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# MAXIMAL COLUMN RANK PRESERVERS OF INTEGER MATRICES

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Abstract. The maximal column rank of an $m \times n$ matrix $A$ over the ring of integers, is the maximal number of the columns of $A$ that are weakly independent. We characterize the linear operators that preserve the maximal column ranks of integer matrices.

## 1. Introduction

There are many research papers on the rank of matrices and their preserver over semirings([1], [3]). Also the integer matrices are important topics of many researchers. Beasley and Pullman ([2]) compared the rank and column rank of a matrix over various semirings. And Song et al. ([5], [6], [7]) defined the maximal column rank of a matrix over semirings and compared the column rank and maximal column rank. Moreover they characterized the linear operators that preserve the maximal column rank over semirings. Motivated from these researches, we study the difference between column rank and maximal column rank of matrices over $\mathbb{Z}$, the ring of integers. In this paper, we show how much different they are, and characterize the linear operators that preserve the maximal column rank of matrices over ring of integers.

## 2. Preliminaries and definitions

Let $\mathbb{Z}$ be the ring of integers, and denote $\mathbb{M}_{m\times n}(\mathbb{Z})$ as the set of all $m \times n$ matrices over $\mathbb{Z}$. Algebraic operations on matrices over $\mathbb{Z}$ and such notions as *linearity* and *invertibility* are also defined as if $\mathbb{Z}$ were a field.

If $\mathbb{V}$ is a nonempty subset of $\mathbb{Z}^n \equiv \mathbb{M}_{n\times 1}(\mathbb{Z})$ which is closed under addition and multiplication by scalars in $\mathbb{Z}$, then $\mathbb{V}$ is called a *vector space* over $\mathbb{Z}$. A subset $S$ of a vector space $\mathbb{V}$ is a *spanning set* if each vector in $\mathbb{V}$ can be written as a linear combination of elements of $S$.

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Let $E_{i,j}$ denote the $m \times n$ matrix whose $(i,j)^{\text{th}}$ entry is 1 and whose other entries are 0. We call $E_{i,j}$ a *cell.* The set of all cells spans $\mathbb{M}_{m\times n}(\mathbb{Z})$.

The *column space* of a matrix $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is the vector space that is spanned by its columns. Since the column space of $A$ is spanned by a finite set of vectors, it contains a spanning set of minimum cardinality; that cardinality is the *column rank*, $c(A)$, of $A$. The column rank of a zero matrix is zero.

Lowercase, boldface letters will represent vectors, and all vector $\boldsymbol{u}$ are column vectors ($\boldsymbol{u}^t$ is a row vector). A nonzero vector $\boldsymbol{p} = [p_1\ p_2\ \cdots\ p_n]^t$ in $\mathbb{Z}^n$ is *irreducible* if the greatest common divisor of $p_i$'s is 1 (that is, $\gcd(p_1,\ldots,p_n) = 1$).

For nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{Z}^n$, we say that $\boldsymbol{a}$ is *similar* to $\boldsymbol{b}$, which is denoted $\boldsymbol{a} \simeq \boldsymbol{b}$, if $\boldsymbol{a}$ and $\boldsymbol{b}$ have a common irreducible factor. That is, $\boldsymbol{a} \simeq \boldsymbol{b}$ if and only if there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{a} = \alpha\boldsymbol{p}$ and $\boldsymbol{b} = \beta\boldsymbol{p}$ for some nonzero integers $\alpha$ and $\beta$. Then it is trivial that this relation "$\simeq$" is an equivalent relation in $\mathbb{Z}^n$.

**Lemma 2.1.** *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero vectors in $\mathbb{Z}^n$ with $\alpha\boldsymbol{a} = \beta\boldsymbol{b}$ for some nonzero integers $\alpha$ and $\beta$, then we have $\boldsymbol{a} \simeq \boldsymbol{b}$.*

*Proof.* Let $\boldsymbol{a} = [a_1\ \cdots\ a_n]^t$, $\boldsymbol{b} = [b_1\ \cdots\ b_n]^t$ and $\alpha' = \gcd(a_1,\ldots,a_n)$. Then there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{a} = \alpha'\boldsymbol{p}$. Thus $\alpha\boldsymbol{a} = \beta\boldsymbol{b}$ becomes

$$\alpha\alpha'\boldsymbol{p} = \beta\boldsymbol{b}. \tag{2.1}$$

Let $\gamma = \gcd(\alpha\alpha',\beta)$, $\gamma_1 = \frac{\alpha\alpha'}{\gamma}$ and $\gamma_2 = \frac{\beta}{\gamma}$. Then $\gamma_1$ and $\gamma_2$ are nonzero integers, and (2.1) becomes

