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ENDPOINT ESTIMATES FOR MULTILINEAR INTEGRAL OPERATORS

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ABSTRACT. In this paper, the endpoint estimates for some multilinear operators related to certain integral operators are obtained. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

1. Introduction and Notations

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-6]). In [10], the boundedness properties of the commutators for the extreme values of p are obtained. The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type integral operators. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

First, let us introduce some notations (see [7-9], [14-16]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y)-f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#}\in L^\infty$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$; We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f\in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1}|Q(0,r)|^{-1}\int_Q |f(y)-f_Q|dy < \infty.$$

It is well-known that (see [8], [9])

$$||f||_{CMO} \approx \sup_{r>1}\inf_{c\in C}|Q(0,r)|^{-1}\int_Q |f(x)-c|dx.$$

Also, we give the concepts of the atom and H^1 space. A function a is called as H^1 atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty}\le |Q|^{-1}$

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and $\int a(x)dx = 0$. It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [9]).

For $k \in Z$, define $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \geq 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1. Let $0 < p < \infty$ and $\alpha \in R$.

(1) The homogeneous Herz space $\dot{K}_p^\alpha(R^n)$ is defined by

$$\dot{K}_p^\alpha(R^n) = \{f \in L^p_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_p^\alpha} < \infty\},$$

where

$$||f||_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha} ||f\chi_k||_{L^p};$$

(2) The nonhomogeneous Herz space $K_p^\alpha(R^n)$ is defined by

$$K_p^\alpha(R^n) = \{f \in L^p_{loc}(R^n) : ||f||_{K_p^\alpha} < \infty\},$$

where

$$||f||_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} ||f\tilde{\chi}_k||_{L^p}.$$

If $\alpha = n(1 - 1/p)$, we denote that $\dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n)$, $K_p^\alpha(R^n) = K_p(R^n)$.

Definition 2. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

Definition 3. Let $1 < p < \infty$.

(1) The homogeneous Herz type Hardy space $H\dot{K}_p(R^n)$ is defined by

$$H\dot{K}_p(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(R^n)\},$$

where

$$||f||_{H\dot{K}_p} = ||G(f)||_{\dot{K}_p};$$

(2) The nonhomogeneous Herz type Hardy space $HK_p(R^n)$ is defined by

$$HK_p(R^n) = \{f \in S'(R^n) : G(f) \in K_p(R^n)\},$$

where

$$||f||_{HK_p} = ||G(f)||_{K_p};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $1 < p < \infty$. A function $a(x)$ on R^n is called a central $(n(1-1/p),p)$-atom (or a central $(n(1-1/p),p)$-atom of restrict type), if

1) Supp$a \subset B(0,r)$ for some $r > 0$ (or for some $r \geq 1$);
2) $||a||_{L^p} \leq |B(0,r)|^{1/p-1}$,
3) $\int_{R^n} a(x)dx = 0$.

Lemma 1. (see [8], [15]). *Let $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(R^n)$ (or $HK_p(R^n)$) if and only if there exist central $(n(1-1/p),p)$-atoms (or central $(n(1-1/p),p)$-atoms of restrict type) a_j supported on $B_j = B(0,2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_p} (or \; ||f||_{HK_p}) \sim \sum_j |\lambda_j|.$$

2. Theorems

In this paper, we will study a class of multilinear operators related to some non-convolution type integral operators, whose definition are following.

Let m_j be the positive integers($j = 1,\ldots,l$), $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n ($j = 1,\ldots,l$). Set

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\beta|\leq m_j} \frac{1}{\beta!} D^\beta A_j(y)(x-y)^\beta$$

and

$$Q_{m_j+1}(A_j;x,y) = R_{m_j}(A_j;x,y) - \sum_{|\beta|=m_j} \frac{1}{\beta!} D^\beta A_j(x)(x-y)^\beta.$$

Fixed $0 \leq \delta < n$. Let $F_t(x,y)$ define on $R^n \times R^n \times [0,+\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x,y)f(y)dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} F_t(x,y)f(y)dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operator related to F_t is defined by

$$T_\delta^A(f)(x) = ||F_t^A(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$,

$$||F_t(x,y)|| \leq C|x-y|^{-n+\delta}$$

and

$$||F_t(y,x) - F_t(z,x)|| \leq C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \le |x-z|$. We define that $T_\delta(f)(x) = ||F_t(f)(x)||$. We also consider the variant of T_δ^A, which is defined by

$$\tilde{T}_\delta^A(f)(x) = ||\tilde{F}_t^A(f)(x)||,$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} F_t(x,y)f(y)dy.$$

Note that when $m=0$, T_δ^A is just the multilinear commutators of T_δ and A (see [1], [11-13], [18]). It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3-6]). In [2], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the endpoint continuity properties of the multilinear operators T_δ^A and $\tilde{T}_\delta^A$. In Section 4, we will give some applications of Theorems in this paper.

