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ENERGY FINITE p-HARMONIC FUNCTIONS ON GRAPHS AND ROUGH ISOMETRIES

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Abstract. We prove that if a graph G of bounded degree has finitely many p-hyperbolic ends $(1 < p < \infty)$ in which every bounded energy finite p-harmonic function is asymptotically constant for almost every path, then the set $\mathcal{HBD}_p(G)$ of all bounded energy finite p-harmonic functions on G is in one to one corresponding to $\mathbf{R}^l$, where l is the number of p-hyperbolic ends of G. Furthermore, we prove that if a graph G' is roughly isometric to G, then $\mathcal{HBD}_p(G')$ is also in an one to one correspondence with $\mathbf{R}^l$.

1. Introduction

We say that a graph G has the Liouville property if every bounded harmonic function on G is constant. Thus the set of all bounded harmonic functions on G having Liouville property is in one to one correspondence with the real line $\mathbf{R}$. With this view point, given an operator $\mathcal{A}$ on a graph, it seems natural to regard a class $\mathcal{S}$ of solutions of $\mathcal{A}$ which is in an one to one correspondence with the Euclidean space $\mathbf{R}^l$ for some positive integer l as a generalized version of the Liouville property of the pair $(\mathcal{A}, \mathcal{S})$. In this paper, we study case of the p-Laplacian operator $(1 < p < \infty)$ and the bounded p-harmonic functions on a graph G of bounded degree. If $p = 2$, then we obtain harmonic functions on G as a special case. (See [6] and [8].) In Section 3, we study a sort of an asymptotic behavior of p-harmonic functions which enables us to identify a subset of the set of the bounded p-harmonic functions on G. To be precise, if a graph G has a finite number of p-hyperbolic ends and every bounded energy finite p-harmonic function on G satisfies such an behavior, then we have the following theorem:

Theorem 1.1. *Let G be a graph with l $(l \geq 1)$ p-hyperbolic ends. Suppose that every p-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for p-almost every path in each p-hyperbolic end, where $\mathcal{HBD}_p(G)$ denotes the set of all*

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bounded energy finite p-harmonic functions on G. Then given any real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, *there exists a unique p-harmonic function* $v \in \mathcal{HBD}_p(G)$ *such that*

$$v(\mathbf{p}) = a_i \text{ for } p\text{-almost every path } \mathbf{p} \in \mathbf{P}_{E_i} \tag{1}$$

for each $i = 1, 2, \ldots, l$, *where* $E_1, E_2, \ldots, E_l$ *are p-hyperbolic ends of G, and* $\mathbf{P}_{E_i}$ *denotes a family of paths lying in* E_i *to be explained in Section 3.*

In Section 4, we extend our result to graphs being roughly isometric to those satisfying the assumption of Theorem 1.1:

Theorem 1.2. *Let G be a graph with* l $(l \geq 1)$ *p-hyperbolic ends. Suppose that every p-harmonic function in* $\mathcal{HBD}_p(G)$ *is asymptotically constant for p-almost every path in each p-hyperbolic end. Let* G' *be a graph being roughly isometric to G. Then given any real numbers* $a_1, a_2, \ldots, a_l \in \mathbf{R}$, *there exists a unique p-harmonic function* $v \in \mathcal{HBD}_p(G')$ *such that*

$$v(\mathbf{p}) = a_i \text{ for } p\text{-almost every path } \mathbf{p} \in \mathbf{P}_{E_i}$$

for each $i = 1, 2, \ldots, l$, *where* $E_1, E_2, \ldots, E_l$ *are p-hyperbolic ends of* G'.

2. Preliminaries

Let $G = (V_G, E_G)$ be a graph, where V_G and E_G denote the vertex set and the edge set, respectively, of G. If vertices x and y are the endpoints of the same edge, then we say that x and y are neighbors and write $y \in N_x$ and $x \in N_y$. The degree of x is the number of all neighbors of x and it is denoted by $\sharp N_x$. A graph G is said to be *of bounded degree* if there exists a number $\nu < \infty$ such that $\sharp N_x \leq \nu$ for all $x \in V_G$. A sequence $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ of vertices in V_G is called a path from x_0 to x_r with the length r if x_k is an element of $N_{x_{k-1}}$ for each $k = 1, 2, \ldots, r$. We say that a graph G is *connected* if any two points of V_G can be joined by a path. Throughout this paper, G is a connected infinite graph with no self-loops and is of bounded degree.

