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GROUP DETERMINANT FORMULAS AND CLASS NUMBERS OF CYCLOTOMIC FIELDS

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ABSTRACT. Let m, n be positive integers or monic polynomials in $\mathbb{F}_q[T]$ with $n|m$. Let K_m and K_m^+ be the m-th cyclotomic field and its maximal real subfield, respectively. In this paper we define two matrices $\mathcal{D}_{m,n}^+$ and $\mathcal{D}_{m,n}^-$ whose determinants give us the ratios $\frac{h(\mathcal{O}_{K_m^+})}{h(\mathcal{O}_{K_n^+})}$ and $\frac{h^-(\mathcal{O}_{K_m})}{h^-(\mathcal{O}_{K_n})}$ with some factors, respectively.

1. Introduction

In [3], Girstmair gave a recursion formula for the relative class numbers in the tower of cyclotomic number fields with prime power conductor. Adopting the idea of Girstmair, Jung and Ahn [4] gave an analogous result in the tower of cyclotomic function fields with prime power conductor. In this paper we generalize the results of Girstmair and Jung-Ahn to the tower of any two cyclotomic fields with arbitrary conductor. We treat both cyclotomic number field and cyclotomic function field cases in a unique way.

The layout of this paper is as follows. In section 2, we obtain a group determinant formula (Theorem 2.4) which is a generalization of [6, Lemma 2], [5, Lemma 2.1] and [1, Prop. 2.1]. Let m, n be positive integers or monic polynomials in $\mathbb{F}_q[T]$ with $n|m$. In section 3, we define two matrices $\mathcal{D}_{m,n}^+$ and $\mathcal{D}_{m,n}^-$ whose determinants give us the ratios $\frac{h(\mathcal{O}_{K_m^+})}{h(\mathcal{O}_{K_n^+})}$ and $\frac{h^-(\mathcal{O}_{K_m})}{h^-(\mathcal{O}_{K_n})}$ with some factors, respectively. In section 4, we restrict ourselves to the prime power conductor case that $m = p^\ell$ and $n = p^{\ell-1}$. We show that the recursion formulas of Girstmair ([3, Theorem]) and Jung-Ahn ([4, Thm. 4.1]) are immediately obtained from Theorem 3.3. We also obtain the recursion formula between $h(\mathcal{O}_{K_m^+})$ and $h(\mathcal{O}_{K_n^+})$, and the one between $h(K_m^+)$ and $h(K_n^+)$ in function field case.

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2. Group determinant formulas

Let G be a finite abelian group of order n. Let $L^2(G)$ be the n-dimensional vector space of complex-valued functions on G. Let $\widehat{G}$ be the character group of G with values in $\mathbb{C}$. Let $\chi_0 \in \widehat{G}$ be the trivial character. For any subgroup H of G, we define $L^2(G)^H = \{f \in L^2(G) : f(\sigma\tau) = f(\sigma) \text{ for all } \sigma \in G \text{ and } \tau \in H\}$ and $\widehat{G}^H = \{\chi \in \widehat{G} : \chi(\sigma) = 1 \text{ for all } \sigma \in H\}$. For any $f \in L^2(G)$, we define $s_f^H \in L^2(G)^H$ by $s_f^H(\sigma) = \sum_{\tau \in H} f(\sigma\tau)$ for $\sigma \in G$. For any subset M of G, let $s(M) = \sum_{\sigma \in M} \sigma$ and $e_M = s(M)/|M|$. Let $e_\chi = \frac{1}{|G|}\sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathbb{C}[G]$ be the idempotent element associated to $\chi \in \widehat{G}$.

For a subspace L of $\mathbb{C}[G]$ and a subgroup H of G, we define two subspaces $L^H = \{x \in L : e_H x = x\}$ and $L_H = \{x \in L : e_H x = 0\}$. For any two subgroups H, H' of G, we denote $(L^H)_{H'}$ by $L^H_{H'}$ and $(L_H)_{H'}$ by $L_{H,H'}$. It is easy to see that $L^H_{H'} = L^H_{HH'}$. We are interested in the structures of $\mathbb{C}[G]^H_{H'}$ and $\mathbb{C}[G]_{H,H'}$ as $\mathbb{C}$-vector spaces.

