f_1 = 1 =$ deg f_2.*

Proof. The fact that *deg* $f = 1$ implies that $f_* : H_k(N_1 \times N_2) \to H_k(N_1 \times N_2)$ is an isomorphism for every $k \geq 0$. Since $\beta_i(N_1) = 0$ for $1 \leq i \leq \min\{m, \frac{n}{2}\}$ and $H_n(N_1 \times N_2) = [H_n(N_1) \otimes H_0(N_2)] \oplus [\sum_{i=1}^{m} H_{n-i}(N_1) \otimes H_i(N_2)] \oplus Torsions$ and $H_m(N_1 \times N_2) = [H_0(N_1) \otimes H_m(N_2)] \oplus [\sum_{k=1}^{m} H_k(N_1) \otimes H_{m-k}(N_2)] \oplus Torsions$, we have that the free part of $H_n(N_1 \times N_2)$ is $H_n(N_1) \otimes H_0(N_2) \cong \mathbb{Z}$ and $H_m(N_1 \times N_2)$ is $H_0(N_1) \otimes H_m(N_2) \cong \mathbb{Z}$. Therefore, deg $f_1 = 1 =$ deg f_2. □

Remark. The argument of the proof above cannot be applied for the case *dim* $N_1 = n$, *dim* $N_2 = m$ with $m = n$, since $H_n(N_1 \times N_2) = [H_n(N_1) \otimes H_0(N_2)] \oplus [\sum_{i=1}^{n-1} H_{n-i}(N_1) \otimes H_i(N_2)] \oplus [H_0(N_1) \otimes H_n(N_2)] \oplus Torsions \cong \mathbb{Z} \oplus [\sum_{i=1}^{n-1} H_{n-i}(N_1) \otimes H_i(N_2)] \oplus \mathbb{Z} \oplus Torsions$.

Lemma 2.4. *If f_1 and f_2 homotopy equivalences and for all k, $f_{\pi_k}(\pi_k(N_1) \times 1) \subset \pi_k(N_1) \times 1$, then f is a homotopy equivalence.*

Proof. Since $\pi_k(N_1 \times N_2) \cong \pi_k(N_1) \times \pi_k(N_2)$, by the Five Lemma $f_{\pi_k} : \pi_k(N_1 \times N_2) \to \pi_k(N_1 \times N_2)$ is an isomorphism for all k. Apply the Whitehead Theorem. □

Theorem 2.5. *Suppose that N_1^n, N_2^m are hopfian manifolds with $m < n$ and $\pi_1(N_1)$ is incommensurable with $\pi_1(N_2)$. If $\beta_i(N_1) = 0$ for $1 \leq i \leq \min\{m, \frac{n}{2}\}$, $H^{m-1}(\widetilde{N_1}) = 0$, where $\widetilde{N_1}$ is the universal covering of N_1, and $\pi_k(N_2) = 0$ for $2 \leq k \leq m-2$, then $N_1^n \times N_2^m$ is a hopfian manifold.*

Proof. Let deg f =1 and f induce $\pi-$isomorphism. By Lemma 2.3, deg f_1 = 1 = deg f_2. From the incommensurability of $\pi_1(N_1)$ of $\pi_1(N_2)$, f_1 and f_2 induce π_1-isomorphisms. The homotopy classes of maps $\widetilde{N_1} \to N_2$ are in 1-1 correspondence with those of maps $\widetilde{N_1} \to \widetilde{N_2}$, where $\widetilde{N_2}$ is the universal covering of N_2. By the obstruction theory [12, Corollary VI. 16.4], the latter are in 1-1 correspondence with $H^{m-1}(\widetilde{N_1}; \pi_{m-1}(\widetilde{N_2})) \cong H^{m-1}(\widetilde{N_1}; \mathbb{Z}) \otimes \pi_{m-1}(\widetilde{N_2}) \cong 0$. Hence, all maps $\widetilde{N_1} \to N_2$ are null-homotopic. Lemma 2.4 assures that $N_1 \times N_2$ is hopfian. □

We state the following main result of this section.