For any vertices x and y, we define $d(x, y)$ to be the length of the shortest path joining x to y. Then d defines a metric on V_G. For this metric d and $r \in \mathbf{N}$, define an r-neighborhood $N_r(x) = \{y \in V_G : d(x, y) \leq r\}$ for each $x \in V_G$. Given any subset $S \subset V_G$, the outer boundary ∂S and the inner boundary δS of S are defined by

$$\partial S = \{x \in V_G : d(x, S) = 1\} \text{ and } \delta S = \{x \in V_G : d(x, V_G \setminus S) = 1\},$$

respectively.

For each real valued function u on $S \cup \partial S$, define the *norm of p-gradient*, the *p-Dirichlet sum*, and the *p-Laplacian* of u at a point $x \in S$, where $1 < p < \infty$, in such a way that

$$\begin{aligned} |Du|(x) &= \Big(\sum_{y\in N_x} |u(y)-u(x)|^p\Big)^{1/p}, \\ I_p(u,S) &= \sum_{x\in S} |Du|^p(x), \\ \Delta_p u(x) &= \sum_{y\in N_x} \operatorname{sign}(u(y)-u(x))|u(y)-u(x)|^{p-1} \\ &= \sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x)), \end{aligned}$$

respectively.

We say that u is *p-harmonic* on S if $\Delta_p u(x) = 0$ for all $x \in S$. We introduce some useful properties of p-harmonic functions on graphs in [1]. If a subset $S \subset V_G$ is finite, then the following conditions are equivalent:

(i) A function u is p-harmonic on S.

(ii) A function u satisfies p-Laplacian equation in a weak form. That is,

$$\sum_{x\in S}\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x))(w(y)-w(x)) = 0$$

for any real valued function w on $S \cup \partial S$ such that $w = 0$ on ∂S.

(iii) A function u is a minimizer of p-Dirichlet sum $I_p(\cdot, S)$ among functions on $S \cup \partial S$ with the same values on ∂S. That is,

$$\sum_{x\in S} |Du|^p(x) \le \sum_{x\in S} |Dv|^p(x)$$

for every function v on $S \cup \partial S$ such that $v = u$ on ∂S.

Let us set $T(u,w;x,y) = |u(y)-u(x)|^{p-2}(u(y)-u(x))(w(y)-w(x))$ whenever functions u and w are defined at x and y. Then it is easy to check that

$$T(v, v-u; x, y) \ge T(u, v-u; x, y) \tag{2}$$

if u and v are defined at x and y. The equality occurs only if $v(x) - u(x) = v(y) - u(y)$. By (2), the following comparison principle holds on S: Suppose there exist p-harmonic functions u and v on a finite set $S \subset V_G$ such that $u \ge v$ on ∂S. Then $u \ge v$ on S.

Let S be a finite subset of V_G. Suppose that $\{u_i\}$ is a sequence of functions on $S \cup \partial S$ converging to a function u pointwisely. Then for each point $x \in S$,

$$|Du_i|^p(x) \to |Du|^p(x) \quad \text{and} \quad \Delta_p u_i(x) \to \Delta_p u(x)$$

and

$$I_p(u_i, S) \to I_p(u, S)$$

as $i \to \infty$. By these facts together with the comparison principle, the following existence and uniqueness result holds: Let S be a finite subset of V_G. For any function v on ∂S, there exists a unique function on $S \cup \partial S$ which is p-harmonic on S and equal to v on ∂S.

Let $\{S_i\}$ be an increasing sequence of finite connected subsets of V_G and $S = \bigcup S_i$. Let $\{u_i\}$ be a sequence of functions on $S \cup \partial S$ such that each u_i is p-harmonic on S_i and $u_i(x) \to u(x) < \infty$ as $i \to \infty$ for all $x \in S \cup \partial S$. Then the limit function u is p-harmonic on S.

We say that a real valued function u is *energy finite* if it has finite p-Dirichlet sum on the whole set V_G, i.e., $I_p(u, V_G) < \infty$. Let $\mathcal{BD}_p(G)$ denote the set of all bounded energy finite functions on V_G. Then, $\mathcal{BD}_p(G)$ is a Banach space with the norm

$$||u||_p = \sup_{V_G} |u| + I_p(u, V_G)^{1/p}.$$

We denote by $\mathcal{BD}_{p,0}(G)$ the closure of the set of all finitely supported functions on V_G in $\mathcal{BD}_p(G)$ with respect to the norm $|| \cdot ||_p$. The subset of all bounded p-harmonic functions in $\mathcal{BD}_p(G)$ is denoted by $\mathcal{HBD}_p(G)$.