For any subgroup H of G, we denote by $\mathcal{R}_{G/H}$ any system of representatives of G/H and when $\mathcal{R}_{G/H}$ is fixed, we define a function $r_H : G \to G$ such that $r_H(\sigma)H = \sigma H$ with $r_H(\sigma) \in \mathcal{R}_{G/H}$ for each $\sigma \in G$. From [1, Prop. 2.1], we know that

Proposition 2.1. *For any two subgroups H, H' of G, we take $\mathcal{R}_{G/H}$ and $\mathcal{R}_{G/HH'}$ with $\mathcal{R}_{G/HH'} \subseteq \mathcal{R}_{G/H}$. Then we have*

(i) $\dim_{\mathbb{C}} \mathbb{C}[G]^H_{H'} = |G/H| - |G/HH'|$,

(ii) $\{e_\chi : \chi \in \widehat{G}^H \backslash \widehat{G}^{HH'}\}$ *and* $\{(\sigma - r_{HH'}(\sigma))s(H) : \sigma \in \mathcal{R}_{G/H} \backslash \mathcal{R}_{G/HH'}\}$ *are two $\mathbb{C}$-bases of $\mathbb{C}[G]^H_{H'}$, and*

(iii) *for any $f \in L^2(G)$,*

$$\prod_{\chi \in \widehat{G}^H \backslash \widehat{G}^{HH'}} \sum_{\sigma \in G} \chi(\sigma) f(\sigma) = \det\left(\varepsilon_{\sigma,\tau} : \sigma, \tau \in \mathcal{R}_{G/H} \backslash \mathcal{R}_{G/HH'}\right), \tag{2.1}$$

where $\varepsilon_{\sigma,\tau} = s_f^H(\sigma\tau^{-1}) - s_f^H(\sigma r_{HH'}(\tau)^{-1})$.

Since $\mathbb{C}[G]_{H,H'} = \mathbb{C}[G]_H \cap \mathbb{C}[G]_{H'}$ and $\mathbb{C}[G]_H + \mathbb{C}[G]_{H'} = \mathbb{C}[G]_{HH'}$, we have $\dim_{\mathbb{C}} \mathbb{C}[G]_{H,H'} = |G| - |G/H| - |G/H'| + |G/HH'|$. In fact, it is easy to see that $\{e_\chi : \chi \in \widehat{G} \backslash (\widehat{G}^H \cup \widehat{G}^{H'})\}$ is a basis for $\mathbb{C}[G]_{H,H'}$. We need another basis of $\mathbb{C}[G]_{H,H'}$ to obtain a group determinant formula for $\prod_{\chi \notin \widehat{G}^H \cup \widehat{G}^{H'}} \sum_{\sigma \in G} \chi(\sigma) f(\sigma)$.

We need the following lemma. The proof is an elementary one, so we leave it to the readers.

Lemma 2.2. *Let H, H' be two subgroups of G. Let $\mathcal{R}_{G/H}, \mathcal{R}_{G/H'}$ and $\mathcal{R}_{G/HH'}$ be systems of representatives of $G/H, G/H'$ and G/HH' respectively. Then the following conditions are equivalent.*

(i) $\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma) = 0$ *for* $\sigma \in \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}$.

(ii) $r_{H'}(\sigma) = r_{HH'}(\sigma)$ *for* $\sigma \in \mathcal{R}_{G/H}$ *and* $r_H(\sigma) = r_{HH'}(\sigma)$ *for* $\sigma \in \mathcal{R}_{G/H'}$.

(iii) $r_H \circ r_{H'} = r_{H'} \circ r_H = r_{HH'}$ *as functions from G to G.*

(iv) *For any* $\sigma \in \mathcal{R}_{G/H}$ *(resp.* $\tau \in \mathcal{R}_{G/H'}$*) there exists* $h' \in H'$ *(resp.* $h \in H$*) such that* $\sigma h' \in \mathcal{R}_{G/HH'}$ *(resp.* $\tau h \in \mathcal{R}_{G/HH'}$*) and* $\mathcal{R}_{G/HH'} \subset \mathcal{R}_{G/H} \cap \mathcal{R}_{G/H'}$.

Throughout the paper, we say that $\mathcal{R}_{G/H}, \mathcal{R}_{G/H'}$ and $\mathcal{R}_{G/HH'}$ satisfy the condition $(*)$ if they satisfy any (thus all) conditions of Lemma 2.2.