Now we state our results as following.

Theorem 1. *Let $0 \le \delta < n$ and $D^\beta A_j \in BMO(R^n)$ for all β with $|\beta| = m_j$ and $j=1,\dots,l$. Suppose that T_δ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r,s \in (1,+\infty]$ with $1<r<n/\delta$ and $1/s = 1/r - \delta/n$. Then T_δ^A is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.*

Theorem 2. *Let $0 \le \delta < n$ and $D^\beta A_j \in BMO(R^n)$ for all β with $|\beta| = m_j$ and $j=1,\dots,l$. Suppose that $\tilde{T}_\delta^A$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r,s \in (1,+\infty]$ with $1<r<n/\delta$ and $1/s = 1/r - \delta/n$. Then $\tilde{T}_\delta^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.*

Theorem 3. *Let $0 \le \delta < n$, $1<p<n/\delta$ and $D^\beta A_j \in BMO(R^n)$ for all β with $|\beta| = m_j$ and $j=1,\dots,l$. Suppose that T_δ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r,s \in (1,+\infty]$ with $1<r<n/\delta$ and $1/s = 1/r - \delta/n$. Then T_δ^A is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.*

Theorem 4. *Let $0 \le \delta < n$, $1<p<n/\delta$, $1/q = 1/p - \delta/n$ and $D^\beta A_j \in BMO(R^n)$ for all β with $|\beta| = m_j$ and $j=1,\dots,l$. Suppose that $\tilde{T}_\delta^A$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any $r,s \in (1,+\infty]$ with $1<r<n/\delta$ and $1/s = 1/r - \delta/n$. Then $\tilde{T}_\delta^A$ is bounded from $H\dot{K}_p(R^n)$ to $\dot{K}_q^\alpha(R^n)$ with $\alpha = n(1-1/p)$.*

Remark. Theorem 4 is also hold for nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 2. (see [6]) *Let A be a function on R^n and $D^\beta A \in L^q(R^n)$ for $|\beta| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T_\delta^A(f)(x) - C_Q| dx \le C||f||_{L^{n/\delta}}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A_j)_{\tilde{Q}} x^\beta$, then $R_m(A_j;x,y) = R_m(\tilde{A}_j;x,y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)_{\tilde{Q}}$ for $|\beta| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
& F_t^A(f)(x) \\
= & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f(y) dy \\
= & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_2(y) dy \\
& + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_1(y) dy \\
& - \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m} \\
& \times D^{\beta_1} \tilde{A}_1(y) F_t(x,y) f_1(y) dy \\
& - \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m} \\
& \times D^{\beta_2} \tilde{A}_2(y) F_t(x,y) f_1(y) dy \\
& + \sum_{|\beta_1|=m_1,\ |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!} \int_{R^n} \frac{(x-y)^{\beta_1+\beta_2} D^{\beta_1}\tilde{A}_1(y) D^{\beta_2}\tilde{A}_2(y)}{|x-y|^m} \\
& \times F_t(x,y) f_1(y) dy.
\end{aligned}$$

Then

$$\begin{aligned}
& \left| T_\delta^A(f)(x) - T_\delta^{\tilde{A}}(f_2)(x_0) \right| \\
= & \left| ||F_t^A(f)(x)|| - ||F_t^{\tilde{A}}(f_2)(x_0)|| \right|
\end{aligned}$$

$$
\begin{aligned}
&\le ||F_t^A(f)(x) - F_t^{\tilde{A}}(f_2)(x_0)|| \\
&\le \left|\left|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y)f_1(y)dy\right|\right| \\
&\quad + \left|\left|\sum_{|\beta_1|=m_1} \frac{1}{\beta_1!}\int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m} D^{\beta_1}\tilde{A}_1(y)F_t(x,y)f_1(y)dy\right|\right| \\
&\quad + \left|\left|\sum_{|\beta_2|=m_2} \frac{1}{\beta_2!}\int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m} D^{\beta_2}\tilde{A}_2(y)F_t(x,y)f_1(y)dy\right|\right| \\
&\quad + \left|\left|\sum_{|\beta_1|=m_1,\ |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!}\right.\right. \\
&\qquad \times \left.\left.\int_{R^n} \frac{(x-y)^{\beta_1+\beta_2} D^{\beta_1}\tilde{A}_1(y)D^{\beta_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y)f_1(y)dy\right|\right| \\
&\quad + \left|T_\delta^{\tilde{A}}(f_2)(x) - T_\delta^{\tilde{A}}(f_2)(x_0)\right| \\
&:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\end{aligned}
$$

thus,

$$
\begin{aligned}
& \frac{1}{|Q|}\int_Q \left|T_\delta^A(f)(x) - T_\delta^{\tilde{A}}(f_2)(x_0)\right| dx \\
\le\ & \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx \\
& + \frac{1}{|Q|}\int_Q I_4(x)dx + \frac{1}{|Q|}\int_Q I_5(x)dx \\
:=\ & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. For I_1, by Lemma 2, we get, for $x \in Q$ and $y \in \tilde{Q}$,

$$
R_{m_j}(\tilde{A}_j;x,y) \le C|x-y|^{m_j} \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO},
$$

thus, by $(L^{n/\delta}, L^\infty)$-boundedness of T_δ, we get

$$
\begin{aligned}
I_1 &\le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) \frac{1}{|Q|}\int_Q |T_\delta(f_1)(x)|dx \\
&\le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) ||T_\delta(f_1)||_{L^\infty}
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};$$