$$\gamma_1\boldsymbol{p} = \gamma_2\boldsymbol{b}. \tag{2.2}$$

Therefore $\gamma_1$ divides every $\gamma_2 b_i$ for all $i = 1,\ldots,n$. Since $\gamma_1$ is relatively prime to $\gamma_2$, it follows that $\gamma_1$ divides every entry in $\boldsymbol{b}$. Thus we have $\boldsymbol{b} = \gamma_1\boldsymbol{c}$ for some nonzero vector $\boldsymbol{c}$ in $\mathbb{Z}^n$. By the cancellation, (2.2) becomes $\boldsymbol{p} = \gamma_2\boldsymbol{c}$. Then $\gamma_2$ is a unit in $\mathbb{Z}$ because $\gamma_2$ divides every entry in the irreducible vector $\boldsymbol{p}$. That is, $\gamma_2 = \pm 1$ so that $\boldsymbol{b} = \pm\gamma_1\boldsymbol{p}$. Therefore $\boldsymbol{a}$ and $\boldsymbol{b}$ have a common irreducible factor $\boldsymbol{p}$, and thus $\boldsymbol{a} \simeq \boldsymbol{b}$. □

**Proposition 2.2.** *If $\boldsymbol{a} = [a_1\ \cdots\ a_n]^t$ and $\boldsymbol{b} = [b_1\ \cdots\ b_n]^t$ are nonzero vectors in $\mathbb{Z}^n$ such that $a_ib_j = a_jb_i$ for all $i,j \in \{1,\ldots,n\}$, then $\boldsymbol{a} \simeq \boldsymbol{b}$.*

*Proof.* If $n = 1$ or 2, the result is obvious. So, we may assume that $n \geq 3$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero vectors, it follows that $a_ib_j \neq 0$ for some $i,j \in \{1,\ldots,n\}$. If $b_i = 0$, then $i \neq j$. Since $a_ib_j = a_jb_i = 0$, it follows that $b_j = 0$, a contradiction. Hence $b_i \neq 0$ and there exist nonzero integers $c$ and $d$ such that

$$ca_i = db_i. \tag{2.3}$$

It follows from $a_ib_i \neq 0$ and $a_ib_k = a_kb_i$ that $a_k = 0$ if and only if $b_k = 0$ for all $k \in \{1,\ldots,n\}$. Now, we will show that $ca_l = db_l$ for all $l \in \{1,\ldots,n\}$. From

$a_i b_l = a_l b_i$, we have $cda_i b_l = cda_l b_i$. By (2.3), $ca_l = db_l$ for all $l \in \{1, \ldots, n\}$, and hence $c\boldsymbol{a} = d\boldsymbol{b}$. It follows from Lemma 2.1 that $\boldsymbol{a} \simeq \boldsymbol{b}$. □

A subset $S$ of $\mathbb{Z}^n$ is called *weakly dependent* if for some $\boldsymbol{x}$ in $S$, $\boldsymbol{x}$ is a linear combination of elements in $S \setminus \{\boldsymbol{x}\}$; $S$ is called *weakly independent* if it is not weakly dependent.

The *maximal column rank*, $mc(A)$, of $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is the maximal number of the columns of $A$ which are weakly independent over $\mathbb{Z}$. The maximal column rank of a zero matrix is zero.

It follows that

$$0 \leq c(A) \leq mc(A) \leq n \tag{2.4}$$

for all matrices $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.

Over a field $\mathbb{F}$ we have $c(A) = mc(A)$ for all $A$ in $\mathbb{M}_{m\times n}(\mathbb{F})$. For, if $c(A) = k$, then the column space of $A$ has dimension $k$. So any $r$ columns of $A$ are weakly dependent for $r$ with $r > k$. Hence $mc(A) \leq k = c(A)$. Therefore the column rank and maximal column rank of any matrix $A$ in $\mathbb{M}_{m\times n}(\mathbb{F})$ are equal by (2.4). But the following Lemma shows that they may differ over the ring of integers and hence the inequality in (2.4) may be strict over $\mathbb{Z}$.

**Lemma 2.3.** *Let $n$ be an integer with $n \geq 2$. Then there exists a $1 \times n$ matrix $A$ such that $c(A) = 1$ and $mc(A) = n$.*

*Proof.* Let $p_1, p_2, \ldots, p_n$ be distinct positive prime integers, and let

$$N = p_1 p_2 \cdots p_n \quad \text{and} \quad N_i = \frac{N}{p_i}$$

for all $i \in \{1, 2, \ldots, n\}$. Then we can easily show that $\{N_1, N_2, \ldots, N_n\}$ is weakly independent. Therefore the $1 \times n$ matrix $A \equiv [N_1\ N_2\ \cdots\ N_n]$ must have maximal column rank $n$. Since $\gcd(N_1, N_2, \ldots, N_n) = 1$, and "1" is a linear combination of $N_1, N_2, \ldots, N_n$ over $\mathbb{Z}$, $\{1\}$ spans the column space of $A$ over $\mathbb{Z}$, and hence $c(A) = 1$. □

For any $2 \times 2$ integer matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the *determinant* of $A$ is defined as $\det(A) = ad - bc$.

**Lemma 2.4.** *Let $A = [a_{ij}]$ be a nonzero matrix in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $\min(m, n) \geq 2$. If $mc(A) = 1$, then $\det(A') = 0$ for every $2 \times 2$ submatrix $A'$ of $A$.*

*Proof.* The proof is straightforward. □

The converse of Lemma 2.4 may not be true. For example, consider a matrix $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{Z})$. Then we can easily show that $\det(A) = 0$, while $mc(A) = 2$.