The subgraph Γ induced by a set $S \subset V_G$ is the graph $\Gamma = (S, E_\Gamma)$, where E_Γ is the set of all edges in E_G with both ends points in S. In particular, that a subset $S \subset V_G$ is connected means that the subgraph $\Gamma = (S, E_\Gamma)$ induced by S is connected. A connected subset $S \subset V_G$ with $\partial S \neq \emptyset$ is called *D_p-massive* if there exists a nonnegative p-harmonic function u on S such that $u = 0$ on ∂S, $\sup_S u = 1$ and $I_p(u, S) < \infty$. We say that a connected infinite set $S \subset V_G$ is *p-hyperbolic* if there exists a nonempty finite set $A \subset S$ such that

$$\mathrm{Cap}_p(A, \infty, S) = \inf_u I_p(u, S) > 0,$$

where the infimum is taken over all finitely supported function u on $S \cup \partial S$ such that $u = 1$ on A. Otherwise, S is called *p-parabolic.*

We now introduce the p-Royden decomposition: (See [9].)

Proposition 2.1. *If a graph G is p-hyperbolic, then for each function $u \in \mathcal{BD}_p(G)$, there exist unique functions $h \in \mathcal{HBD}_p(G)$ and $g \in \mathcal{BD}_{p,0}(G)$ such that $u = h + g$.*

For each nonnegative real valued function w on E_G, define

$$\mathcal{E}_p(w) = \sum_{e \in E_G} w^p(e).$$

Let $\mathbf{P}$ be a family of infinite paths in G. The *p-extremal length* $\lambda_p(\mathbf{P})$ of $\mathbf{P}$ is defined by

$$\lambda_p(\mathbf{P}) = \left(\inf_w \ \mathcal{E}_p(w)\right)^{-1},$$

where the infimum is taken over the set of all nonnegative functions w on E_G such that $\mathcal{E}_p(w) < \infty$ and $\sum_{e \in E_{\mathbf{x}}} w(e) \geq 1$ for each path $\mathbf{x} \in \mathbf{P}$, where $E_{\mathbf{x}}$ denotes the edge set of $\mathbf{x}$. The following proposition gives some fundamental properties of the extremal length. (See [4].)

Proposition 2.2. *Let $\mathbf{P}_n, n = 1, 2, \ldots,$ be families of paths in a graph G.*

(i) *If* $\mathbf{P}_1 \subset \mathbf{P}_2$*, then* $\lambda_p(\mathbf{P}_1) \geq \lambda_p(\mathbf{P}_2)$.
(ii) $\sum_{n=1}^{\infty} \lambda_p(\mathbf{P}_n)^{-1} \geq \lambda_p(\cup_{n=1}^{\infty}\mathbf{P}_n)^{-1}$.

On the other hand, the p-extremal length is closely related to the p-capacity: Let $S \subset V_G$ be a connected infinite subset. For a nonempty finite subset $A \subset S$, let $\mathbf{P}_{S,A}$ be the set of all non-self-intersecting infinite paths in S starting from a vertex in A. Then we have

$$\lambda_p(\mathbf{P}_{S,A}) = \mathrm{Cap}_p(A, \infty, S)^{-1}. \tag{3}$$

(See [9] and [7].) Furthermore, if $S \subset V_G$ is p-hyperbolic, then by (3),

$$\lambda_p(\mathbf{P}_{S,A}) = \mathrm{Cap}_p(A, \infty, S)^{-1} < \infty. \tag{4}$$

We say that a property holds for *p-almost every path* in $\mathbf{P}$ if the subset of all paths for which the property is not true has p-extremal length ∞.

The following proposition gives some p-almost every path properties of energy finite functions: (See [4] and [9].)

Proposition 2.3. *Let* $\mathbf{P}_o$ *be the family of all non-self-intersecting infinite paths from a fixed point* $o \in V_G$.

(i) *If* $u \in \mathcal{BD}_p(G)$*, then* $u(\mathbf{x})$ *exists and is finite for p-almost every path* $\mathbf{x} \in \mathbf{P}_o$*, where* $u(\mathbf{x}) = \lim u(x)$ *as* $x \to \infty$ *along the vertices of* $\mathbf{x}$.
(ii) $u \in \mathcal{BD}_{p,0}(G)$ *if and only if* $u(\mathbf{x}) = 0$ *for p-almost every path* $\mathbf{x} \in \mathbf{P}_o$.