For I_2, by (L^p, L^q)-boundedness of T_δ for $1/q = 1/p - \delta/n$, $n/\delta > p > 1$ and Hölder'inequality, we get

$$\begin{aligned}
I_2 &\leq C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\sum_{|\beta_1|=m_1}\frac{1}{|Q|}\int_Q|T_\delta(D^{\beta_1}\tilde{A}_1f_1)(x)|dx\\
&\leq C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\\
&\quad\times\sum_{|\beta_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\beta_1}\tilde{A}_1f_1)(x)|^qdx\right)^{1/q}\\
&\leq C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\\
&\quad\times\sum_{|\beta_1|=m_1}|Q|^{-1/q}\left(\int_{R^n}|D^{\beta_1}\tilde{A}_1(x)f_1(x)|^pdx\right)^{1/p}\\
&\leq C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\\
&\quad\times\sum_{|\beta_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\beta_1}A_1(x)-(D^{\beta_1}A_1)_{\tilde{Q}}|^qdx\right)^{1/q}||f||_{L^{n/\delta}}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};$$

Similarly, for I_4, choose $1 < p < n/\delta$ and $q, r_1, r_2 > 1$ such that $1/q = 1/p-\delta/n$ and $1/r_1 + 1/r_2 + p\delta/n = 1$, we obtain, by Hölder'inequality,

$$\begin{aligned}
&I_4\\
&\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\frac{1}{|Q|}\int_Q|T_\delta(D^{\beta_1}\tilde{A}_1D^{\beta_2}\tilde{A}_2f_1)(x)|dx\\
&\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\beta_1}\tilde{A}_1D^{\beta_2}\tilde{A}_2f_1)(x)|^qdx\right)^{1/q}
\end{aligned}$$

$$\le C \sum_{|\beta_1|=m_1,|\beta_2|=m_2} |Q|^{-1/q} \left(\int_{R^n} |D^{\beta_1}\tilde{A}_1(x)D^{\beta_2}\tilde{A}_2(x)f_1(x)|^p dx\right)^{1/p}$$

$$\le C \sum_{|\beta_1|=m_1,|\beta_2|=m_2} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\beta_1}\tilde{A}_1(x)|^{pr_1} dx\right)^{1/pr_1}$$

$$\times \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\beta_2}\tilde{A}_2(x)|^{pr_2} dx\right)^{1/pr_2} ||f||_{L^{n/\delta}}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) ||f||_{L^{n/\delta}};$$

For I_5, we write

$$F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)$$

$$= \int_{R^n} \left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x_0,y)}{|x_0-y|^m}\right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y) f_2(y)dy$$

$$+ \int_{R^n} \left(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\right) \frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m} F_t(x_0,y) f_2(y)dy$$

$$+ \int_{R^n} \left(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\right) \frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m} F_t(x_0,y) f_2(y)dy$$

$$- \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m} F_t(x,y) - \frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\beta_1}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\beta_1}\tilde{A}_1(y) f_2(y)dy$$

$$- \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m} F_t(x,y) - \frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\beta_2}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\beta_2}\tilde{A}_2(y) f_2(y)dy$$

$$+ \sum_{|\beta_1|=m_1,\ |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!} \int_{R^n} \left[\frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m} F_t(x,y) - \frac{(x_0-y)^{\beta_1+\beta_2}}{|x_0-y|^m} F_t(x_0,y)\right] D^{\beta_1}\tilde{A}_1(y) D^{\beta_2}\tilde{A}_2(y) f_2(y)dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 1 and the following inequality(see [16])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned} |R_{m_j}(\tilde{A}_j; x, y)| &\le C|x-y|^{m_j} \sum_{|\beta|=m_j} (||D^\beta A_j||_{BMO} \\ &\quad +|(D^\beta A_j)_{\tilde{Q}(x,y)} - (D^\beta A_j)_{\tilde{Q}}|) \\ &\le Ck|x-y|^{m_j} \sum_{|\beta|=m_j} ||D^\beta A_j||_{BMO}. \end{aligned}$$

Note that $|x-y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of F_t,

$$\begin{aligned} &||I_5^{(1)}|| \\ \le\ & C\int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \\ & \times \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)||f_2(y)|dy \\ \le\ & C\prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \\ & \times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f(y)|dy \\ \le\ & C\prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k})||f||_{L^{n/\delta}} \\ \le\ & C\prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}; \end{aligned}$$

