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ISOMORPHISM OF MODULAR GROUP ALGEBRAS OF ABELIAN GROUPS WITH SEMI-COMPLETE p-PRIMARY COMPONENTS

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ABSTRACT. Let G be a p-mixed abelian group with semi-complete torsion subgroup G_t such that G is splitting or is of torsion-free rank one, and let R be a commutative unitary ring of prime characteristic p. It is proved that the group algebras RG and RH are R-isomorphic for any group H if and only if G and H are isomorphic. This isomorphism relationship extends our earlier results in (Southeast Asian Bull. Math., 2002), (Proc. Amer. Math. Soc., 2002) and (Bull. Korean Math. Soc., 2005) as well as completely settles a problem posed by W. May in (Proc. Amer. Math. Soc., 1979).

1. Introduction and Preliminaries

As a point of departure, let us recall some relevant information and briefly outline our program. Suppose G is an abelian group, written multiplicatively as is customary when discussing group algebras, with torsion part G_t and p-component G_p and suppose R is a commutative ring with 1 of prime characteristic, for instance, p. We shall use the traditional symbol RG to denote the group algebra of G over R with group of all normalized units $V(RG)$ and its p-component $V_p(RG)$.

The other notions and notation not explicitly defined herein are at most standard and follow either [10] or the references listed in the bibliography. For example, since $(G^{p^\alpha})_p = (G_p)^{p^\alpha}$ for any ordinal α, we unambiguously write $G_p^{p^\alpha}$ to denote both of these expressions.

The leitmotif in our discussion is created by the following two related and exceptionally difficult conjectures.

(a) *The Generalized Direct Factor Problem:* For an abelian group G and a perfect field F of $char(F) = p > 0$, the quotient $V_p(FG)/G_p$ is simply presented

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or, in other terms, $V_p(FG)/G_p$ has a nice composition series (that are smooth unions consisting of nice subgroups with countable factors).

(b) *The Generalized Isomorphism Problem:* For a p-mixed abelian group G (i.e., $G_t = G_p$) and a field F of $char(F) = p \neq 0$, the F-isomorphism $FH \cong FG$ for an arbitrary group H yields $H \cong G$.

We shall show below that the positive answering of the first problem serves the second one; actually the same observation was demonstrated in our subsequent papers (e.g. [2], [4], [5], [7], [8] and [9] plus [15]). The question of determining the isomorphism class of G by RG is extremely complicated. Although so far none of these two conjectures have been resolved in full generality, a great deal of progress has been achieved under certain additional assumptions on the structure of G, especially in the cases where G is simply presented ([15]) or is semi-complete ([5] and [8]).

It is the purpose of this manuscript to bring the reader to the frontier of our knowledge of RG and $V_p(RG)$ by investigating in details the previous problems for a special sort of abelian groups, termed by Kolettis in [12] as semi-complete groups, presenting many of the best results known and necessary as well as examining the difficulties that arise.

In order to avoid excessive verbosity it will be convenient and friendly to the reader to introduce the following terminology and attainment due to Kolettis ([12]).

Definition. The abelian p-group A is named semi-complete whenever A is the direct product of a torsion-complete group and a coproduct of cyclic groups.

Theorem ([12]). *Suppose that* $A = T \times C$ *and that* $B = T' \times C'$, *where* T *and* T' *are torsion-complete p-groups, and* C *and* C' *are coproducts of cyclic p-groups. Then* $A \cong B \iff A$ *and* B *have the same Ulm-Kaplansky invariants and there is a natural number n with the properties* $T^{p^n} \cong T'^{p^n}$ *and* $C^{p^n} \cong C'^{p^n}$.

We continue with a Direct Factor Theorem established in [9] (see also [6]).

Theorem (Direct Factor). *Given that* G_p *is a separable group and* F *is a (perfect) field of nonzero characteristic p. Then* $V_p(FG)/G_p$ *is a coproduct of cyclic groups, and thus* G_p *is a direct factor of* $V_p(FG)$ *with complement isomorphic to* $V_p(FG)/G_p$.

2. The main result and the proof

The following assertion is our crucial tool. It is needed for proving the central affirmation of this article, but is of independent interest too; (see cf. [8]).

Proposition (Isomorphism). *Let* G *be an abelian group for which* G_p *is semi-complete (in particular, torsion-complete) and let* R *be a commutative ring with identity of prime characteristic p such that the R-isomorphism* $RG \cong RH$ *for another group* H *holds. Then* $G_p \cong H_p$. *In particular,* $G \cong H$, *provided* G *is a p-group.*