Theorem 2.6. *Suppose N_1 and N_2 are t-aspherical closed, PL manifolds whose fundamental groups are normally cohopfian and $N = N_1 \times N_2$ is a hopfian PL manifold which is a codimension-2 PL fibrator. Suppose $\pi_1(N_1)$ is hereditary incommensurable with $\pi_1(N_2)$, $\pi_1(N_1)$ is sparsely abelian and $\pi_1(N_2)$ has no nontrivial abelian normal subgroup. Then N is a codimension-$(t+1)$ PL fibrator.*

Proof. By Lemma 2.1 and Lemma 2.2, N has Property NCSA. Since N_1 and N_2 are t-aspherical, so is N. According to [9, corollary 2.6], N is a codimension-$(t+1)$ PL fibrator. □

Corollary 2.7. *Suppose that N_1^n is a hopfian t-aspherical manifold with $\chi(N_1) \neq 0$ and $\beta_i(N_1) = 0$ for $1 \leq i \leq \min\{m, \frac{n}{2}\}$, and that N_2^m is aspherical manifold with $\chi(N_2) \neq 0$ such that $m < n$. If $\pi_1(N_1)$ has Property NCSA and is the hopfian group, and is incommensurable with the hopfian group $\pi_1(N_2)$. Then $N_1 \times N_2$ is a codimension-$(t+1)$ PL fibrator.*

Proof. Let $G = \pi_1(N_1)$ and $K = \pi_1(N_2)$. It suffices show that $G \times K$ is a hopfian group; then $N_1 \times N_2$ is a codimension-2 PL fibrator.

Suppose that $\phi : G \times K \to G \times K$ is an epimorphism. Since G is incommensurable with K, as before we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{j_1} & G \times K & \xrightarrow{pr_2} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_2} & & \\ 0 & \longrightarrow & G & \xrightarrow{j_1} & G \times K & \xrightarrow{pr_2} & K & \longrightarrow & 0, \end{array}$$

where the horizontal sequences are exact sequences of groups. Here ϕ_2 must be an epimorphism since ϕ is onto, and then ϕ_2 is an isomorphism since K is hopfian. By simple diagram chasing, we have that ϕ_1 is an epimorphism. But since G is hopfian, ϕ_1 is an isomorphism. Now apply the Five Lemma. □

Lemma 2.8. [7, Corollary 3.8] *Suppose that N_1^n, N_2^m are hopfian manifolds with $\pi_k(N_2) = 0$ for all $k \geq 2$. If (1) $\pi_1(N_1)$ is finite, or if (2) $\pi_1(N_1)$ is solvable and $\chi(N_2) \neq 0$, then $N_1^n \times N_2^m$ is a hopfian manifold.*

Corollary 2.9. *Suppose N_1 is a closed, orientable t-aspherical $n-$manifold with finite, sparsely abelian fundamental group and $\chi(N_1) \neq 0$, and N_2 is a closed, orientable aspherical $m-$manifold with hopfian fundamental group and $\chi(N_2) \neq 0$. Then $N_1 \times N_2$ is a codimension-$(t+1)$ PL fibrator.*

Proof. First, note that $N_1 \times N_2$ is a codimension-2 fibrator [4, Theorem 5.10]. Work of Rosset [16] implies $\pi_1(N_2)$ has no nontrivial abelian normal subgroup. Lemma 2.1 and Lemma 2.2 confirm that $\pi_1(N_1 \times N_2)$ has Property NCSA. Therefore, the conclusion follows from Theorem 2.6. □

Corollary 2.10. *Suppose N_1 is a closed,orientable t-aspherical $n-$manifold with nilpotent, sparsely abelian fundamental group and $\chi(N_1) \neq 0$, and N_2 is a closed, orientable aspherical $m-$manifold with residually finite fundamental group and $\chi(N_2) \neq 0$. Then $N_1 \times N_2$ is a codimension-$(t+1)$ PL fibrator.*