3. Asymptotically constant for p-almost every path on ends

We now define ends of a graph G with its vertex set V_G: Fix a point $o \in V_G$. For each $r \in \mathbf{N}$, we denote by $\sharp(r)$ the number of infinite connected components of $V_G \setminus N_r(o)$. Let $\lim_{r\to\infty}\sharp(r) = l$, where l may be infinity, then we say that the number of ends of G is l. If l is finite, then we can choose $r_0 \in \mathbf{N}$ such that $\sharp(r) = l$ for all $r \geq r_0$.

Using the p-hyperbolicity, we can divide ends of G into two classes as follows: An end E of G is called p-hyperbolic if

$$\mathrm{Cap}_p(\partial E, \infty, E) = \inf_u I_p(u, E) > 0,$$

where the infimum is taken over all finitely supported function u on $E \cup \partial E$ such that $u = 1$ on ∂E. Otherwise, the end is called p-parabolic.

From the definition of a p-hyperbolic end, we have the following lemma:

Lemma 3.1. *If E is a p-hyperbolic end, then there exists a p-harmonic function* u_E *on E, called a p-harmonic measure of E, with the following properties:*

(i) $0 \leq u_E \leq 1$ *on* E;
(ii) $u_E = 0$ *on* ∂E;
(iii) $\limsup_{x\in E} u_E(x) = 1$;
(iv) u_E *has finite p-Dirichlet sum over* E.

Let us denote $\mathbf{P}_G$ to be the family of all non-self-intersecting infinite paths lying in $V_G \setminus N_{r_1}(o)$ starting from a vertex in $\delta N_{r_1}(o)$ for some large $r_1 \in \mathbf{N}$. For each end E of G, let us denote $\mathbf{P}_E \subset \mathbf{P}_G$ to be the family of all paths lying in $E \setminus N_{r_1}(o)$ starting from a vertex in $\delta N_{r_1}(o) \cap E$. We say that a real valued function u on V_G is asymptotically constant for p-almost every path in E if there exists a constant c such that

$$u(\mathbf{x}) = c \text{ for } p\text{-almost every path } \mathbf{x} \in \mathbf{P}_E,$$

where $u(\mathbf{x}) = \lim u(x)$ as x goes to ∞ along vertices on $\mathbf{x}$.

Lemma 3.2. *Let E be a p-hyperbolic end of a graph G and u be a nonconstant function in $\mathcal{HBD}_p(G)$ such that $0 \le u \le 1$. Suppose that u is asymptotically constant for p-almost every path in E. If $\limsup_{x \to \infty, x \in E} u = 1$, then $u(\mathbf{x}) = 1$ for p-almost every path $\mathbf{x} \in \mathbf{P}_E$.*

Proof. Suppose the lemma is not true. Then by assumption, there exists a constant c such that $u(\mathbf{x}) = c$ for p-almost every path $\mathbf{x} \in \mathbf{P}_E$ and $0 \le c < 1$. Since u is nonconstant, there exists a proper subset Ω of E such that $\Omega = \{x \in E : u(x) > 1 - \epsilon\}$, where ϵ is a positive constant so small that $1 - \epsilon > c$. Clearly, Ω is a D_p-massive subset. By (4), there exists a subfamily $\mathbf{P}_\Omega$ of $\mathbf{P}_E$ such that $\lambda_p(\mathbf{P}_\Omega) < \infty$. But from the definition of Ω, one can conclude that $u(\mathbf{x}) > c$ for all paths $\mathbf{x} \in \mathbf{P}_\Omega$. This contradicts the fact that $u(\mathbf{x}) = c$ for p-almost every path $\mathbf{x} \in \mathbf{P}_E$. This completes the proof. □

Proof of Theorem 1.1. For each $i = 1, 2, \ldots, l$, extend u_{E_i} to be zero outside E_i and then construct a sequence of real valued functions $\{u_{r,i}\}_{r > r_0}$ on V_G such that

$$\begin{cases} \Delta_p u_{r,i} = 0 & \text{on } N_r(o); \\ u_{r,i} = u_{E_i} & \text{on } V_G \setminus N_r(o), \end{cases}$$

where u_{E_i} is a p-harmonic measure of E_i constructed in Lemma 3.1 for each i. By the comparison principle, $u_{E_i} \le u_{r,i} \le 1$ on $N_r(o)$ for each i. Thus there exists a convergent subsequence, and its limit function u_i satisfies that

$$\begin{cases} \Delta_p u_i = 0 \quad \text{on } V_G; \\ 0 \le u_i \le 1; \\ \limsup_{x \to \infty, x \in E_i} u_i = 1. \end{cases}$$

By the minimizing property of p-harmonic functions, u_i is energy finite for each i.