Hwang, Kim and Song [5] obtained characterizations of the linear operators that preserve maximal column ranks of matrices over Boolean algebra. They also compared the column rank and the maximal column rank for matrices over certain semirings. Now we compare the column rank and the maximal column rank for matrices over $\mathbb{Z}$.

**Theorem 2.5.** *Let $\alpha(\mathbb{Z}, m, n)$ be the largest $k$ such that for all $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, $c(A) = mc(A)$ if $c(A) \leq k$ and there is at least one matrix $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $c(A) = k$. Then for $m \geq 1$,*

$$\alpha(\mathbb{Z}, m, n) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n \geq 2. \end{cases}$$

*Proof.* For $n = 1$, clearly $\alpha(\mathbb{Z}, m, 1) = 1$. So we may assume that $n \geq 2$. Consider a matrix $A = [p \;\; q]$ in $\mathbb{M}_{1\times 2}(\mathbb{Z})$, where $p$ and $q$ are relatively prime positive integers. Then there exist nonzero integers $x$ and $y$ such that $xp+yq = 1$. Thus $\{1\}$ spans the column space of $A$, and hence $c(A) = 1$. Since $\{p, q\}$ is weakly independent, it follows that $mc(A) = 2$. Let $X = A \oplus O$ be the matrix in $\mathbb{M}_{m\times n}(\mathbb{Z})$, where $O$ is the $(m-1) \times (n-2)$ zero matrix. Then we have $c(X) = 1$, while $mc(X) = 2$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$. This means $\alpha(\mathbb{Z}, m, n) < 1$, equivalently $\alpha(\mathbb{Z}, m, n) = 0$. Thus the result follows. □

**Lemma 2.6.** *For $A \in \mathbb{M}_{m\times n}(\mathbb{Z})$, $mc(A) = 1$ if and only if $A$ can be factored as $A = \boldsymbol{x}\boldsymbol{a}^t$ for some nonzero vectors $\boldsymbol{x}$ in $\mathbb{Z}^m$ and $\boldsymbol{a}$ in $\mathbb{Z}^n$ with $mc(\boldsymbol{a}^t) = 1$.*

*Proof.* It is straightforward to see that $mc(A) = mc(\boldsymbol{x}\boldsymbol{a}^t) = 1$ for nonzero vectors $\boldsymbol{x}$ in $\mathbb{Z}^m$ and $\boldsymbol{a}$ in $\mathbb{Z}^n$ with $mc(\boldsymbol{a}^t) = 1$. Conversely, assume that $mc(A) = 1$. Then any two columns of $A$ are weakly dependent. Thus there exists one nonzero column $\boldsymbol{a}_k$ of $A$ such that all the other columns $\boldsymbol{a}_i$ are expressed as scalar multiples of $\boldsymbol{a}_k$; that is, $\boldsymbol{a}_i = \alpha_i\boldsymbol{a}_k$ for some integers $\alpha_i$ and for all $i \in \{1, \ldots, n\}$. Therefore $A = \boldsymbol{a}_k[\alpha_1 \; \cdots \; \alpha_n]$. If we let $\boldsymbol{x} = \boldsymbol{a}_k$ and $\boldsymbol{a} = [\alpha_1 \; \cdots \; \alpha_n]^t$, then $A = \boldsymbol{x}\boldsymbol{a}^t$ and the fact that $mc(\boldsymbol{a}^t) = 1$ follows from $mc(A) = 1$. □

For any matrix $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, let $A_i$ denote the $i^{\text{th}}$ row of $A$ for all $i \in \{1, \ldots, m\}$. Then Lemma 2.6 implies the following corollary.

**Corollary 2.7.** *If $A$ is a matrix in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $mc(A) = 1$, then the maximal column rank of $A_i$ has either 0 or 1 for all $i \in \{1, \ldots, m\}$.*

In general, the converse of Corollary 2.7 is not true. For example, consider a matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ in $\mathbb{M}_{2\times 2}(\mathbb{Z})$. Then we have $mc(A) = 2$ by Lemma 2.4, while $mc(A_1) = mc(A_2) = 1$.

## 3. Maximal column rank preservers

In this section we have characterizations of the linear operators that preserve the maximal column rank of matrices over the ring of integers.

Suppose that $T$ is an operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Say that $T$ is a

(i) *linear operator* if $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $\alpha, \beta$ in $\mathbb{Z}$ and for all $X, Y$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.
(ii) *maximal column rank preserver* if $mc(T(X)) = mc(X)$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.
(iii) *maximal column rank $r$ preserver* if $mc(T(X)) = r$ whenever $mc(X) = r$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.

