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FUZZY SUBALGEBRAS WITH THRESHOLDS IN BCK/BCI-ALGEBRAS

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ABSTRACT. Using the *belongs to* relation ($\in$) and *quasi-coincidence with* relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subalgebras where α, β are any two of $\{\in,\ \mathrm{q},\ \in\vee\,\mathrm{q},\ \in\wedge\,\mathrm{q}\}$ with $\alpha \neq \in\wedge\,\mathrm{q}$ was introduced, and related properties were investigated in [3]. As a continuation of the paper [3], in this paper, the notion of a fuzzy subalgebra with thresholds is introduced, and its characterizations are obtained. Relations between a fuzzy subalgebra with thresholds and an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra are provided.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [5], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [2]. In particular, $(\in, \in\vee\,\mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, the author [3] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. This paper is a continuation of the paper [3]. We introduce the notion of a fuzzy subalgebra with thresholds, and obtain its characterizations. We investigate relations between a fuzzy subalgebra with thresholds and an $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
(iii) $(\forall x \in X)\,(x * x = 0)$,

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(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. In what follows let X denote a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the book [4] for further information regarding BCK/BCI-algebras.

A fuzzy set $\mathscr{A}$ in a set X of the form

$$\mathscr{A}(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set $\mathscr{A}$ in a set X, Pu and Liu [5] gave meaning to the symbol $x_t \alpha \mathscr{A}$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$.

To say that $x_t \in \mathscr{A}$ (resp. $x_t \mathrm{q} \mathscr{A}$) means that $\mathscr{A}(x) \geq t$ (resp. $\mathscr{A}(x) + t > 1$.), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mathscr{A}$.

To say that $x_t \in\vee\,\mathrm{q} \mathscr{A}$ (resp. $x_t \in\wedge\,\mathrm{q} \mathscr{A}$) means that $x_t \in \mathscr{A}$ or $x_t \mathrm{q} \mathscr{A}$ (resp. $x_t \in \mathscr{A}$ and $x_t \mathrm{q} \mathscr{A}$). For all $t_1, t_2 \in [0,1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $m(t_1, t_2)$ and $M(t_1, t_2)$, respectively.

A fuzzy set $\mathscr{A}$ in X is called a *fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\,(\mathscr{A}(x * y) \geq m(\mathscr{A}(x), \mathscr{A}(y))). \tag{2.1}$$

Proposition 2.1. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy subalgebra of X if and only if $U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$ is a subalgebra of X for all $t \in (0,1]$, for our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.*

3. Fuzzy subalgebras with thresholds

In what follows let α, $\beta \in \{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$ unless otherwise specified. To say that $x_t \overline{\alpha} \mathscr{A}$ means that $x_t \alpha \mathscr{A}$ does not hold.

Definition 3.1. [3] A fuzzy set $\mathscr{A}$ in X is said to be an *(α, β)-fuzzy subalgebra* of X, where $\alpha \neq \in\wedge\,\mathrm{q}$, if it satisfies the following condition:

$$x_{t_1} \alpha \mathscr{A},\, y_{t_2} \alpha \mathscr{A} \Rightarrow (x * y)_{\min\{t_1, t_2\}} \beta \mathscr{A}. \tag{3.1}$$

for all $x, y \in X$ and $t_1, t_2 \in (0,1]$.

Let $\mathscr{A}$ be a fuzzy set in X such that $\mathscr{A}(x) \leq 0.5$ for all $x \in X$. Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in\wedge\,\mathrm{q} \mathscr{A}$. Then $\mathscr{A}(x) \geq t$ and $\mathscr{A}(x) + t > 1$. It follows that

$$1 < \mathscr{A}(x) + t \leq \mathscr{A}(x) + \mathscr{A}(x) = 2\mathscr{A}(x)$$

so that $\mathscr{A}(x) > 0.5$. This means that $\{x_t \mid x_t \in\wedge\,\mathrm{q} \mathscr{A}\} = \emptyset$. Therefore the case $\alpha = \in\wedge\,\mathrm{q}$ in Definition 3.1 will be omitted.