Without loss of generality, we may assume that $0 < a_1 \le a_2 \le \cdots \le a_l \le 2a_1$. Let us construct a sequence of real valued functions $\{v_r\}_{r > r_0}$ such that

$$\begin{cases} \Delta_p v_r = 0 & \text{on } N_r(o); \\ v_r = a_i & \text{on } E_i \setminus N_r(o); \\ v_r = 0 & \text{on } V_G \setminus (\cup_{k=1}^l E_k \cup N_r(o)), \end{cases}$$

where $i = 1, 2, \ldots, l$. Then

$$a_i u_i \leq v_r \leq a_i(2 - u_i) \text{ on } (\delta N_{r_0}(o) \cup \partial N_r(o)) \cap E_i,$$

where u_i is the p-harmonic function constructed above. Hence by the comparison principle, we conclude that

$$a_i u_i \leq v_r \leq a_i(2 - u_i) \text{ on } N_r(o) \cap E_i.$$

There exists a subsequence, denoted by $\{v_{r_m}\}$, converging to a p-harmonic function v on V_G. By Lemma 3.2, $u_i(\mathbf{x}) = 1$ for p-almost every path $\mathbf{x} \in \mathbf{P}_{E_i}$ for each i. Hence v satisfies (1). By the minimizing property of p-harmonic function, v has finite p-Dirichlet sum.

Suppose that there exists a p-harmonic function $w \in \mathcal{HBD}_p(G)$ satisfying (1). Put $\mathbf{P}_{E_i} = \mathbf{P}_{i,w,1} \cup \mathbf{P}_{i,w,2}$ for each i, where

$$\mathbf{P}_{i,w,1} = \{\mathbf{x} \in \mathbf{P}_{E_i} : w(\mathbf{x}) = a_i\} \text{ and } \mathbf{P}_{i,w,2} = \{\mathbf{x} \in \mathbf{P}_{E_i} : w(\mathbf{x}) \neq a_i\}.$$

Then we have $\lambda_p(\mathbf{P}_{i,w,1}) < \infty$ and $\lambda_p(\mathbf{P}_{i,w,2}) = \infty$ for each i. Similarly, let us set $\mathbf{P}_{E_i} = \mathbf{P}_{i,v,1} \cup \mathbf{P}_{i,v,2}$ for each i, where

$$\mathbf{P}_{i,v,1} = \{\mathbf{x} \in \mathbf{P}_{E_i} : v(\mathbf{x}) = a_i\} \text{ and } \mathbf{P}_{i,v,2} = \{\mathbf{x} \in \mathbf{P}_{E_i} : v(\mathbf{x}) \neq a_i\}.$$

Then we have $\lambda_p(\mathbf{P}_{i,v,1}) < \infty$ and $\lambda_p(\mathbf{P}_{i,v,2}) = \infty$ for each i. From Proposition 2.2 and Proposition 2.3, we conclude that

$$\begin{aligned} \lambda_p(\mathbf{P}_{E_i} \setminus (\mathbf{P}_{i,w,1} \cap \mathbf{P}_{i,v,1})) &= \lambda_p((\mathbf{P}_{E_i} \setminus \mathbf{P}_{i,w,1}) \cup (\mathbf{P}_{E_i} \setminus \mathbf{P}_{i,v,1})) \\ &\geq 1/(\lambda_p(\mathbf{P}_{E_i} \setminus \mathbf{P}_{i,w,1})^{-1} + \lambda_p(\mathbf{P}_{E_i} \setminus \mathbf{P}_{i,v,1})^{-1}) \\ &= \infty \end{aligned}$$

for each i. This implies that

$$(v - w)(\mathbf{x}) = 0 \text{ for } p\text{-almost every path } \mathbf{x} \in \mathbf{P}_{E_i}$$

for each $i = 1, 2, \ldots, l$. On the other hand, since $\lambda_p(\mathbf{P}_G \setminus \cup_{i=1}^{l} \mathbf{P}_{E_i}) = \infty$, we have

$$(v - w)(\mathbf{x}) = 0 \text{ for } p\text{-almost every path } \mathbf{x} \in \mathbf{P}_G. \tag{5}$$