Proposition 2.3. *Let* H, H' *be two subgroups of* G. *Suppose that* $\mathcal{R}_{G/H}$, $\mathcal{R}_{G/H'}$ *and* $\mathcal{R}_{G/HH'}$ *satisfy the condition* $(*)$. *Then* $\{\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma) : \sigma \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}\}$ *is a* $\mathbb{C}$*-basis of* $\mathbb{C}[G]_{H,H'}$.

Proof. Let $X = \{\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma) : \sigma \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}\}$. By Lemma 2.2 (iii), we have $\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma) = \sigma - r_{H'}(\sigma) - (r_H(\sigma) - r_{H'}(r_H(\sigma)))$ and so $e_H(\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma)) = e_{H'}(\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma)) = 0$. Thus $X \subset \mathbb{C}[G]_{H,H'}$. Since $\dim_{\mathbb{C}} \mathbb{C}[G]_{H,H'} = |G| - |G/H| - |G/H'| + |G/HH'| = |X|$, it suffices to show that X generates $\mathbb{C}[G]_{H,H'}$. Let $x = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{C}[G]_{H,H'}$. Since $e_H x = e_{H'} x = 0$, we have that $\sum_{\tau \in H} a_{\sigma\tau} = \sum_{\tau' \in H'} a_{\sigma\tau'} = 0$ for any $\sigma \in G$. Let $\bar{x} = \sum_{\sigma \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}} a_\sigma(\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma))$. By Lemma 2.2 (i), we have

$$\begin{aligned}
\bar{x} &= \sum_{\sigma \in G} a_\sigma(\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma)) \\
&= x - \sum_{\sigma \in \mathcal{R}_{G/H}} \Big(\sum_{\tau \in H} a_{\sigma\tau}\Big)\sigma - \sum_{\sigma \in \mathcal{R}_{G/H'}} \Big(\sum_{\tau' \in H'} a_{\sigma\tau'}\Big)\sigma \\
&\quad + \sum_{\sigma \in \mathcal{R}_{G/HH'}} \Big(\sum_{\mu \in HH'} a_{\sigma\mu}\Big)\sigma \\
&= x.
\end{aligned}$$

Thus X generates $\mathbb{C}[G]_{H,H'}$. □

Theorem 2.4. *For any two subgroups* H, H' *of* G, *suppose that* $\mathcal{R}_{G/H}$, $\mathcal{R}_{G/H'}$ *and* $\mathcal{R}_{G/HH'}$ *satisfy the condition* $(*)$. *For any* $f \in L^2(G)$, *we have*

$$\prod_{\chi \notin \widehat{G}^H \cup \widehat{G}^{H'}} \sum_{\sigma \in G} \chi(\sigma) f(\sigma) = \det\left(\eta_{\sigma,\tau} : \sigma, \tau \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}\right), \tag{2.2}$$

where $\eta_{\sigma,\tau} = f(\sigma\tau^{-1}) - f(\sigma r_H(\tau)^{-1}) - f(\sigma r_{H'}(\tau)^{-1}) + f(\sigma r_{HH'}(\tau)^{-1})$.

Proof. Let Ψ be a linear transformation on $\mathbb{C}[G]_{H,H'}$ defined as multiplication by $\theta = \sum_{\sigma \in G} f(\sigma)\sigma$. Clearly the matrix of Ψ with respect to the basis $X_1 = \{e_\chi : \chi \notin \widehat{G}^H \cup \widehat{G}^{H'}\}$ is a diagonal matrix $\operatorname{diag}\left(\sum_{\sigma \in G} \chi(\sigma) f(\sigma) : \chi \notin \widehat{G}^H \cup \widehat{G}^{H'}\right)$. To compute the matrix of Ψ with respect to the basis $X_2 = \{\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma) : \sigma \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}\}$, we consider $\theta(\sigma - r_H(\sigma) - r_{H'}(\sigma) +$

$r_{HH'}(\sigma))$ with $\sigma \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}$. Then we have

$$\begin{aligned}
& \theta(\sigma - r_H(\sigma) - r_{H'}(\sigma) + r_{HH'}(\sigma)) \\
= & \sum_{g \in G} f(g)(g\sigma - g r_H(\sigma) - g r_{H'}(\sigma) + g r_{HH'}(\sigma)) \\
= & \sum_{\tau \in G} (f(\sigma^{-1}\tau) - f(r_H(\sigma)^{-1}\tau) - f(r_{H'}(\sigma)^{-1}\tau) + f(r_{HH'}(\sigma)^{-1}\tau))\tau \\
= & \sum_{\tau \notin \mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'}} (f(\sigma^{-1}\tau) - f(r_H(\sigma)^{-1}\tau) - f(r_{H'}(\sigma)^{-1}\tau) \\
& + f(r_{HH'}(\sigma)^{-1}\tau)) \times \ (\tau - r_H(\tau) - r_{H'}(\tau) + r_{HH'}(\tau)).
\end{aligned}$$