Lemma 3.2. [3] *A fuzzy set* $\mathscr{A}$ *in* X *is an* $(\in, \in\vee\mathrm{q})$*-fuzzy subalgebra of* X *if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mathscr{A}(x * y) \geq m(\mathscr{A}(x), \mathscr{A}(y), 0.5)). \tag{3.2}$$

Theorem 3.3. *Let* $\mathscr{A}$ *be a fuzzy set in* X*. Then* $U(\mathscr{A}; t)$ *is a subalgebra of* X *for all* $t \in (0.5, 1]$ *if and only if* $\mathscr{A}$ *satisfies the following assertions:*

$$(\forall x, y \in X)\,(M(\mathscr{A}(x * y), 0.5) \geq m(\mathscr{A}(x), \mathscr{A}(y))). \tag{3.3}$$

Proof. $(\Rightarrow)$ If there exist $x, y \in X$ such that

$$M(\mathscr{A}(x * y), 0.5) < t = m(\mathscr{A}(x), \mathscr{A}(y)),$$

then $t \in (0.5, 1]$, $\mathscr{A}(x*y) < t$ and $x, y \in U(\mathscr{A}; t)$. Since $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (0.5, 1]$, it follows that $x * y \in U(\mathscr{A}; t)$ so that $\mathscr{A}(x * y) \geq t$. This is a contradiction, and therefore (3.3) is valid.

$(\Leftarrow)$ Let $t \in (0.5, 1]$ and $x, y \in U(\mathscr{A}; t)$. Then

$$M(\mathscr{A}(x * y), 0.5) \geq m(\mathscr{A}(x), \mathscr{A}(y)) \geq t > 0.5$$

and so $\mathscr{A}(x * y) \geq t$. Hence $x * y \in U(\mathscr{A}; t)$, and $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (0.5, 1]$. □

From the above theorem, we know that a fuzzy set $\mathscr{A}$ in X may satisfy the condition that $U(\mathscr{A}; t)$ is a subalgebra of X for some $t \in (0, 1]$, but $U(\mathscr{A}; t)$ is not a subalgebra of X for some $t \in (0, 1]$. Let

$$X^* := \{t \in (0, 1] \mid U(\mathscr{A}; t) \text{ is a subalgebra of } X\}.$$

If $X^* = (0, 1]$, then $\mathscr{A}$ is a fuzzy subalgebra of X (see Proposition 2.1). If $X^* = (0, 0.5]$, then $\mathscr{A}$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X. Our question is whether $\mathscr{A}$ is a kind of fuzzy subalgebra of X or not when $X^* \neq \emptyset$ (for example, $X^* = (0.5, 1]$, $(\varepsilon, \delta]$ where $\varepsilon, \delta \in (0, 1]$ with $\varepsilon < \delta$)?

Based on this question, we introduce the concept of a fuzzy subalgebra with thresholds as follows.

Definition 3.4. A fuzzy set $\mathscr{A}$ in X is called a *fuzzy subalgebra with thresholds* ε *and* δ of X, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if it satisfies the following condition:

$$(\forall x, y \in X)\,(M(\mathscr{A}(x * y), \varepsilon) \geq m(\mathscr{A}(x), \mathscr{A}(y), \delta)). \tag{3.4}$$

Example 3.5. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table (see [1]):

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(i) Let $\mathscr{A}$ be a fuzzy set in X defined by $\mathscr{A}(0) = 0.6$, $\mathscr{A}(a) = 0.7$, and $\mathscr{A}(b) = \mathscr{A}(c) = 0.3$. Then $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.4$ and $\delta = 0.65$ of X. But

(a) $\mathscr{A}$ is not an $(\in, \in)$-fuzzy subalgebra of X since $a_{0.62} \in \mathscr{A}$ and $a_{0.66} \in \mathscr{A}$, but $(a * a)_{m(0.62,\, 0.66)} = 0_{0.62} \overline{\in} \mathscr{A}$.
(b) $\mathscr{A}$ is not a $(\mathrm{q}, \in \vee\, \mathrm{q})$-fuzzy subalgebra of X since $a_{0.41}\, \mathrm{q}\, \mathscr{A}$ and $b_{0.77}\, \mathrm{q}\, \mathscr{A}$, but $(a * b)_{m(0.41,\, 0.77)} = c_{0.41} \overline{\in \vee\, \mathrm{q}}\, \mathscr{A}$.
(c) $\mathscr{A}$ is not an $(\in \vee\, \mathrm{q}, \in \vee\, \mathrm{q})$-fuzzy subalgebra of X since $a_{0.5} \in \vee\, \mathrm{q}\, \mathscr{A}$ and $c_{0.8} \in \vee\, \mathrm{q}\, \mathscr{A}$, but $(a * c)_{m(0.5,\, 0.8)} = b_{0.5} \overline{\in \vee\, \mathrm{q}}\, \mathscr{A}$.

