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SOME IDENTITIES INVOLVING THE LEGENDRE'S CHI-FUNCTION

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ABSTRACT. Since the time of Euler, the dilogarithm and polylogarithm functions have been studied by many mathematicians who used various notations for the dilogarithm function $\mathrm{Li}_2(z)$. These functions are related to many other mathematical functions and have a variety of application. The main objective of this paper is to present corrected versions of two equivalent factorization formulas involving the Legendre's Chi-function χ_2 and an evaluation of a class of integrals which is useful to evaluate some integrals associated with the dilogarithm function.

1. Introduction and preliminaries

The dilogarithm function $\mathrm{Li}_2(z)$ is defined by

$$\begin{aligned} \mathrm{Li}_2(z) :&= \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| \leq 1) \\ &= -\int_0^z \frac{\log(1-t)}{t}\, dt. \end{aligned} \tag{1.1}$$

As noted in Lewin [6], the series in (1.1) was discussed by Euler in 1768, the function $\mathrm{Li}_2(z)$ was later called the dilogarithm by Hill in 1828.

Polylogarithm functions $\mathrm{Li}_n(z)$ ($n \in \mathbb{N} := \{1, 2, 3, \ldots\}$) are defined by

$$\begin{aligned} \mathrm{Li}_n(z) :&= \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (|z| \leq 1;\ n \in \mathbb{N} \setminus \{1\}) \\ &= \int_0^z \mathrm{Li}_{n-1}(t)\, \frac{dt}{t} \quad (n \in \mathbb{N} \setminus \{1, 2\}). \end{aligned} \tag{1.2}$$

Since the time of Euler, the dilogarithm and polylogarithm functions have been studied by many mathematicians who used various notations for $\mathrm{Li}_2(z)$ (see, for example, [7]). Lewin [6] has introduced the present convenient notation $\mathrm{Li}_n(z)$ ($n \in \mathbb{N}$) for polylogarithm functions. Lewin [6] also has studied them

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extensively and remarked their relationships to other mathematical functions and their applications to other research areas. Recently Bowman and David [2] surveyed various results and conjectures concerning the multiple polylogarithms of a generalized form of (1.2) and the multiple zeta function. Some integrals related to polylogarithm functions have also been studied (see, for example, [1], [4]).

Legendre [5] studied the function $\chi_2(x)$ defined by

$$\chi_2(x) := \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)^2} = \tfrac{1}{2}\,\mathrm{Li}_2(x) - \tfrac{1}{2}\,\mathrm{Li}_2(-x) \quad (-1 \le x \le 1), \tag{1.3}$$

which is called the Legendre's chi-function. Lewin [6] noted that Legendre [5] used the notation $\phi(x)$ for the $\chi_2(x)$. We recall here two known identities (see [6, p. 19]):

$$\chi_2(x) = \tfrac{1}{2} \int_0^x \log\left(\frac{1+t}{1-t}\right) \frac{dt}{t} \tag{1.4}$$

and

$$\chi_2\left(\frac{1-x}{1+x}\right) + \chi_2(x) = \frac{\pi^2}{8} + \tfrac{1}{2}\log x \log\frac{1+x}{1-x}. \tag{1.5}$$

Lewin also recorded two equivalent factorization formulas [6, p. 20, Equations (1.72) and (1.73)] for the Legendre's chi-function χ_2 as follows:

$$\begin{aligned} \chi_2\left(x^{2n+1}\right) &= \frac{2n+1}{2} \int_0^x \log\left[\frac{1+t}{1-t} \prod_{r=1}^{n} \frac{1-t\,e^{-\frac{ir\pi}{2n+1}}}{1+t\,e^{-\frac{ir\pi}{2n+1}}} \cdot \frac{1-t\,e^{\frac{ir\pi}{2n+1}}}{1+t\,e^{\frac{ir\pi}{2n+1}}}\right] \frac{dt}{t} \\ &= (2n+1)\left[\chi_2(x) - \sum_{r=1}^{n} \chi_2\left(x\,e^{-\frac{ir\pi}{2n+1}}\right) - \sum_{r=1}^{n} \chi_2\left(x\,e^{\frac{ir\pi}{2n+1}}\right)\right] \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &\chi_2\left[i\tan\{(2n+1)\theta\}\right] \\ &= (2n+1)\Bigg[\chi_2(i\tan\theta) + \sum_{r=1}^{n} \chi_2\left\{i\tan\left(\frac{\frac{1}{2}r\pi}{2n+1} - \theta\right)\right\} \\ &\quad - \sum_{r=1}^{n} \chi_2\left\{i\tan\left(\frac{\frac{1}{2}r\pi}{2n+1} + \theta\right)\right\} - \frac{\pi^2}{8} - i\theta\log(i\tan\theta) + \frac{n\pi^2}{4} \\ &\quad - \sum_{r=1}^{n} i\left(\frac{\frac{1}{2}r\pi}{2n+1} - \theta\right) \log\left\{-i\tan\left(\frac{\frac{1}{2}r\pi}{2n+1} - \theta\right)\right\} \\ &\quad + \sum_{r=1}^{n} i\left(\frac{\frac{1}{2}r\pi}{2n+1} + \theta\right) \log\left\{i\tan\left(\frac{\frac{1}{2}r\pi}{2n+1} + \theta\right)\right\}\Bigg] \\ &\quad + \frac{\pi^2}{8} + i(2n+1)\,\theta\log\left[i\tan\{(2n+1)\theta\}\right]. \end{aligned} \tag{1.7}$$