The last equality follows from the same argument in proof of Proposition 2.3. By comparing determinants of matrices of Ψ with respect to X_1 and X_2, we prove the theorem. □

Remark 2.5. For a subset M of G, we denote $M^{-1} = \{\sigma^{-1} : \sigma \in M\}$. For any $\mathcal{R}_{G/H}, \mathcal{R}_{G/H'}$ and $\mathcal{R}_{G/HH'}$ satisfying the condition $(*)$, $\mathcal{R}_{G/H}^{-1}, \mathcal{R}_{G/H'}^{-1}$ and $\mathcal{R}_{G/HH'}^{-1}$ also satisfy the condition $(*)$. As in the proof of [5, Lemma 2.1], for any $f \in L^2(G)$, we have

$$\prod_{\chi \notin \widehat{G}^H \cup \widehat{G}^{H'}} \sum_{\sigma \in G} \chi(\sigma) f(\sigma) = \pm \det \left(\tilde{\eta}_{\sigma,\tau} : \sigma, \tau \in G \backslash (\mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'})\right),$$

where $\tilde{\eta}_{\sigma,\tau} = f(\sigma\tau) - f(\sigma r_H(\tau)) - f(\sigma r_{H'}(\tau)) + f(\sigma r_{HH'}(\tau))$.

3. Ratio of class numbers of cyclotomic fields

In this section we give determinant formulas for the ratio of class numbers of cyclotomic fields in both number field and function field cases. We handle both number field and function field cases in a unique way.

First we denote $k = \mathbb{Q}, \mathbb{A} = \mathbb{Z}$ or $k = \mathbb{F}_q(T), \mathbb{A} = \mathbb{F}_q[T]$, where $\mathbb{F}_q$ is a finite field with q elements. Let $\mathbb{A}^+ = \{a \in \mathbb{A} : a$ is positive if $\mathbb{A} = \mathbb{Z}$ or a is monic if $\mathbb{A} = \mathbb{F}_q[T]\}$. For any $m \in \mathbb{A}^+$, we denote by K_m the m-th cyclotomic field in both number field and function field cases and by K_m^+ its maximal real subfield. Let $G_m = \mathrm{Gal}(K_m/k), G_m^+ = \mathrm{Gal}(K_m^+/k)$ and $J = \mathrm{Gal}(K_m/K_m^+)$. For the cyclotomic theory of function field, we refer the reader Rosen's book ([8, Chap. 12]). It is well known that $G_m \cong (\mathbb{A}/m\mathbb{A})^*$ in both number field and function field cases. For any $a \in \mathbb{A}$ with $(a, m) = 1$, we denote by σ_a the element of G_m which corresponds to a mod m under the above isomorphism. For any subset S of $\mathbb{A}$, we define $\sigma(S) = \{\sigma_a \in G_m : a \in S$ with $(a, m) = 1\}$. Let $\widehat{G}_m$ be the group of characters of G_m with values in $\mathbb{C}$. A character $\chi \in \widehat{G}_m$ is called *real* if $\chi(J) = \{1\}$ and *non-real* otherwise. Thus $\widehat{G}_m^J$ is the subgroup of all real characters of G_m and $\widehat{G}_m \backslash \widehat{G}_m^J$ is the set of all non-real characters of G_m. For $\chi \in \widehat{G}_m$ and $a \in \mathbb{A}$, we define $\chi(a)$ as follows. Let $f_\chi \in \mathbb{A}^+$

be the conductor of χ. If $(a, f_\chi) = 1$, we define $\chi(a) = \chi((a, K_{f_\chi}/k))$, where $(a, K_{f_\chi}/k)$ is the Artin symbol. If $(a, f_\chi) \neq 1$, we put $\chi(a) = 0$.