Consequently, by Proposition 2.3, we conclude that $v - w \in \mathcal{BD}_{p,0}(G)$. Thus there exists a sequence of finitely supported functions converging to $v - w$ in $\mathcal{BD}_p(G)$. By this fact together with the Hölder inequality, since v and w are p-harmonic functions on V_G, it is easy to see that

$$\sum_{x \in V_G} \sum_{y \in N_x} |v(y) - v(x)|^{p-2}(v(y) - v(x))((v - w)(y) - (v - w)(x)) = 0$$

and

$$\sum_{x \in V_G} \sum_{y \in N_x} |w(y) - w(x)|^{p-2}(w(y) - w(x))((v - w)(y) - (v - w)(x)) = 0.$$

Thus by (2), we conclude that $v - w$ is constant function on N_x for all points $x \in V_G$. Since V_G is connected, by (5), we conclude that $v \equiv w$ on V_G. □

4. Asymptotically constant for p-almost every path and rough isometries

We begin with introducing rough isometries between metric spaces. A map $\varphi : X \to Y$ is called a *rough isometry* between metric spaces X and Y if it satisfies the following condition:

(R) for some constant $\tau > 0$, the τ-neighborhood of the image $\varphi(X)$ covers Y;
there exist constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b$$

for all points $x_1, x_2 \in X$, where d denotes the distances of X and Y induced from their metrics, respectively.

If such a map exists, then X is said to be roughly isometric to Y. Being roughly isometric is an equivalent relation. (See [2].) In particular, if $\varphi : X \to Y$ is a rough isometry satisfying (R), then for any point $y \in Y$, there exists at least one point $x \in X$ such that $d(\varphi(x), y) < \tau$. If we set $\varphi^-(y) = x$, then φ^- satisfies (R) with constants τ', a' and b', where $\tau' = a(b + \tau), a' = a$ and $b' = a(b + 2\tau)$.

On the other hand, since the vertex set of each graph is a metric space, we can define rough isometries between the vertex sets of graphs similarly as above. Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be graphs, and $\varphi : V_{G'} \to V_G$ be a rough isometry. For convenience' sake, we prefer to write the rough isometry $\varphi : G' \to G$ rather than $\varphi : V_{G'} \to V_G$.

Slightly modifying the proof of [5, 3], the number of ends of a graph is a rough isometric invariant. In fact, the rough isometry between graphs gives a one to one correspondence between ends of the graphs and, furthermore, it induces the rough isometry between each end and its corresponding end. On the other hand, the p-parabolicity of ends is preserved under rough isometries between ends. Also, we can prove that the property of asymptotically constant for p-almost every path is invariant under rough isometries between ends as follows:

Theorem 4.1. *Let G and G' be graphs with finitely many ends and roughly isometric to each other. Suppose that every p-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G. Then every p-harmonic function in $\mathcal{HBD}_p(G')$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G'.*

To prove Theorem 4.1, we need the following lemmas:

Lemma 4.2. *Let G and G' be graphs with finitely many ends, and $\varphi : G' \to G$ be a rough isometry. Suppose that every p-harmonic function in $\mathcal{HBD}_p(G)$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G. Then for each $u \in \mathcal{HBD}_p(G')$, $u \circ \varphi^-$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G.*

Proof. For each $u \in \mathcal{HBD}_p(G')$, it is easy to check that $u \circ \varphi^- \in \mathcal{BD}_p(G)$. So, by Proposition 2.1, there exist unique $h \in \mathcal{HBD}_p(G)$ and $g \in \mathcal{D}_{p,0}(G)$ such that

$$u \circ \varphi^- = h + g.$$

By the assumption, h is asymptotically constant for p-almost every path in each p-hyperbolic end of G. On the other hand, by Proposition 2.3, g is asymptotically constant 0 for p-almost every path in each p-hyperbolic end of G.

Let $E_1, E_2, \ldots, E_l$ be p-hyperbolic ends of G. Then there exist constants $c_1, c_2, \ldots, c_l$ such that

$$h(\mathbf{y}) = c_i \text{ for } p\text{-almost every path } \mathbf{y} \in \mathbf{P}_{E_i}$$

for each $i = 1, 2, \ldots, l$. Put $\mathbf{P}_{E_i} = \mathbf{P}_{i,h,1} \cup \mathbf{P}_{i,h,2}$ for each i, where

$$\mathbf{P}_{i,h,1} = \{\mathbf{y} \in \mathbf{P}_{E_i} : h(\mathbf{y}) = c_i\} \text{ and } \mathbf{P}_{i,h,2} = \{\mathbf{y} \in \mathbf{P}_{E_i} : h(\mathbf{y}) \neq c_i\}.$$

