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CRITICALITY OF CHARACTERISTIC VECTOR FIELDS ON ALMOST COSYMPLECTIC MANIFOLDS

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Abstract. Main interest of the present paper is to investigate the criticality of characteristic vector fields on almost cosymplectic manifolds. Killing critical characteristic vector fields are absolute minima. This paper contains some examples of non-Killing critical characteristic vector fields.

1. Introduction

Let $(\tilde{M},\tilde{g})$ be a Riemannian manifold and $\tilde{\nabla}$ be its Levi-Civita connection. Consider an immersion $f : M \longrightarrow (\tilde{M},\tilde{g})$. The second fundamental form B is defined by $B(X,Y) := (\tilde{\nabla}_X Y)^{\perp}$ for $X, Y \in \Gamma(T\tilde{M})$, where $X^{\perp}$ denotes the component of X normal to $M \subseteq \tilde{M}$. The mean curvature vector field is given by $H = \mathrm{tr}B$. f is said to be critical or minimal if $H = 0$. Indeed, H is characterized by the gradient of the volume and energy functionals for closed manifolds.

This paper specially considers immersions determined by unit vector fields. Let (M,g) be a Riemannian manifold and $Z \in \Gamma(T^1M)$ (T^1M denotes the tangent unit sphere bundle of M). Then $Z : M \longrightarrow T^1M$ is an embedding. T^1M is endowed with the restriction of the Sasaki metric G determined by g and the Levi-Civita connection map $\kappa : TTM \longrightarrow TM$, also denoted by G. Recall the Sasaki metric given by

$$G(X,Y) := g(\pi_* X, \pi_* Y) + g(\kappa X, \kappa Y), \tag{1.1}$$

where $\pi : TM \longrightarrow M$ is the natural projection ([25]). Z is said to be critical or minimal if $Z : (M,g) \longrightarrow (T^1M,G)$ is a minimal embedding. Parallel vector fields, when they exist, are absolute minima for the volume and energy functionals. Since there are manifolds without parallel vector fields, it is natural to look for the next best thing - critical unit vector fields. For instance, in the three-dimensional sphere the Hopf vector fields are absolute minima ([2], [12]).

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However, in higher dimensions, these vector fields are not minima anymore, but they are still critical ([22]).

The problem of determining the criticality of unit vector fields of the volume and energy functionals was studied in [10], [11], [24]. The criticality problem for characteristic vector fields on contact metric manifolds was discussed in [22].

This paper mainly treats with almost cosymplectic manifolds. Killing critical characteristic vector fields are absolute minima. We describe some examples of non-Killing critical characteristic vector fields.

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2. Almost cosymplectic manifolds

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost contact metric manifold, that is, ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form, g is a Riemannian metric satisfying the conditions

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for $X, Y \in \Gamma(TM)$. The fundamental 2-form Φ of M is defined by $\Phi(X, Y) := g(\phi X, Y)$.

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$. Topological and geometrical properties of almost cosymplectic manifolds have been studied by many mathematicians ([4], [5], [13], [15], [17], [20]). The products of almost Kähler manifolds and the real line $\mathbb{R}$ or the circle S^1 are the simplest examples of almost cosymplectic manifolds. In general the converse does not hold, even locally (for the converse, see Corollary 2.3 below).

Proposition 2.1. *On an almost cosymplectic manifold (M, ϕ, ξ, η, g) it holds that $h = \frac{1}{2}L_\xi\phi$ if and only if $\nabla_X\xi = -\phi hX$ for any $X \in \Gamma(TM)$. In this case, the operator h is symmetric and satisfies $h\xi = 0$ and $h\phi + \phi h = 0$.*

Proof. Let $h = \frac{1}{2}L_\xi\phi$. It is obvious that $h\xi = 0$. A general formula for $\nabla\phi$ in an almost contact metric manifold ([1]) yields

$$\nabla_\xi\phi = 0. \tag{2.1}$$

Since $d\eta = 0$, (2.1) implies for $X, Y \in \Gamma(TM)$

$$\begin{aligned} 2g(hX, Y) &= g(\phi\nabla_X\xi - \nabla_{\phi X}\xi, Y) = \eta(\nabla_X\phi Y) + \eta(\nabla_{\phi X}Y) \\ &= \eta(\nabla_{\phi Y}X) + \eta(\nabla_Y\phi X) = 2g(X, hY), \end{aligned} \tag{2.2}$$

which means that h is symmetric. Moreover,

$$2g((\nabla_X\phi)\xi, Y) = -g(\phi(L_\xi\phi)Y, \phi X) = -2g(hX, Y),$$

so that $\nabla_X \xi = -\phi h X$. Finally, from the formula ([17])

$$\nabla_{\phi X} \xi = -\phi \nabla_X \xi, \tag{2.3}$$

we see that $h\phi + \phi h = 0$.