Proof. It is well-known that $RG \cong RH$ insures $FG \cong FH$ over some field F of $char(F) = p$ that depends on R. Thus, with no loss of generality, we may assume that $FG = FH$ whence $V_p(FG) = V_p(FH)$. Thereby, according to [13], we infer at once that the Ulm-Kaplansky invariants of G_p and H_p are equal. Moreover, since obviously $length(G_p) = length(H_p) \leq \omega$, referring to the foregoing Direct Factor Theorem we can write $V_p(FG) = G_p \times V_p(FG)/G_p = H_p \times V_p(FH)/H_p = V_p(FH)$, where the latter direct factors are coproducts of cyclic groups. Consequently, appealing to [11], we conclude that H_p must also be semi-complete, hence we obtain the standard decompositions $G_p = T \times C$ and $H_p = T' \times C'$ where both T and T' are torsion-complete groups and both C and C' are coproducts of cyclic groups. Therefore, we deduce that $T \times C \times V_p(FG)/G_p = T' \times C' \times V_p(FH)/H_p$. Then, T being torsion-complete implies that its projection into the coproduct of cyclic groups $C' \times V_p(FH)/H_p$ would be bounded. Thus there exists a natural number, say m, with the property that $T^{p^m} \subseteq T'$. Because of the purity of T' in $V_p(FH)$ we derive $T^{p^m} \subseteq T' \cap V_p^{p^m}(FH) = T'^{p^m}$. By a reason of symmetry we have $T^{p^m} = T'^{p^m}$. Furthermore, $FT^{p^m} = FT'^{p^m}$ along with $FG^{p^m} = F.(F^{p^m}G^{p^m}) = F.(FG)^{p^m} = F.(FH)^{p^m} = F.(F^{p^m}H^{p^m}) = FH^{p^m}$ ensure that $F(G^{p^m}/T^{p^m}) \cong F(H^{p^m}/T'^{p^m})$. Since $(G^{p^m}/T^{p^m})_p = G_p^{p^m}/T^{p^m} \cong C^{p^m} \subseteq C$ is a coproduct of cyclic groups as being an isomorphic copy of a subgroup of a coproduct of cyclic groups, ([1], Proposition 6) is applicable to get that $C^{p^m} \cong C'^{p^m}$. Finally, by what we have already shown, the alluded to above criterion due to Kolettis leads us to $G_p \cong H_p$. So, the proposition is true. □

We have at our disposal all the information necessary to prove the following statement, which is our main result. It unambiguously shows that, under certain circumstances on R and G, the complete set of invariants for RG consists of G. Other claims of such a type for different large classes of abelian groups were established in ([3], [4] and [7]), respectively.

Theorem (Invariants). *Suppose that G is an abelian group whose G_t is a semi-complete p-group (in particular, a torsion-complete p-group), and suppose that R is a commutative ring with identity of prime characteristic p. If*

(1) *G is splitting,*

or

(2) *G is of torsion-free rank one,*

then $RG \cong RH$ as R-algebras for some arbitrary group H precisely when $G \cong H$.

Proof. As in the foregoing Proposition, we have $FG \cong FH$ as F-algebras. The last isomorphism plainly assures that H_t must also be a p-group. Consequently, the Proposition gives that $G_t \cong H_t$. Moreover, since by a classical theorem of Steinitz every field is contained in an algebraically closed (whence perfect) field of the same characteristic, without harm of generality, we may presume that F is perfect.

Now, we concentrate on the first condition for G. And so, write $G \cong G_t \times G/G_t$. Invoking [13] we infer that $G/G_t \cong H/H_t$. We are thus left to check that H is also splitting. To this end, we first observe via [6] that $V(FG) \cong V_p(FG) \times V(FG)/V_p(FG)$ whence the algebra isomorphism implies $V(FH) \cong V_p(FH) \times V(FH)/V_p(FH)$. Besides, because H_t is just semi-complete, hence it is of length not exceeding ω, an appeal to the previous Direct Factor Theorem from [9] (see [6] too) riches us that H_p is a direct factor of $V_p(FH)$ whence immediately it is a direct factor even of $V(FH)$ and H. Finally, H is really a splitting p-mixed group. Hence $G \cong G_t \times G/G_t \cong H_t \times H/H_t \cong H$, completing the proof of this point. These conclusions may also be described in a more compact form like this: Since G_p being a direct factor of G obviously implies that $V_p(FG) = V_p(FH)$ is a direct factor of $V(FG) = V(FH)$ and since H_p is separable, hence by [9] it is a direct factor of $V_p(FH)$, one can infer that H_p is a direct factor of $V(FH)$, whence of H as wanted.

Next, we deal with the second restriction on G. And so, utilizing [13], we detect that $G/G_t \cong H/H_t$, so H is of torsion-free rank one as well. On the other hand, the application of ([3], Lemma 13) is a guarantor that the Ulm p-height matrices of G and H are equivalent. It therefore follows by combining the proofs of the central results in [16] and the above quoted theorem from [12] that $G \cong H$. So, the theorem is verified in all generality. □

Since the torsion-complete groups are obviously semi-complete, an important special case of the preceding proposition or theorem is the following consequence.

Corollary ([5]). *Let G be a torsion-complete abelian p-group and R a commutative ring with unity of prime characteristic p. Then FG and FH are isomorphic F-group algebras for any group H only when G and H are isomorphic groups.*

Remark. This corollary was a part of the major result from [5]. Comparing with the evidence there, which possesses a topological accent, the proof here is purely algebraic. We note that the foregoing consequence solves in a positive way a long-standing problem by May ([14]), but by exploiting another idea as early mentioned.

We close by quoting the following challenging question.

Problem. Prove that the Theorem remains valid for the class of all coproducts of torsion-complete abelian p-groups.

This problem was examined in [2] and [4] but under extra restrictions on the cardinality either of the direct factors or of the whole coproduct, respectively.

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