(ii) Let $\mathscr{B}$ be a fuzzy set in X defined by $\mathscr{B}(0) = 0.5$, $\mathscr{B}(a) = 0.7$, $\mathscr{B}(b) = 0.4$, and $\mathscr{B}(c) = 0.3$. Then $\mathscr{B}$ is not a fuzzy subalgebra with thresholds $\varepsilon = 0.6$ and $\delta = 0.8$ of X since

$$M(\mathscr{B}(a * a), 0.6) = 0.6 < 0.7 = m(\mathscr{B}(a), \mathscr{B}(a), 0.8).$$

Moreover $\mathscr{B}$ is not a fuzzy subalgebra of X because

$$\mathscr{B}(a * b) = \mathscr{B}(c) = 0.3 < 0.4 = m(\mathscr{B}(a), \mathscr{B}(b)).$$

But we know that $\mathscr{B}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.77$ and $\delta = 0.88$ of X.

In what follows let $\varepsilon, \delta \in [0, 1]$ and $\varepsilon < \delta$ unless otherwise specified.

Observation 3.6. (i) A fuzzy subalgebra (resp., $(\in, \in \vee\, \mathrm{q})$-fuzzy subalgebra) is a fuzzy subalgebra with some thresholds.

(ii) Every fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 1$ is a fuzzy subalgebra.

(iii) Every fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 0.5$ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy subalgebra.

(iv) Every fuzzy subalgebra $\mathscr{A}$ of X with thresholds $\varepsilon < \mathscr{A}(x) \leq \delta$ or $\varepsilon \leq \mathscr{A}(x) < \delta$ for all $x \in X$ is a fuzzy subalgebra of X.

(v) If $\mathscr{A}$ is a fuzzy set in X and $\varepsilon \geq \mathscr{A}(x)$ for all $x \in X$, then $\mathscr{A}$ is a fuzzy subalgebra with thresholds ε and δ of X.

(vi) Let $\mathscr{A}$ be a fuzzy set in X and let $\mathscr{A}(0) \leq \varepsilon < \mathscr{A}(x) \leq \delta$, $\mathscr{A}(0) \leq \varepsilon < \delta \leq \mathscr{A}(x)$, $\varepsilon \leq \mathscr{A}(0) < \delta \leq \mathscr{A}(x)$, or $\varepsilon \leq \mathscr{A}(0) < \mathscr{A}(x) \leq \delta$ for some $x (\neq 0) \in X$. Then $\mathscr{A}$ can't be a fuzzy subalgebra with thresholds ε and δ of X.

Theorem 3.7. *A fuzzy set $\mathscr{A}$ in X is a fuzzy subalgebra with thresholds ε and δ of X if and only if $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (\varepsilon, \delta]$.*

Proof. Assume that $\mathscr{A}$ is a fuzzy subalgebra with thresholds ε and δ of X. Let $t \in (\varepsilon, \delta]$ and $x, y \in U(\mathscr{A}; t)$. Then

$$M(\mathscr{A}(x * y), \varepsilon) \geq m(\mathscr{A}(x), \mathscr{A}(y), \delta) \geq t > \varepsilon,$$

and so $\mathscr{A}(x * y) \geq t$. Thus $x * y \in U(\mathscr{A}; t)$, which shows that $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (\varepsilon, \delta]$.

Conversely, suppose that $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (\varepsilon, \delta]$. If there exist $x, y \in X$ such that

$$M(\mathscr{A}(x * y), \varepsilon) < t = m(\mathscr{A}(x), \mathscr{A}(y), \delta),$$

then $x, y \in U(\mathscr{A}; t)$, $t \in (\varepsilon, \delta]$ and $\mathscr{A}(x * y) < t$. Since $U(\mathscr{A}; t)$ is a subalgebra of X for all $t \in (\varepsilon, \delta]$, we have $x * y \in U(\mathscr{A}; t)$ and so $\mathscr{A}(x * y) \geq t$ which is a contradiction. Therefore $\mathscr{A}$ satisfies the condition (3.4). This completes the proof. □

The following example shows that there is a fuzzy subalgebra with some thresholds which is neither a fuzzy subalgebra nor an $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebra.