For $I_5^{(2)}$, by the formula (see [6]):

$$R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y) = \sum_{|\gamma|<m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{A}_j; x, x_0)(x-y)^\gamma$$

and Lemma 1, we have

$$\begin{aligned} &|R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y)| \\ &\le C \sum_{|\gamma|<m_j} \sum_{|\beta|=m_j} |x-x_0|^{m_j-|\gamma|}|x-y|^{|\gamma|}||D^\beta A_j||_{BMO}, \end{aligned}$$

thus

$$\begin{aligned}||I_5^{(2)}|| &\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\\ &\quad\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|dy\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};\end{aligned}$$

Similarly,

$$||I_5^{(3)}||\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};$$

For $I_5^{(4)}$, taking $r>1$ such that $1/r+\delta/n=1$, then

$$\begin{aligned}||I_5^{(4)}|| &\le C\sum_{|\beta_1|=m_1}\int_{R^n}\left|\left|\frac{(x-y)^{\beta_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\beta_1}F_t(x_0,y)}{|x_0-y|^m}\right|\right|\\ &\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\beta_1}\tilde{A}_1(y)||f_2(y)|dy\\ &\quad+C\sum_{|\beta_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\ &\quad\times\frac{||(x_0-y)^{\beta_1}F_t(x_0,y)||}{|x_0-y|^m}|D^{\beta_1}\tilde{A}_1(y)||f_2(y)|dy\\ &\le C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\sum_{|\beta_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\\ &\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\beta_1}\tilde{A}_1(y)|^r dy\right)^{1/r}||f||_{L^{n/\delta}}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};\end{aligned}$$

Similarly,

$$||I_5^{(5)}||\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{L^{n/\delta}};$$

For $I_5^{(6)}$, taking $r_1, r_2 > 1$ such that $\delta/n + 1/r_1 + 1/r_2 = 1$, then

$$\begin{aligned}
&||I_5^{(6)}|| \\
&\le C \sum_{|\beta_1|=m_1,|\beta_2|=m_2} \int_{R^n} \left| \left| \frac{(x-y)^{\beta_1+\beta_2} F_t(x,y)}{|x-y|^m} \right. \right. \\
&\qquad \left. \left. - \frac{(x_0-y)^{\beta_1+\beta_2} F_t(x_0,y)}{|x_0-y|^m} \right| \right| |D^{\beta_1} \tilde{A}_1(y)| |D^{\beta_2} \tilde{A}_2(y)| |f_2(y)| dy \\
&\le C \sum_{|\beta_1|=m_1,|\beta_2|=m_2} \sum_{k=1}^{\infty} (2^{-k} + 2^{-\varepsilon k}) ||f||_{L^{n/\delta}} \\
&\times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\beta_1} \tilde{A}_1(y)|^{r_1} dy \right)^{1/r_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\beta_2} \tilde{A}_2(y)|^{r_2} dy \right)^{1/r_2} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}};
\end{aligned}$$

Thus

$$I_5 \le C \prod_{j=1}^{2} \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. It is only to show that there exists a constant $C > 0$ such that for every H^1-atom a (that is that a satisfies: supp$a \subset Q = Q(x_0, d)$, $||a||_{L^\infty} \le |Q|^{-1}$ and $\int a(y)dy = 0$ (see [9])), the following holds:

$$||\tilde{T}_\delta^A(a)||_{L^{n/(n-\delta)}} \le C.$$

Without loss of generality, we may assume $l = 2$. Write

$$\begin{aligned}
&\int_{R^n} \left[\tilde{T}_\delta^A(a)(x) \right]^{n/n-\delta} dx \\
&= \left[\int_{|x-x_0|\le 2d} + \int_{|x-x_0|>2d} \right] \left[\tilde{T}_\delta^A(a)(x) \right]^{n/(n-\delta)} dx := J_1 + J_2.
\end{aligned}$$

For J_1, by the (L^p, L^q)-boundedness of $\tilde{T}_\delta^A$ for $1/q = 1/p - \delta/n$, $n/\delta > p > 1$, we get

$$\begin{aligned}
J_1 &\le C ||\tilde{T}_\delta^A(a)||_{L^q}^{n/((n-\delta)q)} |2Q|^{1-n/((n-\delta)q)} \\
&\le C ||a||_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \le C.
\end{aligned}$$

To obtain the estimate of J_2, we denote that

$$\tilde{A}_j(x) = A_j(x) - \sum_{|\beta_j|=m_j} \frac{1}{\beta!} (D^\beta A_j)_{2Q} x^\beta.$$