Conversely, let $\nabla_X \xi = -\phi h X$. Then $0 = \nabla_\xi \xi = -\phi h \xi$, which induces $h\xi = 0$. (2.2) and (2.3) show that h is symmetric and satisfies $h\phi + \phi h = 0$ respectively. Furthermore, an easy computation together with (2.1) gives rise to

$$(L_\xi \phi)X = \phi \nabla_X \xi - \nabla_{\phi X} \xi = hX + \phi h \phi X = 2hX,$$

which completes the proof. □

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost cosymplectic manifold. The condition $d\eta = 0$ allows us to admit on M a $2n$-dimensional foliation $\mathcal{F}$ of codimension 1 which is defined by the contact distribution $D := \ker \eta$. Let $D^\perp$ be the orthogonal complement to D and $\mathcal{F}^\perp$ be the corresponding flow, which is generated by ξ. In the foliation context, we have

Proposition 2.2. *Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold. Then $\mathcal{F}$ is Riemannian, tangentially almost Kähler and minimal. Moreover, the followings are equivalent.*

(1) *$\mathcal{F}$ is totally geodesic,*
(2) *$h = 0$,*
(3) *ξ is Killing,*
(4) *ξ is parallel.*

Proof. Observe that $\mathcal{F}$ is Riemannian if and only if $\nabla_\xi \xi = 0$, or equivalently $d\eta(\xi, X) = 0$ for $X \in \Gamma(D)$. This observation shows that $\mathcal{F}$ is Riemannian.

We note that

$$\phi \xi = 0, \ \ \iota_\xi \Phi = 0, \tag{2.4}$$

where ι denotes the interior product operator. Hence the restriction (g_D, ϕ_D, Φ_D) of (g, ϕ, Φ) to the contact distribution D inherits an almost Hermitian structure for $\mathcal{F}$. Since $d\Phi_D = 0$, $(\mathcal{F}, g_D, \phi_D, \Phi_D)$ is tangentially almost Kähler.

On the other hand, we find

$$g(\nabla_X Y, \xi) = g(\phi h X, Y) \tag{2.5}$$

for $X, Y \in \Gamma(D)$. (2.5) shows that ϕh plays a role as the second fundamental form for $\mathcal{F}$. In particular, Proposition 2.1 implies that $\mathrm{tr}\phi h = 0$. so that $\mathcal{F}$ is minimal.

Finally, (1) $\Leftrightarrow$ (2) follows from (2.5). (2.4) induces $L_\xi \Phi = 0$. It follows that (2) $\Leftrightarrow$ (3). (3) $\Leftrightarrow$ (4) is due to $d\eta = 0$. □

Remarks. (1) We find that the 1-dimensional foliation $\mathcal{F}^\perp$ is totally geodesic. Furthermore, $\mathcal{F}^\perp$ is Riemannian if and only if $\mathcal{F}$ is totally geodesic. It follows from Proposition 2.2 that;

Corollary 2.3. *An almost cosymplectic manifold is locally a trivial product of an almost Kähler manifold and* $\mathbb{R}$ *or* S^1 *if and only if* $h = 0$.

(2) When M is closed, if the Ricci curvature of M is non-negative then $\mathcal{F}$ is totally geodesic ([19]). Then Corollary 2.3 reads that locally M is the trivial product of a closed almost Kähler manifold and S^1.

The conclusion that $\mathcal{F}$ is totally geodesic can also be shown by applying the usual Bochner technique. Indeed, note that η is the transversal volume form for $\mathcal{F}$. Since $\mathcal{F}$ is Riemannian and minimal, η is a harmonic 1-form ([23]). Then the Bochner technique implies η is parallel, that is, $\mathcal{F}$ is totally geodesic by Proposition 2.2.

By a similar argument, we conclude that the Ricci curvature of a closed almost cosymplectic manifold cannot be positive.

Recall that an almost contact manifold (M, ϕ, ξ, η) is said to be normal if

$$N_\phi(X, Y) := [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] + 2d\eta(X, Y)\xi$$

vanishes for any $X, Y \in \Gamma(TM)$. A normal almost cosymplectic manifold is called a cosymplectic manifold. It was obtained a characterization of an almost cosymplectic manifold to be normal ([20]).