Example 3.8. Let $\mathbb{Z}$ be the set of all integers. Then $(\mathbb{Z}; *, 0)$ is a BCI-algebra where the operation $*$ is the minus operation, i.e., $x * y = x - y$ for all $x, y \in \mathbb{Z}$. Let $\mathscr{A}$ be a fuzzy set in $\mathbb{Z}$ defined by

$$\mathscr{A}(x) := \begin{cases} 0 & \text{if } x \in \{2k+1 \mid k \in \mathbb{Z},\, k < 0\}, \\ 0.3 & \text{if } x \in \{2k-1 \mid k \in \mathbb{Z},\, k > 0\}, \\ 0.5 & \text{if } x \in \{2k \mid k \in \mathbb{Z}\} \setminus \{4k \mid k \in \mathbb{Z}\}, \\ 0.8 & \text{if } x \in \{4k \mid k \in \mathbb{Z}\} \setminus \{8k \mid k \in \mathbb{Z}\}, \\ 0.9 & \text{if } x \in \{8k \mid k \in \mathbb{Z},\, k < 0\}, \\ 1 & \text{if } x \in \{8k \mid k \in \mathbb{Z},\, k \geq 0\}. \end{cases}$$

Then $U(\mathscr{A}; 0) = \mathbb{Z}$. If $t \in (0, 0.3]$, then

$$U(\mathscr{A}; t) = \mathbb{Z} \setminus \{2k+1 \mid k \in \mathbb{Z},\, k < 0\}$$

which is not a subalgebra of $\mathbb{Z}$. If $t \in (0.3, 0.5]$, then $U(\mathscr{A}; t) = 2\mathbb{Z}$ which is a subalgebra of $\mathbb{Z}$. If $t \in (0.5, 0.8]$, then $U(\mathscr{A}; t) = 4\mathbb{Z}$ which is a subalgebra of $\mathbb{Z}$. If $t \in (0.8, 0.9]$, then $U(\mathscr{A}; t) = 8\mathbb{Z}$ which is a subalgebra of $\mathbb{Z}$. If $t \in (0.9, 1]$, then $U(\mathscr{A}; t) = \{8k \mid k \in \mathbb{Z},\, k \geq 0\}$ which is not a subalgebra of $\mathbb{Z}$. Thus $\mathbb{Z}^* = (0.3, 0.9]$, and so $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.3$ and $\delta = 0.9$ of $\mathbb{Z}$ by Theorem 3.7. But $\mathscr{A}$ is not a fuzzy subalgebra of $\mathbb{Z}$ since

$$\mathscr{A}(4 * 7) = \mathscr{A}(-3) = 0 \not\geq 0.3 = \min\{\mathscr{A}(4), \mathscr{A}(7)\}.$$

Now we know that $4_{0.7} \in \mathscr{A}$ and $7_{0.2} \in \mathscr{A}$, but $(4 * 7)_{min\{0.2, 0.7\}} \overline{\in\vee\, \mathrm{q}} \mathscr{A}$. Hence $\mathscr{A}$ is not an $(\in, \in\vee\, \mathrm{q})$-fuzzy subalgebra of $\mathbb{Z}$.

Theorem 3.9. *Let $\mathscr{A}$ be a fuzzy subalgebra with thresholds ε and δ of X. Then*

$$(\forall x \in X)\, (M(\mathscr{A}(0), \varepsilon) \geq m(\mathscr{A}(x), \delta)).$$

In particular, if there exists $y \in X$ such that $\mathscr{A}(y) > \delta$, then $\mathscr{A}(0) > \varepsilon$.