Here we are aiming at presenting the corrected versions of the formulas (1.6) and (1.7). We also give an evaluation of a class of integrals which is useful to get some integral equations satisfied by the dilogarithm.

2. Corrected versions of (1.6) and (1.7)

We first recall four known factorization formulas (see [8, p. 4]): For $n \in \mathbb{N}$,

$$x^{2n+1} - y^{2n+1} = (x-y) \prod_{r=1}^{n} \left(x^2 - 2xy \cos \frac{2r\pi}{2n+1} + y^2 \right); \tag{2.1}$$

$$x^{2n+1} + y^{2n+1} = (x+y) \prod_{r=1}^{n} \left(x^2 + 2xy \cos \frac{2r\pi}{2n+1} + y^2 \right); \tag{2.2}$$

$$x^{2n} - y^{2n} = (x-y)(x+y) \prod_{r=1}^{n-1} \left(x^2 - 2xy \cos \frac{r\pi}{n} + y^2 \right); \tag{2.3}$$

$$x^{2n} + y^{2n} = \prod_{r=1}^{n} \left(x^2 + 2xy \cos \frac{(2r-1)\pi}{2n} + y^2 \right). \tag{2.4}$$

To make our reasoning assure, we need to verify the formula (2.1). Indeed, let $x^{2n+1} - y^{2n+1} = 0$. Then $(x/y)^{2n+1} = 1$ and so we have

$$x = y\, e^{i\frac{2\pi k}{2n+1}} \quad (k = 0, 1, 2, \ldots, 2n).$$

We thus see that

$$\begin{aligned} x^{2n+1} - y^{2n+1} &= (x-y) \prod_{k=1}^{2n} \left(x - y\, e^{i\frac{2\pi k}{2n+1}} \right) \\ &= (x-y) \prod_{k=1}^{n} \left(x - y\, e^{i\frac{2\pi k}{2n+1}} \right) \prod_{k=n+1}^{2n} \left(x - y\, e^{i\frac{2\pi k}{2n+1}} \right). \end{aligned} \tag{2.5}$$

Now we transform the second product of (2.5) as follows:

$$\prod_{k=n+1}^{2n} \left(x - y\, e^{\frac{2\pi i k}{2n+1}} \right) = \prod_{r=1}^{n} \left(x - y\, e^{-\frac{2\pi i (n-r+1)}{2n+1}} \right) = \prod_{k=1}^{n} \left(x - y\, e^{\frac{-2\pi i k}{2n+1}} \right), \tag{2.6}$$

by letting $r := k - n$ and $k := n - r + 1$ for the first and second equalities, respectively.

Substituting (2.6) for the second product in (2.5), we obtain

$$\begin{aligned} x^{2n+1}-y^{2n+1} &= (x-y)\prod_{k=1}^{n}\left(x-y\,e^{\frac{2\pi ik}{2n+1}}\right)\prod_{k=1}^{n}\left(x-y\,e^{\frac{-2\pi ik}{2n+1}}\right) \\ &= (x-y)\prod_{k=1}^{n}\left[x^2-xy\left(e^{\frac{2\pi ik}{2n+1}}+e^{\frac{-2\pi ik}{2n+1}}\right)+y^2\right] \\ &= (x-y)\prod_{k=1}^{n}\left[x^2-2xy\cos\frac{2k\pi}{2n+1}+y^2\right], \end{aligned}$$

which completes the proof of (2.1). Similarly we can prove the formulas (2.2), (2.3), and (2.4).