Then $Q_{m_j}(A_j;x,y)=Q_{m_j}(\tilde{A}_j;x,y)$. We write, by the vanishing moment of a,

$$
\begin{aligned}
&\tilde{F}_t^A(a)(x)\\
=&\int_{R^n}\left[\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x,x_0)}{|x-x_0|^m}\right]R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)a(y)dy\\
&+\int_{R^n}\frac{F_t(x,x_0)}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)\\
&\qquad -R_{m_1}(\tilde{A}_1;x,x_0)R_{m_2}(\tilde{A}_2;x,x_0)]a(y)dy\\
&-\sum_{|\beta_2|=m_2}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_2}}{|x-y|^m}-\frac{F_t(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m}\right]\\
&\qquad\times R_{m_1}(\tilde{A}_1;x,y)D^{\beta_2}\tilde{A}_2(x)a(y)dy\\
&-\sum_{|\beta_2|=m_2}\int_{R^n}\frac{F_t(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x,x_0)]\\
&\qquad\times D^{\beta_2}\tilde{A}_2(x)a(y)dy\\
&-\sum_{|\beta_1|=m_1}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_1}}{|x-y|^m}-\frac{F_t(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m}\right]\\
&\qquad\times R_{m_2}(\tilde{A}_2;x,y)D^{\beta_1}\tilde{A}_1(x)a(y)dy\\
&-\sum_{|\beta_1|=m_1}\int_{R^n}\frac{F_t(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m}[R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x,x_0)]\\
&\qquad\times D^{\beta_1}\tilde{A}_1(x)a(y)dy\\
&+\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m}-\frac{K(x,x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m}\right]\\
&\qquad\times D^{\beta_1}\tilde{A}_1(x)D^{\beta_2}\tilde{A}_2(x)a(y)dy,
\end{aligned}
$$

similar to the proof of Theorem 1, we obtain

$$
\begin{aligned}
J_2\le& C\left[\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\right]^{n/(n-\delta)}\\
&\times\sum_{k=1}^{\infty}k^2[2^{-kn/(n-\delta)}+2^{-kn\varepsilon/(n-\delta)}]\\
\le& C.
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|}\int_Q|T_\delta^A(f)(x)-C_Q|dx\le C||f||_{B_p^\delta}$$

holds for any cube $Q = Q(0,d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A_j)_{\tilde{Q}} x^\beta$, then $R_m(A_j;x,y) = R_m(\tilde{A}_j;x,y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)_{\tilde{Q}}$ for $|\beta| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$
\begin{aligned}
&F_t^A(f)(x)\\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f(y)dy\\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_2(y)dy\\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_1(y)dy\\
&\quad - \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m} D^{\beta_1}\tilde{A}_1(y) F_t(x,y) f_1(y)dy\\
&\quad - \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m} D^{\beta_2}\tilde{A}_2(y) F_t(x,y) f_1(y)dy\\
&\quad + \sum_{|\beta_1|=m_1,\ |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!}\\
&\qquad\qquad \times \int_{R^n} \frac{(x-y)^{\beta_1+\beta_2} D^{\beta_1}\tilde{A}_1(y) D^{\beta_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y) f_1(y)dy,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q \left| T_\delta^A(f)(x) - T_\delta^{\tilde{A}}(f_2)(0) \right| dx\\
&\le \frac{1}{|Q|}\int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m} F_t(x,y) f_1(y)dy \right| dx\\
&\quad + \frac{1}{|Q|}\int_Q \Bigg| \sum_{|\beta_1|=m_1} \frac{1}{\beta_1!}\\
&\qquad\qquad \times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m} D^{\beta_1}\tilde{A}_1(y) F_t(x,y) f_1(y)dy \Bigg| dx\\
&\quad + \frac{1}{|Q|}\int_Q \Bigg| \sum_{|\beta_2|=m_2} \frac{1}{\beta_2!}
\end{aligned}
$$

$$\times \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m} D^{\beta_2}\tilde{A}_2(y)F_t(x,y)f_1(y)dy\Bigg|\Bigg| dx$$

$$+ \frac{1}{|Q|}\int_Q \Bigg|\Bigg| \sum_{|\beta_1|=m_1,\ |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!}$$

$$\times \int_{R^n} \frac{(x-y)^{\beta_1+\beta_2}D^{\beta_1}\tilde{A}_1(y)D^{\beta_2}\tilde{A}_2(y)}{|x-y|^m} F_t(x,y)f_1(y)dy\Bigg|\Bigg| dx$$

$$+ \frac{1}{|Q|}\int_Q \left|T_\delta^{\tilde{A}}(f_2)(x) - T_\delta^{\tilde{A}}(f_2)(0)\right| dx$$

$$:= L_1 + L_2 + L_3 + L_4 + L_5.$$

Similar to the proof of Theorem 1, we get, for $1/s = 1/r - \delta/n$, $1 < r < p$, $1 < t_1, t_2 < \infty$ and $1/t_1 + 1/t_2 + r/p = 1$,

$$L_1 \le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) \frac{1}{|Q|}\int_Q |T_\delta(f_1)(x)|dx$$