An $n \times n$ integer matrix $A$ is called *nonsingular* if for any vector $\boldsymbol{x}$ in $\mathbb{Z}^n$, $A\boldsymbol{x} = \mathbf{0}$ implies that $\boldsymbol{x} = \mathbf{0}$. An $n \times n$ integer matrix $A$ is called *singular* if it is not nonsingular; that is, there exists a nonzero vector $\boldsymbol{x}$ in $\mathbb{Z}^n$ such that $A\boldsymbol{x} = \mathbf{0}$.

**Lemma 3.1.** *For a given matrix $Q$ in $\mathbb{M}_{m\times m}(\mathbb{Z})$, define a linear operator $T$ on $\mathbb{M}_{m\times n}(\mathbb{Z})$ by $T(X) = QX$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then $T$ is a maximal column rank preserver if and only if $Q$ is nonsingular.*

*Proof.* Assume that $Q$ is nonsingular. For any $X = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_n]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, we have $QX = [Q\boldsymbol{x}_1\ \cdots\ Q\boldsymbol{x}_n]$. Let $\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots, \boldsymbol{x}_{i_k}$ be any $k$ columns of $X$. Since $Q$ is nonsingular, we have that $\{\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots, \boldsymbol{x}_{i_k}\}$ is weakly independent if and only if $\{Q\boldsymbol{x}_{i_1}, Q\boldsymbol{x}_{i_2}, \ldots, Q\boldsymbol{x}_{i_k}\}$ is weakly independent. Hence $mc(X) = mc(QX)$.

Conversely, assume that $Q$ is singular in $\mathbb{M}_{m\times m}(\mathbb{Z})$. Then $Q\boldsymbol{x} = \mathbf{0}$ for some nonzero vector $\boldsymbol{x}$ in $\mathbb{Z}^m$. Consider the matrix $X = [\boldsymbol{x}\ \boldsymbol{x}\ \cdots\ \boldsymbol{x}]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then we have $mc(X) = 1$, but $mc(T(X)) = 0$ since $T(X) = QX = [Q\boldsymbol{x}\ Q\boldsymbol{x}\ \cdots\ Q\boldsymbol{x}] = O$. Therefore $T$ does not preserve maximal column rank 1. This contradiction implies that $Q$ must be nonsingular. □

**Lemma 3.2.** *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be nonzero matrices in $\mathbb{M}_{m\times 2}(\mathbb{Z})$ with $mc(A) = mc(B) = 1$. If $a_{i1}a_{i2} \neq 0$ and $a_{i1}b_{i2} \neq a_{i2}b_{i1}$ for some $i \in \{1, \ldots, m\}$, then there exists a nonzero integer $k$ such that $mc(kA + B) = 2$.*

*Proof.* Now, we will show that there exists a nonzero integer $k$ such that $mc(kA_i + B_i) = 2$. If this statement is true, then we have $mc(kA + B) = 2$ by Corollary 2.7.

If $b_{i1} = 0$, then $b_{i2} \neq 0$ by assumption. Let $p$ be a prime integer such that $\gcd(p, b_{i2}) = 1$ and $pa_{i2} + b_{i2} \neq \pm 1$. Then we can easily show that $mc(pA_i + B_i) = [pa_{i1}\ \ pa_{i2} + b_{i2}] = 2$. If $b_{i2} = 0$, then the similar argument holds. So, we assume that $b_{i1}b_{i2} \neq 0$. Since $mc(A_i) = 1$, without loss of generality, we can assume that $A_i = [a_{i1}\ \ m_1a_{i1}]$ for some nonzero integer $m_1$. Similarly, we can write $B_i$ as $B_i = [b_{i1}\ \ m_2b_{i1}]$ or $B_i = [m_2b_{i2}\ \ b_{i2}]$ for some nonzero integer $m_2$. Let $C_i = [1\ \ m_1]$. Then we will show that there exists a nonzero integer $g$ with $a_{i1} \mid g$ such that $mc(gC_i + B_i) = 2$. It follows that $g = a_{i1}k$ for some nonzero integer $k$ and hence $mc(kA_i + B_i) = 2$.

**Case 1**: Let $B_i = [b_{i1}\ \ m_2b_{i1}]$. It follows from $a_{i1}b_{i2} \neq a_{i2}b_{i1}$ that $m_1 \neq m_2$ and hence $mc(gC_i + B_i) \neq 0$ for all integers $g$. Then we have

$$(g-1)b_{i1}C_i + B_i = [gb_{i1}\ \ \ ((g-1)m_1 + m_2)b_{i1}] \tag{3.1}$$

for all integers $g$. Consider arithmetic progressions

$$1, \quad 1 \pm a_{i1}, \quad 1 \pm 2a_{i1}, \quad 1 \pm 3a_{i1}, \ \ldots . \tag{3.2}$$

Since $\gcd(1, a_{i1}) = 1$, it follows from Dirichlet's Theorem ([4], Theorem 3-7) that the progressions in (3.2) contain infinitely many primes. Thus, by this property, we can find a prime integer $p$ with $a_{i1} \mid (p-1)$ such that

$$|p| > 2 \max\{|m_1|, |m_2|\} \tag{3.3}$$

and

$$(p-1)m_1 + m_2 \neq 0, \pm 1. \tag{3.4}$$

Now, we will show that $p \not| \ ((p-1)m_1 + m_2)$ and $((p-1)m_1 + m_2) \not| \ p$. If this is true, then it follows from (3.1) and $a_{i1} \mid (p-1)$ that

$$mc((p-1)b_{i1}C_i + B_i) = mc(kA_i + B_i) = 2$$

with $g = p$ and $(p-1)b_{i1} = ka_{i1}$ for some nonzero integer $k$.