Setting $x = 1$ and $y = t$ in (2.1) and (2.2), and applying the resulting products and using (1.4), we get

(2.7)

$$\begin{aligned} \chi_2\left(x^{2n+1}\right) &= \frac{2n+1}{2}\int_0^x \log\left(\frac{1+t^{2n+1}}{1-t^{2n+1}}\right)\frac{dt}{t} \\ &= \frac{2n+1}{2}\int_0^x \log\left[\frac{1+t}{1-t}\prod_{r=1}^{n}\frac{1+t\,e^{\frac{2\pi ir}{2n+1}}}{1-t\,e^{\frac{2\pi ir}{2n+1}}}\cdot\frac{1+t\,e^{-\frac{2\pi ir}{2n+1}}}{1-t\,e^{-\frac{2\pi ir}{2n+1}}}\right]\frac{dt}{t} \\ &= (2n+1)\left[\chi_2(x)+\sum_{r=1}^{n}\chi_2\left(x\,e^{\frac{2\pi ir}{2n+1}}\right)+\sum_{r=1}^{n}\chi_2\left(x\,e^{-\frac{2\pi ir}{2n+1}}\right)\right]. \end{aligned}$$

We finally have a corrected version of (1.6):

(2.8)

$$\chi_2\left(x^{2n+1}\right) = (2n+1)\left[\chi_2(x)+\sum_{r=1}^{n}\chi_2\left(x\,e^{\frac{2\pi ir}{2n+1}}\right)+\sum_{r=1}^{n}\chi_2\left(x\,e^{-\frac{2\pi ir}{2n+1}}\right)\right].$$

Since the product part of the integrand in (2.7) can easily be rewritten in the form:

$$\frac{1+t^{2n+1}}{1-t^{2n+1}} = \frac{1+t}{1-t}\prod_{r=1}^{n}\frac{1-t\,e^{-\frac{(2r-1)\pi i}{2n+1}}}{1+t\,e^{-\frac{(2r-1)\pi i}{2n+1}}}\cdot\frac{1-t\,e^{\frac{(2r-1)\pi i}{2n+1}}}{1+t\,e^{\frac{(2r-1)\pi i}{2n+1}}},$$

(2.8) can be equivalently expressed as follows:

(2.9)

$$\chi_2\left(x^{2n+1}\right) = (2n+1)\left[\chi_2(x)-\sum_{r=1}^{n}\chi_2\left(x\,e^{-\frac{(2r-1)\pi i}{2n+1}}\right)-\sum_{r=1}^{n}\chi_2\left(x\,e^{\frac{(2r-1)\pi i}{2n+1}}\right)\right].$$

Now, setting $x = i\,\tan\theta$ in (1.5), we obtain (see [6, p. 20])

$$\chi_2\left(e^{-2i\theta}\right) = -\chi_2\left(i\,\tan\theta\right)+\frac{\pi^2}{8}+i\,\theta\,\log\left(i\,\tan\theta\right). \tag{2.10}$$

Taking $x = e^{-2i\theta}$ in (2.8) and using (2.10), we get a corrected version of (1.7):

$$\begin{aligned}
&\chi_2\left[i\tan\{(2n+1)\theta\}\right] \\
&= (2n+1)\left[\chi_2(i\tan\theta) + \sum_{r=1}^{n}\chi_2\left\{i\tan\left(\theta - \frac{\pi r}{2n+1}\right)\right\}\right. \\
&\quad + \sum_{r=1}^{n}\chi_2\left\{i\tan\left(\theta + \frac{\pi r}{2n+1}\right)\right\} - \frac{\pi^2}{8} - i\theta\log(i\tan\theta) - \frac{n\pi^2}{4} \\
&\quad - \sum_{r=1}^{n} i\left(\theta - \frac{\pi r}{2n+1}\right)\log\left\{i\tan\left(\theta - \frac{\pi r}{2n+1}\right)\right\} \\
&\quad \left. - \sum_{r=1}^{n} i\left(\theta + \frac{\pi r}{2n+1}\right)\log\left\{i\tan\left(\theta + \frac{\pi r}{2n+1}\right)\right\}\right] \\
&\quad + \frac{\pi^2}{8} + i(2n+1)\,\theta\,\log\left[i\tan\{(2n+1)\theta\}\right].
\end{aligned} \tag{2.11}$$

Remark. From (1.6) and (2.7), we see respectively that

$$\frac{1+t^{2n+1}}{1-t^{2n+1}} = \frac{1+t}{1-t}\,p_n(t) \tag{2.12}$$

and

$$\frac{1+t^{2n+1}}{1-t^{2n+1}} = \frac{1+t}{1-t}\,q_n(t), \tag{2.13}$$

where

$$p_n(t) = \prod_{r=1}^{n}\frac{1-2t\,\cos\left(\frac{\pi r}{2n+1}\right)+t^2}{1+2t\,\cos\left(\frac{\pi r}{2n+1}\right)+t^2}$$

and

$$q_n(t) = \prod_{r=1}^{n}\frac{1+2t\,\cos\left(\frac{2\pi r}{2n+1}\right)+t^2}{1-2t\,\cos\left(\frac{2\pi r}{2n+1}\right)+t^2}.$$

It follows from (2.12) and (2.13) that $p_n(t) = q_n(t)$ for every $n \in \mathbb{N}$. However, it is easy to check that $p_1(t) = q_1(t)$ but $p_2(t) \neq q_2(t)$. So we conclude that either (2.12) or (2.13) is not correct. Since (2.13) has been verified in an assured way, the formulas (1.6) and (1.7) should be corrected as (2.8) (or (2.9)) and (2.11), respectively.