$$\le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) \left(\frac{1}{|Q|}\int_Q |T_\delta(f_1)(x)|^q dx\right)^{1/q}$$

$$\le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) d^{-n(1/p-\delta/n)}||f\chi_{\tilde{Q}}||_{L^p}$$

$$\le C\prod_{j=1}^2 \left(\sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO}\right) ||f||_{B_p^\delta};$$

$$L_2 \le C\sum_{|\beta_2|=m_2} ||D^{\beta_2}A_2||_{BMO} \sum_{|\beta_1|=m_1} \frac{1}{|Q|}\int_Q |T_\delta(D^{\beta_1}\tilde{A}_1 f_1)(x)|dx$$

$$\le C\sum_{|\beta_2|=m_2} ||D^{\beta_2}A_2||_{BMO} \sum_{|\beta_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} |T_\delta(D^{\beta_1}\tilde{A}_1 f_1)(x)|^s dx\right)^{1/s}$$

$$\le C\sum_{|\beta_2|=m_2} ||D^{\beta_2}A_2||_{BMO} \sum_{|\beta_1|=m_1} |Q|^{-1/s}||(D^{\beta_1}A_1 - (D^{\beta_1}A_1)_{\tilde{Q}})f_1||_{L^r}$$

$$\le C\sum_{|\beta_2|=m_2} ||D^{\beta_2}A_2||_{BMO}$$

$$\times \sum_{|\beta_1|=m_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\beta_1}\tilde{A}_1(y)|^{pr/(p-r)}dy\right)^{(p-r)/pr} |Q|^{\delta/n-1/p}||f_1||_{L^p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)d^{-n(1/p-\delta/n)}||f\chi_{\tilde{Q}}||_{L^p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$L_3\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$L_4\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\frac{1}{|Q|}\int_Q|T_\delta(D^{\beta_1}\tilde{A}_1D^{\beta_2}\tilde{A}_2f_1)(x)|dx$$

$$\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\beta_1}\tilde{A}_1D^{\beta_2}\tilde{A}_2f_1)(x)|^sdx\right)^{1/s}$$

$$\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}|Q|^{-1/s}\left(\int_{R^n}|D^{\beta_1}\tilde{A}_1(x)D^{\beta_2}\tilde{A}_2(x)f_1(x)|^rdx\right)^{1/r}$$

$$\leq C\sum_{|\beta_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\beta_1}\tilde{A}_1(x)|^{rt_1}dx\right)^{1/rt_1}$$

$$\times\sum_{|\beta_2|=m_2}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\beta_2}\tilde{A}_2(x)|^{rt_2}dx\right)^{1/rt_2}|Q|^{\delta/n-1/p}||f_1||_{L^p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

For L_5, we write, for $x\in Q$,

$$F_t^{\tilde{A}}(f_2)(x)-F_t^{\tilde{A}}(f_2)(0)$$

$$=\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(0,y)}{|y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)f_2(y)dy$$

$$+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|y|^m}F_t(0,y)f_2(y)dy$$

$$+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y)\right)\frac{R_{m_1}(\tilde{A}_1;0,y)}{|y|^m}F_t(0,y)f_2(y)dy$$

$$-\sum_{|\beta_1|=m_1}\frac{1}{\beta_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\beta_1}}{|x-y|^m}F_t(x,y)\right.$$

$$-\frac{R_{m_2}(\tilde{A}_2;0,y)(-y)^{\beta_1}}{|y|^m}F_t(0,y)\Bigg]D^{\beta_1}\tilde{A}_1(y)f_2(y)dy$$

$$-\sum_{|\beta_2|=m_2}\frac{1}{\beta_2!}\int_{R^n}\Bigg[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\beta_2}}{|x-y|^m}F_t(x,y)$$

$$-\frac{R_{m_1}(\tilde{A}_1;0,y)(-y)^{\beta_2}}{|y|^m}F_t(0,y)\Bigg]D^{\beta_2}\tilde{A}_2(y)f_2(y)dy$$

$$+\sum_{|\beta_1|=m_1,\ |\beta_2|=m_2}\frac{1}{\beta_1!\beta_2!}\int_{R^n}\left[\frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m}F_t(x,y)-\frac{(-y)^{\beta_1+\beta_2}}{|y|^m}F_t(0,y)\right]$$

$$\times D^{\beta_1}\tilde{A}_1(y)D^{\beta_2}\tilde{A}_2(y)f_2(y)dy$$

$$=L_5^{(1)}+L_5^{(2)}+L_5^{(3)}+L_5^{(4)}+L_5^{(5)}+L_5^{(6)};$$

Similar to the proof of Theorem 1, we get, for $1<r_1,r_2<\infty$ and $1/r_1+1/r_2+1/p=1$,