First, suppose that $p \mid ((p-1)m_1 + m_2)$. Then there exists a nonzero integer $\alpha$ such that $(p-1)m_1 + m_2 = p\alpha$, equivalently $m_1 - m_2 = p(m_1 - \alpha)$. It follows from (3.3) that $-|p| < m_1 - m_2 < |p|$. Thus we have $m_1 - \alpha = 0$, and hence $m_1 = m_2$, a contradiction. Thus, we have $p \not| \ ((p-1)m_1 + m_2)$.

Next, suppose that $((p-1)m_1 + m_2) \mid p$. It follows from (3.4) that $(p-1)m_1 + m_2 = \pm p$ since $p$ is prime. Then we have $m_1 - m_2 = p(m_1 \pm 1)$. It follows from (3.3) that $-|p| < m_1 - m_2 < |p|$ and hence $m_1 \pm 1 = 0$ so that $m_1 = m_2$, a contradiction. Thus, we have $((p-1)m_1 + m_2) \not| \ p$.

**Case 2**: Let $B_i = [m_2 b_{i2} \ \ b_{i2}]$. It follows from $a_{i1}b_{i2} \neq a_{i2}b_{i1}$ that $m_1 m_2 \neq 1$ and hence $mc(gC_i + B_i) \neq 0$ for all integers $g$. Then we have

$$(g - m_2)b_{i2}C_i + B_i = [gb_{i2} \ \ ((g - m_2)m_1 + 1)b_{i2}] \tag{3.5}$$

for all integers $g$. Let $\gcd(a_{i1}, m_2) = d$ so that $a_{i1} = dx$ and $m_2 = dy$ with $\gcd(x, y) = 1$. Consider arithmetic progressions

$$y, \quad y \pm x, \quad y \pm 2x, \quad y \pm 3x, \ \ldots . \tag{3.6}$$

Since $\gcd(x, y) = 1$, it follows from Dirichlet's Theorem ([4], Theorem 3-7) that the progressions in (3.6) contain infinitely many primes. Thus, by this property, we can find a prime integer $p$ with $x \mid (p - y)$, equivalently, $a_{i1} \mid (dp - m_2)$ such that

$$\sqrt{|p|} > \max\{|m_1|, |m_2|\} \quad \text{and} \quad (dp - m_2)m_1 + 1 \neq 0, \pm 1, \pm d.$$

By the similar argument of Case 1), we can easily show that $p \not| ((p - m_2)m_1 + 1)$ and $((p - m_2)m_1 + 1) \not| p$. It follows from (3.5) and $a_{i1} \mid (dp - m_2)$ that

$$mc((dp - m_2)b_{i2}C_i + B_i) = mc(kA_i + B_i) = 2$$

with $g = dp$ and $(dp - m_2)b_{i2} = ka_{i1}$ for some nonzero integer $k$. □

**Proposition 3.3.** *Let $A$ and $B$ be in $\mathbb{M}_{m\times n}(\mathbb{Z})$ with $mc(A) = mc(B) = 1$ and let $A = \boldsymbol{x}\boldsymbol{a}^t$ and $B = \boldsymbol{y}\boldsymbol{b}^t$ be any factorizations. If $mc(A+B) = 1$, then we have $\boldsymbol{x} \simeq \boldsymbol{y}$ or $\boldsymbol{a} \simeq \boldsymbol{b}$.*

*Proof.* If $m = 1$ or $n = 1$, then the result is obvious. Thus we lose no generality in assuming that $\min(m,n) \geq 2$. Put

$$\boldsymbol{x} = [x_1 \cdots x_m]^t, \quad \boldsymbol{y} = [y_1 \cdots y_m]^t, \quad \boldsymbol{a} = [a_1 \cdots a_n]^t \quad \text{and} \quad \boldsymbol{b} = [b_1 \cdots b_n]^t.$$

Suppose that $mc(A+B) = 1$, while $\boldsymbol{x} \not\simeq \boldsymbol{y}$ and $\boldsymbol{a} \not\simeq \boldsymbol{b}$. Then there exist two different indices $i$ and $h$ in $\{1,\ldots,m\}$, and two different indices $j$ and $k$ in $\{1,\ldots,n\}$ such that

$$\boldsymbol{x}' \equiv \begin{bmatrix} x_i \\ x_h \end{bmatrix} \not\simeq \begin{bmatrix} y_i \\ y_h \end{bmatrix} \equiv \boldsymbol{y}' \quad \text{and} \quad \boldsymbol{a}' \equiv \begin{bmatrix} a_j \\ a_k \end{bmatrix} \not\simeq \begin{bmatrix} b_j \\ b_k \end{bmatrix} \equiv \boldsymbol{b}', \tag{3.7}$$

where $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{a}'$ and $\boldsymbol{b}'$ are nonzero vectors. Consider a matrix

$$C = \begin{bmatrix} x_i a_j + y_i b_j & x_i a_k + y_i b_k \\ x_h a_j + y_h b_j & x_h a_k + y_h b_k \end{bmatrix}.$$