Proof. For all $x \in X$ we have

$$M(\mathscr{A}(0), \varepsilon) = M(\mathscr{A}(x * x), \varepsilon) \geq m(\mathscr{A}(x), \mathscr{A}(x), \delta) = m(\mathscr{A}(x), \delta).$$

If there exists $y \in X$ such that $\mathscr{A}(y) > \delta$, then $M(\mathscr{A}(0), \varepsilon) \geq \delta > \varepsilon$. Hence $\mathscr{A}(0) > \varepsilon$. □

Theorem 3.10. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If $\mathscr{A}$ is a fuzzy subalgebra with thresholds ε and δ of X, then $f(\mathscr{A})$ is a fuzzy subalgebra with thresholds ε and δ of Y, where*

$$f(\mathscr{A})(y) := \sup\{\mathscr{A}(x) \mid f(x) = y\},\ \forall y \in Y.$$

Proof. Let $y_1, y_2 \in Y$. Then

$$\begin{aligned} & M(f(\mathscr{A})(y_1 * y_2), \varepsilon) \\ = \;& M(\sup\{\mathscr{A}(x_1 * x_2) \mid f(x_1 * x_2) = y_1 * y_2\}, \varepsilon) \\ = \;& \sup\{M(\mathscr{A}(x_1 * x_2), \varepsilon) \mid f(x_1 * x_2) = y_1 * y_2\} \\ \geq \;& \sup\{m(\mathscr{A}(x_1), \mathscr{A}(x_2), \delta) \mid f(x_1) = y_1,\ f(x_2) = y_2\} \\ = \;& m(\sup\{\mathscr{A}(x_1) \mid f(x_1) = y_1\}, \sup\{\mathscr{A}(x_2) \mid f(x_2) = y_2\}, \delta) \\ = \;& m(f(\mathscr{A})(y_1), f(\mathscr{A})(y_2), \delta). \end{aligned}$$

Hence $f(\mathscr{A})$ is a fuzzy subalgebra with thresholds ε and δ of Y. □

Theorem 3.11. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $\mathscr{B}$ is a fuzzy subalgebra with thresholds ε and δ of Y, then $f^{-1}(\mathscr{B})$ is a fuzzy subalgebra with thresholds ε and δ of X, where*

$$f^{-1}(\mathscr{B})(x) := \mathscr{B}(f(x)),\ \forall x \in X.$$

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned} M(f^{-1}(\mathscr{B})(x_1 * x_2), \varepsilon) &= M(\mathscr{B}(f(x_1 * x_2)), \varepsilon) \\ &= M(\mathscr{B}(f(x_1) * f(x_2)), \varepsilon) \\ &\geq m(\mathscr{B}(f(x_1)), \mathscr{B}(f(x_2)), \delta) \\ &= m(f^{-1}(\mathscr{B})(x_1), f^{-1}(\mathscr{B})(x_2), \delta), \end{aligned}$$

and so $f^{-1}(\mathscr{B})$ is a fuzzy subalgebra with thresholds ε and δ of X. □

4. Implication-based fuzzy subalgebra

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\to$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

(4.1) $\quad [x \in \mathscr{A}] = \mathscr{A}(x),$

(4.2) $\quad [\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\},$

(4.3) $\quad [\Phi \to \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\},$

(4.4) $\quad [\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)],$

(4.5) $\quad \models \Phi$ if and only if $[\Phi] = 1$ for all valuations.

The truth valuation rules given in (4.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{\mathrm{GR}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_{G}):

$$I_{\mathrm{G}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator ($\overline{I}_{\mathrm{G}}$):

$$\overline{I}_{\mathrm{G}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 1-a & \text{otherwise.} \end{cases}$$

Ying [7] introduced the concept of fuzzifying topology. We can expand his/her idea to BCK/BCI-algebras, and we define a fuzzifying subalgebra as follows.

Definition 4.1. A fuzzy set $\mathscr{A}$ in X is called a *fuzzifying subalgebra* of X if it satisfies the following condition:

(4.6) for any $x, y \in X$, $\models [x \in \mathscr{A}] \wedge [y \in \mathscr{A}] \to [x * y \in \mathscr{A}]$.

Obviously, the condition (4.6) is equivalent to the condition (2.1). Therefore a fuzzifying subalgebra is an ordinary fuzzy subalgebra. In [6], the concept of t-tautology is introduced, i.e.,

(4.7) $\models_t \Phi$ if and only if $[\Phi] \ge t$ for all valuations.