$$||L_5^{(1)}||\le C\int_{R^n}\left(\frac{|x|}{|y|^{m+n+1-\delta}}+\frac{|x|^\varepsilon}{|y|^{m+n+\varepsilon-\delta}}\right)\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||f_2(y)|dy$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)$$

$$\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^\varepsilon}{|y|^{n+\varepsilon-\delta}}\right)|f(y)|dy$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)$$

$$\times\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})(2^kd)^{-n(1/p-\delta/n)}||f\chi_{2^k\tilde{Q}}||_{L^p}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})||f||_{B_p^\delta}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$||L_5^{(2)}||\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x|}{|y|^{n+1-\delta}}|f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$||L_5^{(3)}||\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$||L_5^{(4)}||\leq C\sum_{|\beta_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\beta_1}F_t(x,y)}{|x-y|^m}-\frac{(-y)^{\beta_1}F_t(0,y)}{|y|^m}\right|$$
$$\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\beta_1}\tilde{A}_1(y)||f_2(y)|dy$$

$$+C\sum_{|\beta_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y)|$$
$$\times\frac{|(-y)^{\beta_1}F_t(0,y)|}{|y|^m}|D^{\beta_1}\tilde{A}_1(y)||f_2(y)|dy$$
$$\leq C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}$$
$$\times\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})(2^kd)^{-n(1/p-\delta/n)}||f\chi_{2^k\tilde{Q}}||_{L^p}$$
$$\times\sum_{|\beta_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\beta_1}\tilde{A}_1(y)|^{p'}dy\right)^{1/p'}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$||L_5^{(5)}||\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

$$||L_5^{(6)}||\leq C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}\left|\frac{(x-y)^{\beta_1+\beta_2}F_t(x,y)}{|x-y|^m}-\frac{(-y)^{\alpha_1+\alpha_2}F_t(0,y)}{|y|^m}\right|$$
$$\times|D^{\beta_1}\tilde{A}_1(y)||D^{\beta_2}\tilde{A}_2(y)||f_2(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})(2^kd)^{-n(1/p-\delta/n)}||f\chi_{2^k\tilde{Q}}||_{L^p}$$
$$\times\sum_{|\beta_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\beta_1}\tilde{A}_1(y)|^{r_1}dy\right)^{1/r_1}$$
$$\times\sum_{|\beta_2|=m_2}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\beta_2}\tilde{A}_2(y)|^{r_2}dy\right)^{1/r_2}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta};$$

Thus

$$L_5\leq C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{B_p^\delta}.$$

This finishes the proof of Theorem 3. □

Proof of Theorem 4. Without loss of generality, we may assume $l = 2$. Let $f \in H\dot{K}_p(R^n)$, by Lemma 1, $f = \sum_{j=-\infty}^{\infty}\lambda_j a_j$, where $a_j's$ are the central $(n(1-1/p),p)$-atom with supp$a_j \subset B_j = B(0,2^j)$ and $||f||_{H\dot{K}_p} \sim \sum_j |\lambda_j|$. We write

$$\begin{aligned}
||\tilde{T}_\delta^A(f)||_{\dot{K}_q^\alpha} &= \sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}||\chi_k\tilde{T}_\delta^A(f)||_{L^q}\\
&\leq \sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j|||\chi_k\tilde{T}_\delta^A(a_j)||_{L^q}\\
&\quad+\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=k}^{\infty}|\lambda_j|||\chi_k\tilde{T}_\delta^A(a_j)||_{L^q}=M_1+M_2.
\end{aligned}$$

For M_2, by the (L^p,L^q)-boundedness of $\tilde{T}_\delta^A$ for $1/q = 1/p-\delta/n$, we get

$$\begin{aligned}
M_2 &\leq C\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=k}^{\infty}|\lambda_j|||a_j||_{L^p}\\
&\leq C\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=k}^{\infty}|\lambda_j|2^{jn(1/p-1)}\\
&\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=-\infty}^{j}2^{(k-j)n(1-1/p)}\\
&\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|\leq C||f||_{H\dot{K}_p};
\end{aligned}$$

To obtain the estimate of M_1, we denote that

$$\tilde{A}_j(x)=A_j(x)-\sum_{|\beta_j|=m_j}\frac{1}{\beta!}(D^\beta A_j)_{2Q}x^\beta.$$

