Commun. Korean Math. Soc. **22** (2007), No. 2, pp. 235–240

EXPONENT-QUASIADDITIVE PROPERTIES AND APPLICATION

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ABSTRACT. In this paper the authors study the properties of the so-called exponent-quasiadditive functions and an application to the generalized Grötzsch ring function of quasiconformal theory is specified.

1. Introduction

Functional equations and inequalities have been studied extensively. See for instance the monographs [1, 7–10], which also discuss applications to nonlinear analysis, economics and logistic systems. It is easy to observe that $f(x) = \log x$ is one of the solutions of the following functional equation:

$$f(x) + f(y) = f(xy). \tag{1.1}$$

It was observed in [14] that certain transcendental functions defined in terms of generalized elliptic integrals have asymptotic behavior similar to the function $\log x$. But it is clear that these functions do not satisfy the functional equation(1.1). So we may consider whether these functions have relations with the following inequalities:

$$a \le W_f(x,y) \equiv \frac{f(x) + f(y)}{f(xy)} \le b, \tag{1.2}$$

where f is nondecreasing on $(1, \infty)$, a and b are positive constants. Since these transcendental functions are nonlinear, the functions satisfy condition (1.2) are rather surprising, and it yields a useful technique for proving inequalities and bound estimates. This is the motivation of this paper.

In this paper we study the basic properties of the functions which satisfy the condition (1.2) on $(1, \infty)$, and an application to the generalized Grötzsch ring function of quasiconformal theory is specified.

Received May 22, 2006; Revised February 6, 2007.

2000 *Mathematics Subject Classification.* 26D07, 33C05.

Key words and phrases. exponent-quasiadditive, upper bound, lower bound, generalized Grötzsch ring function, quasiconformal theory.

The research in partly supported by 973 project of China under grant 2006CB708304, N. S. Foundation of China under grant 10471039 and Foundation of the Educational Committee of Zhejiang Province under grant 20060306.

2. Notations and Lemmas

In this section, we shall establish and introduce five lemmas, they are crucial in the proof of the main theorem. We first give some necessary notations and definitions.

Throughout the paper we let $r' = \sqrt{1-r^2}$ for $r \in [0,1]$. Given complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C}\backslash[1,\infty)$ of

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1. \tag{2.1}$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the *shifted factorial function* $(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N} := \{1,2,3,\ldots\}$. For $r \in (0,1)$, and $a \in (0,1)$, the *generalized elliptic integrals* (cf. [4, Section 5.5]) are defined by

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) \equiv \frac{\pi}{2} F(a, 1-a; 1; r^2), \\ \mathcal{K}'_a = \mathcal{K}'_a(r) \equiv \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \frac{\pi}{2}, \quad \mathcal{K}_a(1) = \infty, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) \equiv \frac{\pi}{2} F(a-1, 1-a; 1; r^2), \\ \mathcal{E}'_a = \mathcal{E}'_a(r) \equiv \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \frac{\pi}{2}, \quad \mathcal{E}_a(1) = \dfrac{\sin(\pi a)}{2(1-a)}. \end{cases} \tag{2.3}$$

In the particular case $a = 1/2$, the functions $\mathcal{K}_a$ and $\mathcal{E}_a$ reduce to $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively, which are the well-known *complete elliptic integrals of the first and second kind*, respectively (cf. [5, 6]). By symmetry of (2.2), we may assume that $a \in (0, 1/2]$ in the sequel.

Definition 2.1. ([2]) For $a \in (0, 1/2]$ and $r \in (0,1)$, the generalized Grötzsch ring function is defined as

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{\mathcal{K}'_a(r)}{\mathcal{K}_a(r)}. \tag{2.4}$$

When $a = 1/2$, the function $\mu(r) \equiv \mu_{1/2}(r)$ called the Grötzsch ring function, is the modulus of the plane Grötzsch ring $B^2 \setminus [0,r]$, where B^2 is the open unit disk in the complex plane.

Definition 2.2. ([2]) For $a \in (0, 1/2]$ and $r \in (0,1)$, the special function $m_a(r)$ is defined as

$$m_a(r) = \frac{2}{\pi \sin(\pi a)} r'^2 \mathcal{K}_a(r) \mathcal{K}'_a(r). \tag{2.5}$$

It is well known that the functions $\mu_a(r)$ and $m_a(r)$ play very important roles in the study of quasiconformal theory and the theory of Ramanujan's modular equations (cf. [2, 3, 11, 12, 13]).