Proof. By [7, Corollary 3.9], $N_1 \times N_2$ is a hopfian manifold. Also $\pi_1(N_1 \times N_2)$ is a finitely generated residually finite group, and hence hopfian group. □

Corollary 2.11. *Suppose N_1 is a closed t-aspherical, hopfian $n-$manifold with finite, hyperhopfian, sparsely abelian fundamental group, and N_2 is a closed aspherical $m-$manifold with hyperhopfian fundamental group. If $\chi(N_2) \neq 0$, then $N_1 \times N_2$ is a codimension-$(t+1)$ PL fibrator.*

Proof. Since $N = N_1 \times N_2$ is a hopfian manifold with hyperhopfian fundamental group, N is a codimension-2 fibrator [4, Theorem 5.4]. Lemma 2.1 and Lemma 2.2 imply that $\pi_1(N)$ has Property NCSA. By Theorem 2.6, $N_1 \times N_2$ is a codimension-$(t+1)$ PL fibrator. □

References

[1] D. S. Coram and P. F. Duvall, *Approximate fibrations*, Rocky Mountain J. Math. **7** (1977), 275–288.

[2] ———, *Approximate fibrations and a movability condition for maps*, Pacific J. Math. **72** (1977), 41–56.
[3] R. J. Daverman, *PL maps with manifold fibers*, J. London Math. Soc. (2) **45** (1992), no. 1, 180–192.
[4] ———, *Hyperhopfian groups and approximate fibrations*, Compositio Math. **86** (1993), 159–176.
[5] ———, *Manifolds that induce approximate fibrations in the PL category*, Topology Appl. **66** (1995), 267–297.
[6] ———, *Codimension 2 nonfibrators with finite fundamental groups*, Proc. Amer. Math. Soc. **127** (1999), no. 3, 881–888.
[7] R. J. Daverman, Y. H. Im, and Y. Kim, *Products of Hopfian manifolds and codimension-2 fibrators*, Topology Appl. **103** (2000), no. 3, 323–338.
[8] ———, *Connected sums of 4-manifolds as codimension-k fibrators*, J. London Math. Soc. (2) **68** (2003), no. 1, 206–222.
[9] ———, *PL fibrator properties of partially aspherical manifolds*, Topology Appl. **140** (2004), no. 2-3, 181–195.
[10] M. Hall, *Subgroups of finite index in free groups*, Canadian J. Math. **1** (1949), 187–190.
[11] J.-C. Hausmann, *Geometric hopfian and non-hopfian situations*, in Geometry and Topology (C. McCrory and T. Shifrin, eds.), Lecture Notes in Pure Appl. Math., Marcel Decker, Inc., New York, 1987, pp. 157–166.
[12] S.-T. Hu, *Homotopy Theory*, Academic press (1959).
[13] Y. H. Im and Y. Kim, *Necessary and sufficient conditions for s-hopfian manifolds to be codimension-2 fibrators*, Proc. Amer. Math. Soc. **129** (2001), no. 7, 2135–2140.
[14] Y. Kim, *Manifolds with hyperhopfian fundamental group as codimension-2 fibrators*, Topology Appl. **96** (1999), 241–248.
[15] W. Magnus, A. Karass, and D. Solitar, *Combinatorial Group Theory*, Dover Publ., Inc., New York, 1976.
[16] S. Rosset, *A vanishing theorem for Euler characteristics*, Math. Z. **185** (1985), 211–215.

YOUNG HO IM
DEPARTMENT OF MATHEMATICS
PUSAN NATIONAL UNIVERSITY
PUSAN 609-735, KOREA
E-mail address: yhim@pusan.ac.kr

YONGKUK KIM
DEPARTMENT OF MATHEMATICS
KYUNGPOOK NATIONAL UNIVERSITY
TAEGU 702-701, KOREA
E-mail address: yongkuk@knu.ac.kr