Proposition 2.4. *An almost cosymplectic manifold* M *is cosymplectic if and only if the foliation* $\mathcal{F}$ *given by the contact distribution is tangentially Kähler and* $h = 0$.

In view of Proposition 2.4, it is worthwhile to notice the following characterization.

Theorem 2.5. *Let* (M, ϕ, ξ, η, g) *be an almost contact metric manifold. Then* M *is almost cosymplectic with* $h = 0$ *if and only if*

(1) *the contact distribution* D *is integrable,*
(2) $\mathcal{F}$ *is Riemannian,*
(3) $\nabla_\xi \phi = 0$,
(4) $\mathcal{F}$ *is tangentially almost Kähler,*
(5) $\mathcal{F}$ *is totally geodesic.*

Proof. The sufficiency is obvious from (2.1) and Proposition 2.2. Conversely, (1) and (2) means that $d\eta = 0$. This, together with (5), yields ξ is parallel. Then ξ is Killing and $L_\xi \phi = \nabla_\xi \phi = 0$ by means of (3), so $h = 0$. It follows that $L_\xi \Phi = 0$. Hence we see from (2.4) that the fundamental 2-form Φ is a basic form for $\mathcal{F}^\perp$ corresponding to Φ_D on the contact distribution D in the sense of Proposition 2.2. Moreover (4) implies that $d\Phi = 0$. Therefore M is almost cosymplectic with $h = 0$. □

Remarks. (1) In the case where M is an almost Kenmotsu manifold ([16], [15]), $\mathcal{F}$ is Riemannian and tangentially almost Kähler ([20], [21]). In this case, $h = 0$ if and only if $\mathcal{F}$ is totally umbilic with constant mean curvature $-\xi$. In addition, M is Kenmotsu if and only if $\mathcal{F}$ is tangentially Kähler and $h = 0$.

(2) There are several examples of almost cosymplectic manifolds (with $h = 0$) which are not cosymplectic ([3], [5]).

3. Harmonic maps on almost cosymplectic manifolds

Let (M, g) and (M', g') be Riemannian manifolds and $f : M \longrightarrow M'$ be a map. Denote by ∇ and ∇' the Levi-Civita connection of g and g' respectively. The pullback bundle $f^{-1}(TM')$ admits the connection $\tilde{\nabla}$ induced from ∇ and ∇'. Then the second fundamental form α_f of f is defined by

$$\alpha_f(X, Y) := (\tilde{\nabla}_X f_*)(Y)$$

for $X, Y \in \Gamma(TM)$. The tension field is defined by $\tau_f := \mathrm{tr}\alpha_f$. f is said to be harmonic if $\tau_f = 0$.

Let $f : M \longrightarrow M'$ be a map between two almost contact metric manifolds. f is said to be ϕ-holomorphic (resp. ϕ-antiholomorphic) if $f_* \circ \phi = \phi' \circ f_*$ (resp. $f_* \circ \phi = -\phi' \circ f_*$). By a similar way as in [14], we deduce

Theorem 3.1. *Let M and M' be almost cosymplectic manifolds. Then any ϕ-holomorphic or ϕ-antiholomorphic map $f : M \longrightarrow M'$ is harmonic.*

Proof. On an almost cosymplectic manifold $\nabla\phi$ satisfies for $X, Y \in \Gamma(TM)$ ([17])

(3.1) $$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = -\eta(Y)hX.$$

Then for $X \in \Gamma(D)$

(3.2) $$\nabla_X X + \nabla_{\phi X}\phi X = \phi[\phi X, X].$$

On the other hand, since f is ϕ-holomorphic or ϕ-antiholomorphic, (3.2) becomes

(3.3) $$\tilde{\nabla}_X f_*X + \tilde{\nabla}_{\phi X} f_*\phi X = \phi'[\phi' f_*X, f_*X].$$

It follows from (3.2) and (3.3) that

(3.4) $$\alpha_f(X, X) + \alpha_f(\phi X, \phi X) = 0.$$

Moreover, since $\nabla_\xi\xi = \nabla_{\xi'}\xi' = 0$, we find

(3.5) $$\alpha_f(\xi, \xi) = 0.$$

Therefore we conclude from (3.4) and (3.5) that f is harmonic. □

4. Criticality on almost cosymplectic manifolds

Let (M, g) be a Riemannian manifold. The tangent bundle TM is equipped with a symplectic form $\Omega = -d\Theta$, where Θ is the pull-back of the fundamental 1-form on T^*M by the musical isomorphism. The map

(4.1) $$(\pi_*, \kappa) : TTM \longrightarrow TM \oplus TM$$

is a vector bundle isomorphism along $\pi : TM \longrightarrow M$, which determines an almost complex structure J_{TM} on TM such that if $(\pi_*, \kappa)(x) = (u, v)$ then

$(\pi_*, \kappa)(J_{TM}x) = (-v, u)$ ([8]). Then (G, J_{TM}, Ω) is an almost Kähler structure on TM, where G is the Sasaki metric given by (1.1). TM carries the geodesic spray U determined by $\pi_* U(p,v) = v$ and $\kappa U(p,v) = 0$ for any $(p,v) \in TM$. Note that U and Θ are G-duals.