Let H be the class of all positive nondecreasing functions on $(1, \infty)$, and let

$$W_f(x, y) \equiv \frac{f(x) + f(y)}{f(xy)}. \tag{2.6}$$

Definition 2.3. A function $f \in H$ is called exponent-quasiadditive ($f \in EQAD$) if

$$\lambda_f \equiv \inf_{x,y>1} W_f(x, y) > 0.$$

For $f \in H$, every nonnegative number $\lambda \leq \lambda_f$ is called a lower EQAD bound while λ_f is the greatest lower bound. Similarly, every number $\mu \geq \mu_f \equiv \sup_{x,y>1} W_f(x, y)$ is called an upper EQAD bound while μ_f is the least upper bound.

Definition 2.4. For $f \in H$, let

$$\lambda'_f = \inf_{x \in (1,\infty)} W_f(x, x), \tag{2.7}$$

and

$$\mu'_f = \sup_{x \in (1,\infty)} W_f(x, x). \tag{2.8}$$

Lemma 1. ([14]) *For any fixed $t \in (0, 1)$, the function $G(r) \equiv m_a(rt) - m_a(r)$ is strictly increasing from $(0, 1)$ onto $(-\log t, m_a(t))$.*

Lemma 2. ([2]) *Let $a \in (0, 1/2]$. Then the function $f(r) \equiv r'^c \mathcal{K}_a(r)$ is decreasing if and only if $c \geq 2a(1 - a)$, in which case $r'^c \mathcal{K}_a(r)$ is decreasing from $(0, 1)$ onto $(0, \pi/2)$.*

For the proof of the following lemma we also need the two derivative formulas (cf. [2, Theorem 4.1]). For $r \in (0, 1)$ and $a \in (0, 1/2]$,

$$\frac{\mathrm{d}\mu_a(r)}{\mathrm{d}r} = -\frac{\pi^2}{4rr'^2 \mathcal{K}_a(r)^2}. \tag{2.9}$$

$$\frac{\mathrm{d}\mathcal{K}_a(r)}{\mathrm{d}r} = \frac{2(1 - a)[\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)]}{rr'^2}. \tag{2.10}$$

Lemma 3. *For $a \in (0, 1/2)$, the function $f(r) \equiv \mu_a(r)/\mu_a(r^2)$ is increasing from $(0, 1)$ onto $(1/2, 1)$.*

Proof. By differentiation, (2.4), (2.5) and (2.9), we get

$$\begin{aligned} f'(r) =& \frac{1}{\mu_a(r^2)^2} \left[-\frac{\pi^2}{4} \frac{\mu_a(r^2)}{rr'^2 \mathcal{K}_a(r)^2} + \frac{\pi^2}{4} \frac{2r\mu_a(r)}{r^2(1 - r^4)\mathcal{K}_a(r^2)^2} \right] \\ &= C(r)[2m_a(r) - m_a(r^2)], \end{aligned} \tag{2.11}$$

where $C(r) = \pi^2/[4r(1 + r^2)\mathcal{K}_a(r)^2 \mathcal{K}'_a(r^2)^2]$. The monotonicity of f follows from Lemma 1.

By l'Hôpital's Rule, (2.2), (2.9) and Lemma 2, we have

$$f(0^+) = \lim_{r\to 0^+} \frac{\mu_a(r)}{\mu_a(r^2)} = \frac{1}{2} \lim_{r\to 0^+} \frac{(1+r^2)\mathcal{K}_a(r^2)^2}{\mathcal{K}_a(r)^2} = \frac{1}{2}.$$

By l'Hôpital's Rule, (2.3), (2.9) and Lemma 2, we get

$$\lim_{r\to 1^-} \frac{\mathcal{K}_a(r^2)}{\mathcal{K}_a(r)} = \lim_{r\to 1^-} \frac{2}{1+r^2} \frac{\mathcal{E}_a(r^2) - (1-r^4)\mathcal{K}_a(r^2)}{\mathcal{E}_a(r) - (1-r^2)\mathcal{K}_a(r)} = 1. \tag{2.12}$$

By (2.12) it is easy to obtain the limiting value

$$f(1^-) = \lim_{r\to 1^-} \frac{\mu_a(r)}{\mu_a(r^2)} = \lim_{r\to 1^-} \frac{1+r^2}{2} \frac{\mathcal{K}_a(r^2)^2}{\mathcal{K}_a(r)^2} = 1. \tag{2.13}$$