3. Evaluation of a class of integrals

Here, we evaluate a class of integrals of the form:

$$I_n(x) := \int_0^\infty e^{-px}\,(\log p)^n\,dp \quad (x > 0;\ n \in \mathbb{N}).$$

We observe that

$$\begin{aligned}
I_n(x) &= \frac{d^n}{d\alpha^n} \int_0^\infty e^{-px}\, p^\alpha\, dp\bigg|_{\alpha=0} \\
&= \frac{d^n}{d\alpha^n} \left[\Gamma(1+\alpha)\, x^{-(1+\alpha)}\right]\bigg|_{\alpha=0} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma^{(n-k)}(1+\alpha)\, (\log x)^k\, x^{-(1+\alpha)}\bigg|_{\alpha=0} \\
&= \frac{1}{x} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma^{(n-k)}(1)\, (\log x)^k,
\end{aligned}$$

where Γ denotes the well-known Gamma function (see [9, Chapter 1]).

We thus have

(3.1)

$$\int_0^\infty e^{-px}\, (\log p)^n\, dp = \frac{1}{x} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma^{(n-k)}(1)\, (\log x)^k \quad (x > 0;\ n \in \mathbb{N}).$$

To list some special cases of (3.1) for small values of $n \in \mathbb{N}$, we first recall the following known formula of evaluation of higher-order derivatives of the Gamma function Γ (see [3, p. 12]):

(3.2)

$$\Gamma^{(n+1)}(1) = -\gamma\, \Gamma^{(n)}(1) + n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(n-k)!}\, \zeta(k+1)\, \Gamma^{(n-k)}(1) \quad (n \in \mathbb{N} \cup \{0\}),$$

some special cases of which are given:

$$\Gamma'(1) = -\gamma; \quad \Gamma^{(2)}(1) = \gamma^2 + \frac{\pi^2}{6}; \quad \Gamma^{(3)}(1) = -\gamma^3 - \frac{\gamma\, \pi^2}{2} - 2\,\zeta(3);$$

$$\Gamma^{(4)}(1) = \gamma^4 + \gamma^2\, \pi^2 + 8\,\gamma\,\zeta(3) + \frac{3\,\pi^4}{20},$$

where γ denotes the Euler-Mascheroni's constant and $\zeta(s)$ is the Riemann zeta function (see [9, Chapters 1 and 2]).

By means of (3.2), we give some explicit special cases of (3.1):

$$\int_0^\infty e^{-px}\, \log p\, dp = -\frac{1}{x}\, (\gamma + \log x);$$

$$\int_0^\infty e^{-px}\, (\log p)^2\, dp = \frac{1}{x} \left\{\frac{\pi^2}{6} + (\gamma + \log x)^2\right\};$$

$$\int_0^\infty e^{-px}\, (\log p)^3\, dp = -\frac{1}{x} \left\{2\,\zeta(3) + \frac{\pi^2}{2}\, (\gamma + \log x) + (\gamma + \log x)^3\right\};$$

$$\int_0^\infty e^{-px}\, (\log p)^4\, dp = \frac{1}{x}\, \left\{2\left(\gamma\,\pi^2 + 4\,\zeta(3)\right)\, \log x + \pi^2\, (\log x)^2 + (\gamma + \log x)^4\right\}.$$

We conclude this section by giving a known integral formula associated the dilogarithm function evaluated by the aid of some special cases of (3.1) (see [6, p. 26, Eq. (1.108)]:

$$\begin{aligned} &\int_0^{\infty} e^{-px}\left[\mathrm{Li}_2\left(1-\frac{1}{p}\right)+\frac{\pi^2}{6}+\frac{1}{2}(\log p)^2\right]dp \\ &= \frac{e^{-x}}{x}\left[\int_0^x \frac{e^t-1}{t}(\gamma+\log t)\,dt+\frac{1}{2}(\gamma+\log x)^2+\frac{5\,\pi^2}{12}\right]. \end{aligned} \tag{3.3}$$

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