Let $j : T^1M \longrightarrow TM$ be the inclusion of the tangent unit sphere bundle as a hypersurface in TM. Then $\tilde{\eta} := j^*\Theta$ is a contact form on T^1M whose characteristic vector field is U. The kernel of $\tilde{\eta}$ has associated almost complex operator $\tilde{\phi}$ determined by

$$\tilde{\phi}U = 0, \quad \tilde{\phi}X = J_{TM}X$$

for any $X \in \Gamma(T^1M)$ such that $G(X,U) = 0$. Hence $(\tilde{\phi}, U, \tilde{\eta}, G)$ is a contact metric structure on T^1M.

Under the identification (4.1), a vector tangent at $(p,v) \in TM$ is a couple $(u, \nabla_u W)$, where $u \in T_pM$ and W is a vector field on M such that $W(p) = v$. Note that in this case $g(v, \nabla_u W) = 0$.

Now let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold. A vector tangent to $\xi(M) \subseteq T^1M$ at (p,ξ) is a couple $(u, \nabla_u \xi)$, where $u \in T_pM$. Under the identification (4.1) $U \in \Gamma(T^1M)$ is given by $U(p,v) = (v, 0)$.

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. A submanifold N of M is said to be invariant if the characteristic vector field ξ is tangent to N and ϕX is tangent to N whenever X is. Then N admits an almost contact metric structure from M by the restriction.

It is easy to see that if M is almost cosymplectic then N is also almost cosymplectic. Moreover, the inclusion $i : N \longrightarrow M$ is ϕ-holomorphic. Then i is harmonic by means of Theorem 3.1. Since an isometric immersion is minimal if and only if it is harmonic ([9]), we conclude;

Proposition 4.1. *Let N be an invariant submanifold of an almost cosymplectic manifold (M, ϕ, ξ, η, g). Then N is minimal.*

Theorem 4.2. *Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold. Then $\xi(M) \subseteq T^1M$ is invariant if and only if ξ is Killing.*

Proof. It is obvious that the geodesic spray U is tangent to $\xi(M)$. Let $(u, \nabla_u \xi)$ be a tangent vector at $(p, \xi) \in \xi(M)$ such that $G((u, \nabla_u \xi), (\xi, 0)) = 0$, that is, a tangent vector in $\ker \tilde{\eta}$ on T^1M. Then

$$\tilde{\phi}(u, \nabla_u \xi) = (-\nabla_u \xi, u) = (\phi h u, u),$$

so that $\tilde{\phi}(u, \nabla_u \xi)$ is tangent to $\xi(M)$ if and only if

$$u = \nabla_{\phi h u} \xi = -h^2 u$$

by Proposition 2.1. It follows that $h = 0$, namely, ξ is Killing by Proposition 2.2. □

It should be noted that if $\xi(M) \subseteq T^1M$ is invariant then the inclusion $\xi : M \longrightarrow T^1M$ is ϕ-holomorphic. The converse is not true in general. Extending Theorem 4.2, we establish;

Theorem 4.3. *Let (M, ϕ, ξ, η, g) be as in Theorem 4.2. Then $\xi : M \longrightarrow T^1M$ is ϕ-holomorphic if and only if ξ is Killing.*

Proof. Under the identification (4.1), we easily see that ξ_* sends ξ to U. Thus it suffice to consider the commutativity of the almost complex structures ϕ and $\tilde{\phi}$.