Then $C$ is a $2 \times 2$ submatrix of $A + B$. By Lemma 2.4, we have

$$\det(C) = (x_i a_j + y_i b_j)(x_h a_k + y_h b_k) - (x_i a_k + y_i b_k)(x_h a_j + y_h b_j) = 0,$$

equivalently $(x_i y_h - x_h y_i)(a_j b_k - a_k b_j) = 0$, and hence

$$x_i y_h - x_h y_i = 0 \quad \text{or} \quad a_j b_k - a_k b_j = 0.$$

By Proposition 2.2, we have that $\boldsymbol{x}' \simeq \boldsymbol{y}'$ or $\boldsymbol{a}' \simeq \boldsymbol{b}'$, a contradiction to (3.7). Therefore we obtain $\boldsymbol{x} \simeq \boldsymbol{y}$ or $\boldsymbol{a} \simeq \boldsymbol{b}$. □

In general, the converse of Proposition 3.3 is not true. For example, consider two matrices $A = \boldsymbol{x}[1 \;\; 1]$ and $B = \boldsymbol{x}[1 \;\; 2]$ in $\mathbb{M}_{m\times 2}(\mathbb{Z})$, where $\boldsymbol{x}$ is a nonzero vector in $\mathbb{Z}^m$. Then $mc(A) = mc(B) = 1$, while $mc(A+B) = 2$ since $A + B = \boldsymbol{x}[2 \;\; 3]$.

For any index $i$ in $\{1,\ldots,n\}$, we denote $\boldsymbol{e}_i$ as a vector in $\mathbb{Z}^n$ with "1" in the $i^{\text{th}}$ position and zero elsewhere. We say that $A$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$ is a *column matrix* if $A = \boldsymbol{x}\boldsymbol{e}_i^t$ for some nonzero vector $\boldsymbol{x}$ in $\mathbb{Z}^m$ and for some $i \in \{1,\ldots,n\}$.

**Lemma 3.4.** *Let $T$ be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. If $T$ preserves maximal column ranks 1 and 2, then $T$ maps each cell to a column matrix.*

*Proof.* For the contrary, suppose that $T$ maps a cell to a matrix which is not a column matrix. Say $T(E_{1,1})$ has more than one nonzero column. Let $S = \{1,2,\ldots,n\}$ and let

$$S_1 = \{j \; : \text{the } j^{\text{th}} \text{ column of } T(E_{1,i}) \text{ is zero for all } i = 1,\ldots,n\}.$$

Then for each $j \in S - S_1$, there is a $j_i$ such that the $j^{\text{th}}$ column of $T(E_{1,j_i})$ is not zero. Now $T(E_{1,1})$ has at least two nonzero columns, say columns $k_1$ and

$k_2$. Let $S_2 = S - S_1 - \{k_1, k_2\}$, and let

$$A = E_{1,1} + \sum_{j \in S_2} h^{n_j} E_{1,j_i},$$

where $h$ and $n_j$ are positive integers for all $j \in S_2$. Note that for all $k \in S - S_1$, we can find $h$ and $n^j$ such that the $k^{\text{th}}$ column of $T(A)$ is nonzero. Further, since $A$ is consisted of at most $n-1$ distinct summands, each of which is a column matrix, there is at least one zero column in $A$, say the $r^{\text{th}}$. Let $B = E_{1,r}$. Since $T(A)$ has zero columns only corresponding to indices in $S_1$ (where $T(B)$ also must have a zero column), we can restrict our attention to those columns in $T(A)$ that are nonzero; hence we lose no generality in assuming that $T(A)$ has no zero column. Since $T$ preserves maximal column rank 1 and $mc(A) = mc(B) = 1$, it follows from Lemma 2.6 that

$$T(A) = \boldsymbol{u}\boldsymbol{a}^t \quad \text{and} \quad T(B) = \boldsymbol{v}\boldsymbol{b}^t$$

for some nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{Z}^m$, and for some nonzero vectors $\boldsymbol{a} = [a_1 \ \cdots \ a_n]^t$ and $\boldsymbol{b} = [b_1 \ \cdots \ b_n]^t$ in $\mathbb{Z}^n$, where $mc(\boldsymbol{a}^t) = mc(\boldsymbol{b}^t) = 1$ and $a_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$. Since $A + B$, and hence $T(A+B)$ has maximal column rank 1, it follows from Proposition 3.3 that $\boldsymbol{u} \simeq \boldsymbol{v}$ or $\boldsymbol{a} \simeq \boldsymbol{b}$.