Then we have $\lambda_p(\mathbf{P}_{i,h,1}) < \infty$ and $\lambda_p(\mathbf{P}_{i,h,2}) = \infty$ for each i. Similarly, let us set $\mathbf{P}_{E'_i} = \mathbf{P}_{i,g,1} \cup \mathbf{P}_{i,g,2}$ for each i, where

$$\mathbf{P}_{i,g,1} = \{\mathbf{y} \in \mathbf{P}_{E_i} : g(\mathbf{y}) = 0\} \text{ and } \mathbf{P}_{i,g,2} = \{\mathbf{y} \in \mathbf{P}_{E_i} : g(\mathbf{y}) \neq 0\}.$$

Then, by our claim, we have $\lambda_p(\mathbf{P}_{i,g,1}) < \infty$ and $\lambda_p(\mathbf{P}_{i,g,2}) = \infty$ for each i.

Arguing similarly as in the proof of Theorem 1.1, we have

$$\lambda_p(\mathbf{P}_{E_i} \setminus (\mathbf{P}_{i,h,1} \cap \mathbf{P}_{i,g,1})) = \infty$$

for each i. Hence $u \circ \varphi^-$ is asymptotically constant c_i at infinity of E_i for p-almost every path $\mathbf{y} \in \mathbf{P}_{E_i}$ for each i. This completes the proof. □

Lemma 4.3. *Let G and G' be graphs with finitely many ends and $\varphi : G' \to G$ be a rough isometry. Let $u \in \mathcal{HBD}_p(G')$. Suppose that $u \circ \varphi^-$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G. Then u is asymptotically constant for p-almost every path in each p-hyperbolic end of G'.*

Proof. Let E be a p-hyperbolic end of G and E' be the corresponding end of G' under φ. Since $u \in \mathcal{HBD}_p(G')$, by Proposition 2.3,

$$u(\mathbf{x}) \text{ exists and finite for } p\text{-almost every path } \mathbf{x} \in \mathbf{P}_o.$$

Put $\mathbf{P}_{E'} = \mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{P}_3$, where $\mathbf{P}_1 = \{\mathbf{x} \in \mathbf{P}_{E'} : u(\mathbf{x}) = c\}$, $\mathbf{P}_2 = \{\mathbf{x} \in \mathbf{P}_{E'} : u(\mathbf{x}) \neq c\}$ and $\mathbf{P}_3 = \{\mathbf{x} \in \mathbf{P}_{E'} : u(\mathbf{x}) \text{ does not exists.}\}$. Since $\lambda_p(\mathbf{P}_3) = \infty$, we have only to show that $\lambda_p(\mathbf{P}_2) = \infty$.

For each path $\mathbf{x} \in \mathbf{P}_2$, we will assign a suitable path $\mathbf{y} \in \mathbf{P}_{2,\varphi^-}$, where $\mathbf{P}_{2,\varphi^-} = \{\mathbf{y} \in \mathbf{P}_G : (u \circ \varphi^-)(\mathbf{y}) \neq c\}$. Let us choose any path $\mathbf{x} \in \mathbf{P}_2$. We may assume that $\mathbf{x} = (o, x_1, x_2, \ldots, x_n, \ldots)$. By definition of the inverse rough isometry φ^-, there exists a point $y_n \in E$ such that $d(x_n, \varphi^-(y_n)) < a(b + \tau)$ for each positive integer n. Let us choose a positive constant ρ in such a way that $d(y_n, y_{n+1}) \leq \rho$ and $d(\varphi^-(y_n), \varphi^-(y_{n+1})) \leq \rho$.

For each positive integer n, we can choose a minimal path $(z_0^n, z_1^n, \ldots, z_{m_n}^n)$ in such a way that $z_0^n = y_n$, $z_{m_n}^n = y_{n+1}$, and $m_n \leq \rho$. It follows that there exists an infinite path $\mathbf{y} = (o', t_1, t_2, \ldots, t_j, \ldots) \in \mathbf{P}_E$ and a nondecreasing sequence of subscripts $j(n) \to \infty$ as $n \to \infty$ such that $t_{j(n)} = y_n$ and $j(n+1) - j(n) \le \rho$. One can choose a minimal path $(v_0^n, v_1^n, \ldots, v_{l_n}^n)$ in such a way that $s_0^n = x_n$, $s_{l_n}^n = \varphi^-(t_{j(n)})$ and $l_n \le a(b+\tau)$. Let us observe that

$$\begin{aligned} |u(x_n) - u(\varphi^-(t_{j(n)}))| &\le a(b+\tau) \sum_{i=1}^{l_n} |u(s_i^n) - u(s_{i-1}^n)| \\ &\le C \sum_{x' \in N_{a(b+\tau)}(x_n)} |Du|(x'). \end{aligned}$$