For $a, b \in \mathbb{F}_q[T]$, $a < b$ means that $\deg b > \deg a$ if a is nonzero or that b is nonzero if $a = 0$. For any $m \in \mathbb{A}^+$, we define $\mathbb{M}_m = \{a \in \mathbb{A} : 0 < a < m \text{ with } (a,m) = 1\}$ and $\mathbb{M}_m^+ = \{a \in \mathbb{M}_m : a < m/2\}$ if $k = \mathbb{Q}$ and $\mathbb{M}_m^+ = \mathbb{M}_m \cap \mathbb{A}^+$ if $k = \mathbb{F}_q(T)$. We also define $\mathbb{M}_m^- = \mathbb{M}_m \backslash \mathbb{M}_m^+$. Since $\mathbb{M}_m$ is a set of representatives of $(\mathbb{A}/m\mathbb{A})^*$, we have $\sigma(\mathbb{M}_m) = G_m$. It is easy to see that $\sigma(\mathbb{M}_m^+)$ is a set of representatives of G_m/J. For any $a \in \mathbb{A}$ with $(a,m) = 1$, let $(a)_m$ be the unique element of $\mathbb{M}_m$ such that $a \equiv (a)_m \mod m$. We define $\operatorname{sgn}_m(a)$ for $(a,m) = 1$ as follows. If $k = \mathbb{Q}$, define $\operatorname{sgn}_m(a) = 1$ if $(a)_m \in \mathbb{M}_m^+$ and -1 otherwise. If $k = \mathbb{F}_q(T)$, define $\operatorname{sgn}_m(a)$ as the leading coefficient of $(a)_m$. We also define $\deg_m(a) = \deg((a)_m)$.

For any nonzero $a, f \in \mathbb{A}$ with $(a, f) = 1$, we define

$$g_f^+(a) = \begin{cases} \log|1 - \zeta_f^a| & \text{if } k = \mathbb{Q}, \\ \varphi(a/f)/(q-1) & \text{if } k = \mathbb{F}_q(T), \end{cases}$$

and

$$g_f^-(a) = \begin{cases} \frac{1}{2} - \langle \frac{a}{f} \rangle & \text{if } k = \mathbb{Q}, \\ Z_f(0, a) & \text{if } k = \mathbb{F}_q(T), \end{cases}$$

where $\zeta_f = \exp(2\pi i/f), \langle a/f \rangle$ is the fractional part of a/f if $k = \mathbb{Q}$, and $\varphi(a/f) = (q-1)(\deg f - 1 - \deg_f(a)) - 1$, $Z_f(s, a)$ is the partial zeta function associated to the class of a in $Cl(\mathbb{A})$ if $k = \mathbb{F}_q(T)$. The notation "$f|n$" means that f divides n with $f, n \in \mathbb{A}^+$. Assume that we are given $\{a_{f,\sigma} \in \mathbb{C} : f|m \text{ with } f \neq 1 \text{ and } \sigma \in G_m\}$. For any nonzero $a \in \mathbb{A}$, let $\Phi(a)$ be the order of $(\mathbb{A}/a\mathbb{A})^*$. For $\chi \in \widehat{G}_m$ and $1 \neq f \in \mathbb{A}^+$ with $f|m$, we define $b_{f,\chi} = \frac{\Phi(m)}{\Phi(f)} \sum_{\sigma \in G_m} a_{f,\sigma}\chi(\sigma)$ and $c_\chi = \sum_{\substack{1 \neq f|m \\ f_\chi | f}} b_{f,\chi} \prod_{p|f} (1 - \overline{\chi}(p))$ where p ranges over all prime numbers (or all monic irreducible polynomials) dividing f. For any $\sigma \in G_m$, we define

$$f_m^\pm(\sigma) = \sum_{1 \neq f|m} \sum_{a \in \mathbb{M}_m} a_{f, \sigma_a \sigma^{-1}} g_f^\pm(a). \tag{3.1}$$

Lemma 3.1. *For $\chi \in \widehat{G}_m$, we have*

$$\sum_{\sigma \in G_m} \chi(\sigma) f_m^\pm(\sigma) = \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^\pm(a) c_{\overline{\chi}}.$$

Proof. If $k = \mathbb{F}_q(T)$, the proof follows from [5, Thms. 3.1, 3.6]. If $k = \mathbb{Q}$, the proof for f_m^- and f_m^+ follows from [6, Thm. 1] and [9, Lemma 8.4-8.7], respectively. □

We note that $\sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^-(a) = -B_{1,\chi}$ if $k = \mathbb{Q}$ and $\sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^-(a) = L(0, \chi)$ if $k = \mathbb{F}_q(T)$, where $B_{1,\chi}$ is the first generalized Bernoulli number and $L(s, \chi)$ is the Artin L-function associated to character χ.