Assume that $\boldsymbol{a} \simeq \boldsymbol{b}$. Then there exists an irreducible vector $\boldsymbol{\gamma}$ in $\mathbb{Z}^n$ with $mc(\boldsymbol{\gamma}^t) = 1$ such that $\boldsymbol{a} = \alpha\boldsymbol{\gamma}$ and $\boldsymbol{b} = \beta\boldsymbol{\gamma}$ for some nonzero integers $\alpha$ and $\beta$. Therefore $T(pA + qB) = (\alpha p\boldsymbol{u} + \beta q\boldsymbol{v})\boldsymbol{\gamma}^t$ has maximal column rank 1 for arbitrary nonzero integers $p$ and $q$. This contradicts the fact that $T$ preserves maximal column rank 2 since $mc(pA + qB) = 2$ for distinct prime integers $p$ and $q$. Therefore $\boldsymbol{a} \not\simeq \boldsymbol{b}$ and hence $\boldsymbol{u} \simeq \boldsymbol{v}$, equivalently $\boldsymbol{u} = c\boldsymbol{w}$ and $\boldsymbol{v} = d\boldsymbol{w}$ for some irreducible vector $\boldsymbol{w}$ in $\mathbb{Z}^m$ and for some nonzero integers $c$ and $d$. Then

$$T(dA + B) = d\boldsymbol{w}[g_1 + b_1 \ \ g_2 + b_2 \ \ \cdots \ \ g_n + b_n]$$

must have maximal column rank 1, where $g_i = ca_i \neq 0$ for all $i \in \{1, 2, \ldots, n\}$. Since $\boldsymbol{b}$ is not a zero vector, $b_s \neq 0$ for some $s \in \{1, \ldots, n\}$. Suppose that $b_t = 0$ for some $t \in \{1, \ldots, n\} \setminus \{s\}$. Then we have $g_s g_t \neq 0$ and $g_s b_t \neq g_t b_s$. By Lemma 3.2, there exists a nonzero integer $k$ such that

$$mc(k[g_s \ \ g_t] + [b_s \ \ b_t]) = 2.$$

It follows that $T(kdA + B)$ has at least maximal column rank 2. This is a contradiction to the fact that $mc(kdA + B) = 1$. Hence $b_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Furthermore, Lemma 3.2 implies that $g_i b_j = g_j b_i$ for all $i, j \in \{1, 2, \ldots, n\}$. Also, Proposition 2.2 shows that $c\boldsymbol{a} \simeq \boldsymbol{b}$, equivalently $\boldsymbol{a} \simeq \boldsymbol{b}$, a contradiction. Hence $T$ maps each cell to a column matrix. $\square$

Let $P = [p_{ij}]$ be an $n \times n$ matrix over $\mathbb{Z}$. We say that $P$ is an *absolute permutation matrix* if $[\,|p_{ij}|\,]$ is a permutation matrix in $\mathbb{M}_{n \times n}(\mathbb{Z})$, where $|p_{ij}|$ is the absolute value of $p_{ij}$. Thus all nonzero entries of an absolute permutation matrix are either 1 or $-1$.

The following Lemma is an immediate consequence of the definition of an absolute permutation matrix.

**Lemma 3.5.** *Let $P$ be an absolute permutation matrix in $\mathbb{M}_{n\times n}(\mathbb{Z})$. Define a linear operator $T$ on $\mathbb{M}_{m\times n}(\mathbb{Z})$ by $T(X) = XP$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then $T$ is a maximal column rank preserver.*

**Theorem 3.6.** *Let $T$ be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then $T$ preserves maximal column ranks 1 and 2 if and only if there exist a nonsingular matrix $Q$ in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and an absolute permutation matrix $P$ in $\mathbb{M}_{n\times n}(\mathbb{Z})$ such that $T(X) = QXP$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.*

*Proof.* Lemmas 3.1 and 3.5 prove the necessity. To prove the sufficiency, we assume that $T$ preserves maximal column ranks 1 and 2. By Lemma 3.4, for any cell $E_{i,j}$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$, we can write

$$T(E_{i,j}) = \boldsymbol{u}_{ij}\boldsymbol{e}^t_{\pi(j)} \tag{3.8}$$

for some nonzero vectors $\boldsymbol{u}_{ij}$ in $\mathbb{Z}^m$, where $\pi$ maps $\{1,\ldots,n\}$ into itself. Suppose that $\pi$ is not a permutation. Then there exist two distinct indices $j_1, j_2 \in \{1,\ldots,n\}$ such that $xT(E_{i,j_1}) + yT(E_{i,j_2})$ has at most one nonzero column for all $x, y$ in $\mathbb{Z}$. That is, $mc(T(cE_{i,j_1} + dE_{i,j_2})) \leq 1$, a contradiction since $mc(cE_{i,j_1} + dE_{i,j_2}) = 2$ for distinct positive prime integers $c$ and $d$. Thus $\pi$ is a permutation. So we lose no generality in assuming that $\pi$ is the identity permutation. Then (3.8) becomes $T(E_{i,j}) = \boldsymbol{u}_{ij}\boldsymbol{e}^t_j$ for all $j \in \{1,\ldots,n\}$. Now, we will show that for any index $i$ in $\{1,\ldots,m\}$,

$$\boldsymbol{u}_{i1} \simeq \boldsymbol{u}_{i2} \simeq \cdots \simeq \boldsymbol{u}_{in}. \tag{3.9}$$