Definition 4.2. Let $\mathscr{A}$ be a fuzzy set in X and $t \in (0,1]$. $\mathscr{A}$ is called a *t-implication-based subalgebra* of X if it satisfies:

$$\text{for any } x, y \in X,\ \models_t [x \in \mathscr{A}] \wedge [y \in \mathscr{A}] \to [x * y \in \mathscr{A}].$$

Let I be an implication operator. Clearly, $\mathscr{A}$ is a t-implication-based subalgebra of X if and only if it satisfies

$$(\forall x, y \in X)\ (I(m(\mathscr{A}(x), \mathscr{A}(y)), \mathscr{A}(x * y)) \ge t).$$

Theorem 4.3. *For any fuzzy set $\mathscr{A}$ in X, we have*

(i) *If $I = I_{\mathrm{GR}}$, then $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X if and only if $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 1$ of X.*

(ii) *If $I = I_{\mathrm{G}}$, then $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X if and only if $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 0.5$ of X.*

(iii) *If $I = \overline{I}_{\mathrm{G}}$, then $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X if and only if $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.5$ and $\delta = 1$ of X.*

Proof. (i) Straightforward.

(ii) Assume that $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X. Then $I_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) \geq 0.5$, and so

$$\mathscr{A}(x*y) \geq m(\mathscr{A}(x),\mathscr{A}(y))$$

or

$$m(\mathscr{A}(x),\mathscr{A}(y)) > \mathscr{A}(x*y) \geq 0.5.$$

It follows that $M(\mathscr{A}(x*y),0) = \mathscr{A}(x*y) \geq m(\mathscr{A}(x),\mathscr{A}(y),0.5)$ so that $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 0.5$ of X.

Conversely if $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0$ and $\delta = 0.5$ of X, then

$$\mathscr{A}(x*y) = M(\mathscr{A}(x*y),0) \geq m(\mathscr{A}(x),\mathscr{A}(y),0.5).$$

If $m(\mathscr{A}(x),\mathscr{A}(y),0.5) = m(\mathscr{A}(x),\mathscr{A}(y))$, then

$$I_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) = 1 \geq 0.5.$$

Otherwise, $I_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) \geq 0.5$. Therefore $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X.

(iii) Suppose that $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X. Then

$$\overline{I}_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) \geq 0.5,$$

which implies that $m(\mathscr{A}(x),\mathscr{A}(y)) \leq \mathscr{A}(x*y)$ or

$$1 - m(\mathscr{A}(x),\mathscr{A}(y)) \geq 0.5, \text{ i.e.}, m(\mathscr{A}(x),\mathscr{A}(y)) \leq 0.5.$$

Thus $M(\mathscr{A}(x*y),0.5) \geq m(\mathscr{A}(x),\mathscr{A}(y),1)$, and so $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.5$ and $\delta = 1$ of X. Conversely assume that $\mathscr{A}$ is a fuzzy subalgebra with thresholds $\varepsilon = 0.5$ and $\delta = 1$ of X. Then

$$M(\mathscr{A}(x*y),0.5) \geq m(\mathscr{A}(x),\mathscr{A}(y),1).$$

If $m(\mathscr{A}(x),\mathscr{A}(y),1) = 1$, then

$$\overline{I}_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) = 1 \geq 0.5.$$

Assume that $m(\mathscr{A}(x),\mathscr{A}(y),1) = m(\mathscr{A}(x),\mathscr{A}(y))$. Then

$$M(\mathscr{A}(x*y),0.5) \geq m(\mathscr{A}(x),\mathscr{A}(y)).$$

Now if $M(\mathscr{A}(x*y),0.5) = \mathscr{A}(x*y)$, then $\mathscr{A}(x*y) \geq m(\mathscr{A}(x),\mathscr{A}(y))$ and so

$$\overline{I}_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) = 1 \geq 0.5.$$

If $M(\mathscr{A}(x*y),0.5) = 0.5$, then $\mathscr{A}(x*y) \leq 0.5$ and $m(\mathscr{A}(x),\mathscr{A}(y)) \leq 0.5$. In this case we have

$$\overline{I}_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) = 1 \geq 0.5$$

when $\mathscr{A}(x*y) \geq m(\mathscr{A}(x),\mathscr{A}(y))$; and

$$\overline{I}_{\mathrm{G}}(m(\mathscr{A}(x),\mathscr{A}(y)),\mathscr{A}(x*y)) = 1 - m(\mathscr{A}(x),\mathscr{A}(y)) \geq 0.5$$

when $m(\mathscr{A}(x),\mathscr{A}(y)) < \mathscr{A}(x*y)$. Therefore $\mathscr{A}$ is a 0.5-implication-based fuzzy subalgebra of X. □

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