□

Lemma 4. *For $f \in H$, let $a = f(1^+)$ and $b = \lim_{x\to\infty} f(x)$, so that $0 \le a < \infty$, $0 < b \le \infty$.Let $\lambda = \lambda_f$ and $\mu = \mu_f$. Then*
(1) $0 \le \lambda \le 1 + a/b \le \mu \le 2$,
(2) *If $a = 0$ or $b = \infty$, then $\lambda \le 1$,*
(3) *If $\lambda > 1$, then $0 < a \le b \le a/(\lambda - 1)$,*
(4) *If $a > 0$ or $b < \infty$, then $\mu = 2$.*

Proof. (1) By (2.6), $0 < W_f(x,y) \le 2$, so that $0 \le \lambda \le \mu \le 2$. Clearly,

$$\lambda \le \frac{f(x)+f(y)}{f(xy)} \le 1 + \frac{f(y)}{f(x)}, \tag{2.14}$$

for all $x, y \in (1,\infty)$. Now letting $y \to 1^+$ and $x \to \infty$, we get $\lambda \le 1 + (a/b)$.

Next, let x be a point of continuity of f. Then,

$$\mu \ge \lim_{y\to 1^+} W_f(x,y) = 1 + \frac{a}{f(x)} \ge 1 + \frac{a}{b}. \tag{2.15}$$

(2) This follows from (1).
(3) This follows from (1) and (2).
(4) If $a > 0$, then

$$2 \ge \mu > W_f(x,x) = \frac{2f(x)}{f(x^2)} \to 2, \tag{2.16}$$

as $x \to 1^+$.

If $b < \infty$, then letting $x \to \infty$ in the above inequality, we obtain the result.

□

Lemma 5. *If $f \in H$, then*

$$\frac{1}{2}\lambda'_f \le \lambda_f \le \lambda'_f, \tag{2.17}$$

and

$$\mu'_f \le \mu_f \le 1 + (\mu'_f/2). \tag{2.18}$$

Consequently, $\lambda_f > 0$ if and only if $\lambda'_f > 0$, and $\mu_f < 2$ if and only if $\mu'_f < 2$.

Proof. We only need to prove the first inequality in (2.17) and the second inequality in (2.18). By symmetry, we may assume that $1 < x \leq y < \infty$. Then

$$\begin{aligned}(1/2)W_f(y,y) \leq \frac{f(y)}{f(xy)} \leq W_f(x,y) &= \frac{f(xy)+f(x)+[f(y)-f(xy)]}{f(xy)} \\ &\leq 1+\frac{f(x)}{f(xy)} \leq 1+(1/2)W_f(x,x),\end{aligned}$$

from which the results follow. □

3. Application to $\mu_a(r)$

In this section, we give an application to the generalized Grötzsch ring function of quasiconformal theory.

Theorem. *For* $x \in (1,\infty)$ *and* $a \in (0,1/2]$. *Then* $1/2 \leq \lambda_{\mu_a(1/x)} \leq 1$, $\mu_{\mu_a(1/x)} = 2$. *In particular, for* $s,t \in (0,1)$ *and* $a \in (0,1/2]$

$$\frac{1}{2}\mu_a(st) \leq \mu_a(s)+\mu_a(t) \leq 2\mu_a(st) \tag{3.1}$$

the constant 2 in the second inequality in (3.1) *can not be improved.*

Proof. Let $f(x) = \frac{\mu_a(1/x)}{\mu_a(1/x^2)}$, for $x \in (1,\infty)$. Then $W_{\mu_a \circ g}(x,x) = 2f(x)$, here $g(x) = \frac{1}{x}$. By Lemma 3, we can easily obtain

$$\lambda'_{\mu_a \circ g} = 1, \quad \mu'_{\mu_a \circ g} = 2. \tag{3.2}$$

Hence we get $1/2 \leq \lambda_{\mu_a(1/x)} \leq 1$, $\mu_{\mu_a(1/x)} = 2$ by Lemma 4(1) and Lemma 5. The results of the rest are clear. □

Remark. It is easy to know that $\mu_a(1) = 0$ and $\mu_a(0) = \infty$, while $\mu_{\mu_a(1/x)} = 2$ for $x \in (1,\infty)$, so that the converse of Lemma 4 (4) is not true.

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