Let $j$ and $k$ be arbitrary distinct indices in $\{1,2,\ldots,n\}$. Since $mc(E_{i,j}+E_{i,k}) = 1$ and $T(E_{i,j} + E_{i,k}) = \boldsymbol{u}_{ij}\boldsymbol{e}^t_j + \boldsymbol{u}_{ik}\boldsymbol{e}^t_k$, it follows from Proposition 3.3 that $\boldsymbol{u}_{ij} \simeq \boldsymbol{u}_{ik}$ because $\boldsymbol{e}_j \not\simeq \boldsymbol{e}_k$. Thus (3.9) is established. Therefore we can restate (3.9) as follow: For any index $i$ in $\{1,\ldots,m\}$, there exist an irreducible vector $\boldsymbol{q}_i$ in $\mathbb{Z}^m$ and nonzero integers $b_{ij}$ such that $\boldsymbol{u}_{ij} = b_{ij}\boldsymbol{q}_i$. Therefore we have established that

$$T(E_{i,j}) = b_{ij}\boldsymbol{q}_i\boldsymbol{e}^t_j \tag{3.10}$$

for all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, where $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_m$ are irreducible vectors in $\mathbb{Z}^m$ and $b_{ij}$ are nonzero integers.

Now, we will show that for any index $i \in \{1,\ldots,m\}$, $|b_{i1}| = |b_{ij}|$ for all $j \in \{1,\ldots,n\}$. Suppose that there exists an index $j$ in $\{2,3,\ldots,n\}$ such that $|b_{i1}| > |b_{ij}|$ or $|b_{i1}| < |b_{ij}|$. If $|b_{i1}| > |b_{ij}|$, let $p$ be a prime integer with $\gcd(p, b_{i1}) = 1$. Then we can easily show that

$$T(E_{i,1} + pE_{i,j}) = \boldsymbol{q}_i(b_{i1}\boldsymbol{e}^t_1 + p\, b_{ij}\boldsymbol{e}^t_j) = \boldsymbol{q}_i[b_{i1}\ 0\ \cdots\ 0\ p\, b_{ij}\ 0\ \cdots\ 0]$$

must have maximal column rank 2, while $mc(E_{i,1} + pE_{i,j}) = 1$, a contradiction. If $|b_{i1}| < |b_{ij}|$, then the same argument holds. Thus, for any index

$i \in \{1, \ldots, m\}$, we have that $|b_{i1}| = |b_{ij}|$ for all $j \in \{1, \ldots, n\}$. Then we can rewrite (3.10) as follows:

$$T(E_{i,j}) = \boldsymbol{q}'_i \boldsymbol{p}^t_j,$$

where

$$\boldsymbol{q}'_i = b_{i1}\boldsymbol{q}_i \quad \text{and} \quad \boldsymbol{p}_j = \begin{cases} \boldsymbol{e}_j & \text{if } b_{ij} = b_{i1}; \\ -\boldsymbol{e}_j & \text{if } b_{ij} = -b_{i1} \end{cases}$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Let $Q = [\boldsymbol{q}'_1 \ \boldsymbol{q}'_2 \ \cdots \ \boldsymbol{q}'_m]$ and $P^t = [\boldsymbol{p}_1 \ \boldsymbol{p}_2 \ \cdots \ \boldsymbol{p}_n]$. Then $Q$ is a matrix in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and $P$ is an absolute permutation matrix in $\mathbb{M}_{n\times n}(\mathbb{Z})$. Thus for an arbitrary matrix $X = [x_{ij}]$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$,

$$\begin{aligned} T(X) &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\, T(E_{i,j}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\, \boldsymbol{q}'_i \boldsymbol{p}^t_j = QXP. \end{aligned}$$

Finally, we show that $Q$ is nonsingular. Suppose that $Q$ is singular. Then there is a nonzero vector $\boldsymbol{x}$ in $\mathbb{Z}^m$ such that $Q\boldsymbol{x} = \boldsymbol{0}$. Let $X = [\boldsymbol{x} \ \boldsymbol{x} \ \cdots \ \boldsymbol{x}]$. Then we have $T(X) = QXP = O$ so that $mc(T(X)) = 0$, while $mc(X) = 1$, a contradiction to the fact that $T$ preserves maximal column rank 1. Hence $Q$ is nonsingular. □

**Corollary 3.7.** *Let $T$ be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{Z})$. Then $T$ is a maximal column rank preserver if and only if there exist a nonsingular matrix $Q$ in $\mathbb{M}_{m\times m}(\mathbb{Z})$ and an absolute permutation matrix $P$ in $\mathbb{M}_{n\times n}(\mathbb{Z})$ such that $T(X) = QXP$ for all $X$ in $\mathbb{M}_{m\times n}(\mathbb{Z})$.*

Thus we have characterized the linear operators that preserve the maximal column rank of matrices over the ring of integers.

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