Let F be a finite abelian extension of k. We let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F with its unit group $\mathcal{O}_F^*$. Let $h(\mathcal{O}_F)$ be the ideal class number of $\mathcal{O}_F$ and $h^-(\mathcal{O}_F) = h(\mathcal{O}_F)/h(\mathcal{O}_{F^+})$, the relative ideal class number of $\mathcal{O}_F$. Let $R(F)$ be the regulator of F (for $k = \mathbb{F}_q(T)$, we follow the definition in [10, Sect. 1]). Let W_F be the group of all roots of unity in F and $w_F = |W_F|$ be its order. Let $Q_F = [\mathcal{O}_F^* : W_F\mathcal{O}_{F^+}^*]$. Note that $Q_{K_m} = 1$ if m is a prime power and $Q_{K_m} = |J|$ if m is not a prime power (see [9, Coro. 4.13], [2, Prop. 1.14]). We denote by χ_0 the trivial character of G_m.

Proposition 3.2. *For any $m \in \mathbb{A}^+$, we have*

$$\prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a) = \pm |J|^{[K_m^+:k]-1} h(\mathcal{O}_{K_m^+}) R(K_m^+) \tag{3.2}$$

and

$$\prod_{\chi \in \widehat{G}_m \setminus \widehat{G}_m^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^-(a) = \pm \frac{|J|^{[K_m^+:k]} h^-(\mathcal{O}_{K_m})}{Q_{K_m} w_{K_m}}. \tag{3.3}$$

Proof. In case $k = \mathbb{Q}$, it is known ([9, page 42], [9, Thm. 4.9]) that

$$\frac{2^{[K_m^+:\mathbb{Q}]-1} h(\mathcal{O}_{K_m^+}) R(K_m^+)}{\sqrt{|d(K_m^+)|}} = \prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} L(1, \chi),$$

$$\prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} L(1, \chi) = \pm \prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} (\tau(\chi)/f_\chi) \prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a),$$

where $d(K_m^+)$ be the discriminant of K_m^+ and $\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a)\zeta_{f_\chi}^a$ is a Gauss sum. Then (3.2) follows from the fact that $\prod_\chi \tau(\chi)/f_\chi = 1/\sqrt{|d(K_m^+)|}$ ([9, Thm. 3.11, Coro. 4.6]). Since $\sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^-(a) = -B_{1,\chi}$, (3.3) immediately follows from [9, Thm. 4.17].

Suppose that $k = \mathbb{F}_q(T)$. Let $h(K_m)$ and $h(K_m^+)$ be the divisor class numbers of K_m and K_m^+, respectively. Let $h^-(K_m) = h(K_m)/h(K_m^+)$ be the relative divisor class number of K_m. It is known [7, Corollary after Lemma 4.1] that $h(K_m^+) = R(K_m^+) h(\mathcal{O}_{K_m^+})$. Then (3.2) immediately follows from the fact that ([5, (3.4)])

$$h(K_m^+) = (q-1)^{-[K_m^+:k]+1} \prod_{\chi_0 \neq \chi \in \widehat{G}_m^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a).$$

From [12, Lemma 3, Prop. 4], we have

$$\prod_{\chi \in \widehat{G}_m \setminus \widehat{G}_m^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^-(a) = h^-(K_m) = (q-1)^{e_m} h^-(\mathcal{O}_{K_m}),$$

where e_m is $[K_m^+ : k] - 1$ if m is a prime power and $[K_m^+ : k] - 2$ otherwise. Since $w_{K_m} = q - 1 = |J|$, we get (3.3). $\square$

We now fix $n, m \in \mathbb{A}^+$ with $n|m$. Let $H = \mathrm{Gal}(K_m/K_n)$. We identify $\widehat{G}_m^H$ with $\widehat{G}_n$. Then $\widehat{G}_m^{JH}$ is the group of all real characters of G_n. Let $\mathbb{M}_{m,n}^* = \{a \in \mathbb{M}_m : (a)_n \in \mathbb{M}_n^+\}$. For $a \in \mathbb{A}$ with $(a, m) = 1$, we define $f_m^\pm(a) = f_m^\pm(\sigma_a)$.