Since $u \in \mathcal{BD}_p(E')$, we conclude that

$$|u(x_n) - u(\varphi^-(t_{j(n)}))|^p \le C \sum_{x' \in N_{a(b+\tau)}(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.$$

This implies that $(u \circ \varphi^-)(t_{j(n)}) \to u(\mathbf{y}) \ne c$ as $n \to \infty$. On the other hand, we have

$$\begin{aligned} |u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))| &\le \rho \sum_{i=1}^{m_n} |u(\varphi^-(z_i^n)) - u(\varphi^-(z_{i-1}^n))| \\ &\le C \sum_{x' \in N_\rho(x_n)} |Du|(x') \end{aligned}$$

for each subscript $j \in [j(n), j(n+1)]$. Hence we have

$$|u(\varphi^-(t_j)) - u(\varphi^-(t_{j(n)}))|^p \le C \sum_{x' \in N_\rho(x_n)} |Du|^p(x') \to 0 \text{ as } n \to \infty.$$

Thus $(u \circ \varphi^-)(t_j) \to u(\mathbf{x}) \ne c$ as $j \to \infty$. Hence $\mathbf{y}$ belongs to $\mathbf{P}_{2,\varphi^-}$.

Since $\lambda_p(\mathbf{P}_{2,\varphi^-}) = \infty$, by the equivalent condition for a family of paths to have infinite p-extremal length [4], there exists a nonnegative function w on the edge set E_E of E such that $\sum_{\tilde{e} \in E_E} w^p(\tilde{e}) = \mathcal{E}_p(w) < \infty$ and $\sum_{\tilde{e} \in E(\mathbf{y})} w(\tilde{e}) = \infty$ for all paths $\mathbf{y} \in \mathbf{P}_{2,\varphi^-}$. For each positive integer ζ and each edge $e = [z_1, z_2] \in E_{E'}$, let us define a set $U(e,\zeta) = \{\tilde{e} = [a_1, a_2] \in E_E : d(z_i, \varphi^-(a_j)) \le \zeta \text{ for some } i, j = 1, 2\}$. Let us define a function w^* on $E_{E'}$ in the following way: $w^*(e) = \sup_{\tilde{e} \in U(e,\zeta)} w(\tilde{e})$for all edges $e \in E_{E'}$. Since $w^{*s}(e) \le \sum_{\tilde{e} \in U(e,\zeta)} w^s(\tilde{e})$ for each edge $e \in E_{E'}$, we have

$$\mathcal{E}_p(w^*) \le C \sum_{\tilde{e} \in E_E} w^p(\tilde{e}) < \infty,$$

where C is a positive constant depending on ζ. Let us fix a positive integer κ such that $[t_{j-1}, t_j] \in U([x_n, x_{n+1}], \kappa)$ for all $j(n) \le j \le j(n+1)$, where $\mathbf{y} = (o', t_1, t_2, \ldots, t_j, \ldots)$ is a path in $\mathbf{P}_{2,\varphi^-}$ and $\mathbf{x} = (o, x_1, x_2, \ldots, x_n, \ldots)$ is a path in $\mathbf{P}_2$ which are given above. Then for each path $\mathbf{x} \in \mathbf{P}_2$,

$$\sum_{e \in E(\mathbf{x})} w^*(e) \ge \frac{1}{\rho} \sum_{\tilde{e} \in E(\mathbf{y})} w(\tilde{e}) = \infty.$$

Therefore, we have $\lambda_p(\mathbf{P}_2) = \infty$. This completes the proof. □

We are now ready to prove Theorem 4.1:

Proof of Theorem 4.1. Let u be a p-harmonic function in $\mathcal{HBD}_p(G')$. By Lemma 4.2, the function $u \circ \varphi^-$ is asymptotically constant for p-almost every path in each p-hyperbolic end of G. Then, by Lemma 4.3, the function u is asymptotically constant for p-almost every path in each p-hyperbolic end of G'. This completes the proof. □

Combining Theorem 1.1 and Theorem 4.1, we get Theorem 1.2.

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