Let $v \in T_pM$ satisfy $\eta(v) = 0$. Then

$$\tilde{\phi}\xi_*(v) = \tilde{\phi}(\pi_*\xi_*(v), \kappa\xi_*(v)) = \tilde{\phi}(v, \nabla_v\xi) = (\phi hv, v). \tag{4.2}$$

On the other hand,

$$\xi_*(\phi v) = (\phi v, \nabla_{\phi v}\xi) = (\phi v, -hv). \tag{4.3}$$

It follows from (4.2) and (4.3) that ξ is ϕ-holomorphic if and only if $h = 0$, i.e., ξ is Killing. □

Proposition 2.2 say that Killing characteristic vector fields on almost cosymplectic manifold are absolute minima for the volume and energy functionals. Thus it may be interesting to find critical characteristic vector fields which are not absolute minima. In order to do this, we recall that a unit vector field $Z : (M, g) \longrightarrow (T^1M, G)$ is a harmonic map if and only if

$$\mathrm{tr}R(\nabla_* Z, Z)* = 0, \quad \mathrm{tr}\nabla^2 Z = cZ \tag{4.4}$$

for some constant $c \in \mathbb{R}$. First we establish a formula.

Proposition 4.4. *Let (M, ϕ, ξ, η, g) be as in Theorem 4.2. Then the Ricci operator Q satisfies $Q\xi = \mathrm{tr}\nabla^2\xi$. In particular, ξ is Killing if and only if $Q\xi = 0$.*

Proof. Observe from Proposition 2.1 that

$$\mathrm{tr}\nabla^2\xi = -\mathrm{tr}\nabla(\phi h). \tag{4.5}$$

Moreover we have for any $X, Y \in \Gamma(TM)$

$$\begin{aligned} R(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ &= -(\nabla_X(\phi h))Y + (\nabla_Y(\phi h))X. \end{aligned}$$

Now we may take a local orthonormal ϕ-basis $\{e_A\} = \{e_0 = \xi, e_a, e_{n+a} = \phi e_a\}$ consisting of eigenvectors of h with eigenvalues $\{\lambda_0 = 0, \lambda_a, -\lambda_a\}$. Then

$$\sum g(\nabla_X(h\phi)e_a, e_a) = 2\lambda_a g(\nabla_X e_a, \phi e_a)$$

and

$$\sum g(\nabla_X(h\phi)\phi e_a, \phi e_a) = -2\lambda_a g(\nabla_X e_a, \phi e_a).$$

Hence (4.5) implies

$$\begin{aligned} \sum g(R(X, e_A)e_A, \xi) &= -\sum g((\nabla_{e_A}(\phi h))X - (\nabla_X(\phi h))e_A, e_A) \\ &= -g(\mathrm{tr}\nabla(\phi h), X) = g(\mathrm{tr}\nabla^2\xi, X), \end{aligned}$$

which yields $Q\xi = \mathrm{tr}\nabla^2\xi$.

Moreover, it is obvious that ξ is Killing means $Q\xi = 0$. Conversely, if $Q\xi = 0$ then $0 = g(Q\xi, \xi) = -\mathrm{tr}h^2$, so that $h = 0$. That is, ξ is Killing. □

From the harmonicity criterion (4.4) and Proposition 4.4, we can find an example of almost cosymplectic manifolds whose characteristic vector fields are non-Killing, but are still critical.

Example. Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold with $\xi \in \mathcal{N}(l)$, where

$$\mathcal{N}(l) := \{Z \in \Gamma(TM) | R(X,Y)Z = l(g(Y,Z)X - g(X,Z)Y)\} \tag{4.6}$$

for any $X, Y \in \Gamma(TM)$. Then we see that $\xi : M \longrightarrow T^1M$ is a harmonic map.

Indeed, Proposition 2.1 and (4.6) yield for any $Y \in \Gamma(TM)$

$$\begin{aligned}\sum g(R(\nabla_{e_A}\xi, \xi)e_A, Y) &= -\sum g(R(Y, e_A)\xi, \phi h e_A) \\ &= l\{g(Y, \phi h\xi) - \eta(Y)\mathrm{tr}\phi h\} = 0,\end{aligned}$$

so that $\mathrm{tr}R(\nabla_*\xi, \xi)* = 0$. Moreover it was obtained in [6] that

$$Q\xi = 2nl\xi.$$

Thus $\mathrm{tr}\nabla^2\xi = c\xi$ for some constant $c = 2nl$ by Proposition 4.5. It follows from (4.4) that ξ is a harmonic map.

On the other hand, it was proved in [6] that in this situation, $l \leq 0$ and $l = 0$ if and only if ξ is Killing. When $l < 0$, the foliation $\mathcal{F}$ appeared in Proposition 2.2 is tangentially Kähler. Therefore, we conclude that the characteristic vector field ξ on an almost cosymplectic manifold with $\xi \in \mathcal{N}(l), l < 0$ is non-Killing, but still critical. It was obtained several examples of almost cosymplectic manifolds which the foliation $\mathcal{F}$ are tangentially Kähler but not cosymplectic ([7], [18]).

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