Theorem 3.3. *Let $m, n \in \mathbb{A}^+$ with $n|m$. Assume that we are given $\{a_{f,\sigma} \in \mathbb{C} : f|m, f \neq 1, \sigma \in G\}$, and let f_m^+, f_m^- be defined as in (3.1). We define two matrices*

$$\mathcal{D}_{m,n}^+ = \left(f_m^+(ab) - f_m^+(a\,(b)_n) : a, b \in \mathbb{M}_{m,n}^* \backslash \mathbb{M}_n^+\right),$$

$$\mathcal{D}_{m,n}^- = \left(\alpha_{c,d} : c, d \in \mathbb{M}_m \backslash (\mathbb{M}_{m,n}^* \cup \mathbb{M}_n)\right),$$

where $\alpha_{c,d} = f_m^-(cd) - f_m^-(cd/\mathrm{sgn}_n(d)) - f_m^-(c\,(d)_n) + f_m^-(c\,(d)_n/\mathrm{sgn}_n(d))$. Then we have

$$\det(\mathcal{D}_{m,n}^+) = \pm \frac{h(\mathcal{O}_{K_m^+})R(K_m^+)}{h(\mathcal{O}_{K_n^+})R(K_n^+)} \times C_{m,n}^+$$

and

$$\det(\mathcal{D}_{m,n}^-) = \pm |J|^{[K_m^+:k]-[K_n^+:k]} \frac{h^-(\mathcal{O}_{K_m})Q_{K_n}w_{K_n}}{h^-(\mathcal{O}_{K_n})Q_{K_m}w_{K_m}} \times C_{m,n}^-,$$

where $C_{m,n}^+ = \prod_{\chi \in \widehat{G}^J \backslash \widehat{G}^{JH}} c_\chi$ and $C_{m,n}^- = \prod_{\chi \notin \widehat{G}^J \cup \widehat{G}^H} c_\chi$.

Proof. By Lemma 3.1 and Proposition 3.2, we have

$$\begin{aligned}
\prod_{\chi \in \widehat{G}^J \backslash \widehat{G}^{JH}} \sum_{\sigma \in G} \chi(\sigma) f_m^+(\sigma) &= \prod_{\chi \in \widehat{G}^J \backslash \widehat{G}^{JH}} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a) \times C_{m,n}^+ \\
&= \frac{\prod_{\chi_0 \neq \chi \in \widehat{G}^J} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a)}{\prod_{\chi_0 \neq \chi \in \widehat{G}^{JH}} \sum_{a \in \mathbb{M}_{f_\chi}} \chi(a) g_{f_\chi}^+(a)} \times C_{m,n}^+ \qquad (3.4) \\
&= \pm |J|^{[K_m^+:k]-[K_n^+:k]} \frac{h(\mathcal{O}_{K_m^+})R(K_m^+)}{h(\mathcal{O}_{K_n^+})R(K_n^+)} \times C_{m,n}^+.
\end{aligned}$$

Similarly, we have

(3.5)

$$\prod_{\chi \notin \widehat{G}^J \cup \widehat{G}^H} \sum_{\sigma \in G} \chi(\sigma) f_m^-(\sigma) = \pm |J|^{[K_m^+:k]-[K_n^+:k]} \frac{h^-(\mathcal{O}_{K_m})Q_{K_n}w_{K_n}}{h^-(\mathcal{O}_{K_n})Q_{K_m}w_{K_m}} \times C_{m,n}^-.$$

We note that $\sigma(\mathbb{M}_{m,n}^*), \sigma(\mathbb{M}_n)$ and $\sigma(\mathbb{M}_n^+)$ are systems of representatives of $G/J, G/H$ and G/JH, respectively and they satisfy conditions for Proposition 2.1 and Theorem 2.4. It is easy to see that $r_J(\sigma_a) = \sigma_{a/\mathrm{sgn}_n(a)}$ and $r_H(\sigma_a) = \sigma_{(a)_n}$ for any $a \in \mathbb{M}_m$. Since $f_m^+ \in L^2(G)^J$, we have $s_{f_m^+}^J = |J| f_m^+$. Thus by combining (2.1) and (2.2) with (3.4) and (3.5) respectively, we get the results. □