Then $Q_{m_j}(A_j;x,y)=Q_{m_j}(\tilde{A}_j;x,y)$. We write, by the vanishing moment of a,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x)\\
&=\int_{R^n}\left[\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x,0)}{|x|^m}\right]R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)a(y)dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{R^n}\frac{F_t(x,0)}{|x|^m}[R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)\\
&\qquad\qquad - R_{m_1}(\tilde{A}_1;x,0)R_{m_2}(\tilde{A}_2;x,0)]a(y)dy\\
&-\sum_{|\beta_2|=m_2}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_2}}{|x-y|^m}-\frac{F_t(x,0)x^{\beta_2}}{|x|^m}\right]\\
&\qquad \times R_{m_1}(\tilde{A}_1;x,y)D^{\beta_2}\tilde{A}_2(x)a(y)dy\\
&-\sum_{|\beta_2|=m_2}\int_{R^n}\frac{F_t(x,0)x^{\beta_2}}{|x|^m}[R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x,0)]\\
&\qquad \times D^{\beta_2}\tilde{A}_2(x)a(y)dy\\
&-\sum_{|\beta_1|=m_1}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_1}}{|x-y|^m}-\frac{F_t(x,0)x^{\beta_1}}{|x|^m}\right]\\
&\qquad \times R_{m_2}(\tilde{A}_2;x,y)D^{\beta_1}\tilde{A}_1(x)a(y)dy\\
&-\sum_{|\beta_1|=m_1}\int_{R^n}\frac{F_t(x,0)x^{\beta_1}}{|x|^m}[R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x,0)]\\
&\qquad \times D^{\beta_1}\tilde{A}_1(x)a(y)dy\\
&+\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m}-\frac{F_t(x,0)x^{\beta_1+\beta_2}}{|x|^m}\right]\\
&\qquad \times D^{\beta_1}\tilde{A}_1(x)D^{\beta_2}\tilde{A}_2(x)a(y)dy.
\end{aligned}
$$

Similar to the proof of Theorem 1 and Theorem 2, we get

$$
\begin{aligned}
M_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\\
&\quad\times\sum_{k=-\infty}^{\infty}2^{kn(1-\delta/n)}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right]\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=j+1}^{\infty}[2^{j-k}+2^{(j-k)\varepsilon}]\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)\sum_{j=-\infty}^{\infty}|\lambda_j|\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\beta_j|=m_j}||D^{\beta_j}A_j||_{BMO}\right)||f||_{H\dot{K}_p}.
\end{aligned}
$$

This completes the proof of Theorem 4. □

4. Applications

Now we shall apply the theorems of the paper to some particular operators such as Littlewood-Paley operators and Marcinkiewicz operators.

Application 1. Littlewood-Paley operator.

Fixed $0 \le \delta < n$, $\varepsilon > 0$ and $\mu > (3n+2-2\delta)/n$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y) f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. The variants of g_ψ^A, S_ψ^A and g_μ^A are defined by

$$\tilde{g}_\psi^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$\tilde{S}_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$\tilde{g}_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y) f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y)=\int_{R^n}\frac{\prod_{j=1}^{l}Q_{m_j+1}(A_j;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz.$$

Set $F_t(f)(y)=f*\psi_t(y)$. We also define that

$$g_\psi(f)(x)=\left(\int_0^\infty|F_t(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x)=\left(\int\int_{\Gamma(x)}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x)=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [17]). Let H be the space

$$H=\left\{h:||h||=\left(\int_0^\infty|h(t)|^2dt/t\right)^{1/2}<\infty\right\}$$

or

$$H=\left\{h:||h||=\left(\int\int_{R_+^{n+1}}|h(y,t)|^2dydt/t^{n+1}\right)^{1/2}<\infty\right\},$$

then, for each fixed $x\in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x)=||F_t^A(f)(x)||,\quad g_\psi(f)(x)=||F_t(f)(x)||,$$

$$S_\psi^A(f)(x)=\left||\chi_{\Gamma(x)}F_t^A(f)(x,y)\right||,\quad S_\psi(f)(x)=\left||\chi_{\Gamma(x)}F_t(f)(y)\right||$$

and

$$g_\mu^A(f)(x)=\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x)=\left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(f)(y)\right|\right|.$$

It is easily to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorem 1, 2, 3 and 4, thus the conclusions of Theorem 1, 2, 3 and 4 hold for g_ψ^A and $\tilde{g}_\psi^A$, S_ψ^A and $\tilde{S}_\psi^A$, g_μ^A and $\tilde{g}_\mu^A$.

Application 2. Marcinkiewicz operator.

Fixed $0\le\delta<n$, Fix $\lambda>\max(1,2n/(n+2-2\delta))$ and $0<\gamma\le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$. Assume that $\Omega\in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by (see [12], [18])

$$\mu_\Omega^A(f)(x)=\left(\int_0^\infty|F_t^A(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)dz;$$

The variants of μ_Ω^A, μ_S^A and μ_λ^A are defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\tilde{\mu}_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\tilde{\mu}_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators(see [18]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left||\chi_{\Gamma(x)} F_t^A(f)(x,y)\right||, \quad \mu_S(f)(x) = \left||\chi_{\Gamma(x)} F_t(f)(y)\right||$$

and

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|,$$

$$\mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

It is easily to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorem 1, 2, 3 and 4, thus Theorem 1, 2, 3 and 4 hold for μ_Ω^A and $\tilde{\mu}_\Omega^A$, μ_S^A and $\tilde{\mu}_S^A$, μ_λ^A and $\tilde{\mu}_\lambda^A$.

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