Remark 3.4. (i) Let $\mathbb{M}_{m,n}^{**} = \mathbb{M}_m \backslash \mathbb{M}_{m,n}^* = \{a \in \mathbb{M}_m : (a)_n \in \mathbb{M}_n^-\}$. When $k = \mathbb{Q}$, we note that $\sigma(\mathbb{M}_{m,n}^{**}), \sigma(\mathbb{M}_n)$ and $\sigma(\mathbb{M}_n^-)$ are systems of representatives of $G/J, G/H$ and G/JH, respectively and they satisfy conditions for Proposition 2.1 and Theorem 2.4. Thus Theorem 3.3 still holds when we replace $\mathbb{M}_{m,n}^* \backslash \mathbb{M}_n^+$

(resp. $\mathbb{M}_m \backslash (\mathbb{M}^*_{m,n} \cup \mathbb{M}_n)$) by $\mathbb{M}^{**}_{m,n} \backslash \mathbb{M}^-_n$ (resp. $\mathbb{M}_m \backslash (\mathbb{M}^{**}_{m,n} \cup \mathbb{M}_n)$) in the definition of $\mathcal{D}^+_{m,n}$ (resp. $\mathcal{D}^-_{m,n}$).

(ii) Assume that $k = \mathbb{F}_q(T)$. It is known [5, (3.2)] that

$$\frac{|J|^{[K^+_m:k]} h^-(\mathcal{O}_{K_m})}{Q_{K_m} w_{K_m}} = h^-(K_m).$$

Thus by Theorem 3.3, we have

$$\det(\mathcal{D}^-_{m,n}) = \pm \frac{h^-(K_m)}{h^-(K_n)} \times C^-_{m,n}.$$

We note that $C^{\pm}_{m,n}$ is dependent on the choice of $\{a_{f,\sigma} : 1 \neq f|m$ and $\sigma \in G\}$. For some special choices of them, we compute $C^{\pm}_{m,n}$.

Example 3.5. For a prime (or monic irreducible) p in $\mathbb{A}$, let e_p, f_p and g_p (resp. $\tilde{e}_p$, $\tilde{f}_p$ and $\tilde{g}_p$) be the ramification index of p, the residue class degree of p and the number of primes lying above p in K_m (resp. in K_n), respectively. We also define e^+_p, f^+_p and g^+_p (resp. $\tilde{e}^+_p, \tilde{f}^+_p$ and $\tilde{g}^+_p$) the corresponding ones for K^+_m (resp. for K^+_n). Put $a_{f,\sigma} = 1$ if $f = m$, $\sigma = 1$ and 0 otherwise. Then we have $f^{\pm}_m = g^{\pm}_m$ and $c_\chi = \prod_{p|m}(1 - \bar{\chi}(p))$. Let $C^+_{m,n}(z) = \prod_{\chi \in \widehat{G}^J \backslash \widehat{G}^{JH}} \prod_{p|m}(z - \bar{\chi}(p))$ and $C^-_{m,n}(z) = \prod_{\chi \notin \widehat{G}^J \cup \widehat{G}^H} \prod_{p|m}(z - \bar{\chi}(p))$ as in [5, Lemma 3.2, 3.7]. By [9, Thm. 3.7], we have

$$C^+_{m,n}(z) = \prod_{p|m} \frac{(z^{f^+_p} - 1)^{g^+_p}}{(z^{\tilde{f}^+_p} - 1)^{\tilde{g}^+_p}},$$

$$C^-_{m,n}(z) = \prod_{p|m} \frac{(z^{f_p} - 1)^{g_p}}{(z^{f^+_p} - 1)^{g^+_p}} \Big(\frac{(z^{\tilde{f}_p} - 1)^{\tilde{g}_p}}{(z^{\tilde{f}^+_p} - 1)^{\tilde{g}^+_p}} \Big)^{-1}.$$

By considering the number of the factor $(z-1)$ in numerator and denominator of $C^{\pm}_{m,n}(z)$, we have

$$C^+_{m,n} = \begin{cases} \prod_{p|m} (f^+_p / \tilde{f}^+_p)^{g^+_p} & \text{if } g^+_p = \tilde{g}^+_p \text{ for all } p|m, \\ 0 & \text{otherwise} \end{cases}$$

and

$$C^-_{m,n} = \begin{cases} \prod_{p|m} \Big(\frac{f_p^{g_p}}{f_p^{+g^+_p}} \Big) / \Big(\frac{\tilde{f}_p^{\tilde{g}_p}}{\tilde{f}_p^{+\tilde{g}^+_p}} \Big) & \text{if } g_p - g^+_p = \tilde{g}_p - \tilde{g}^+_p \text{ for all } p|m, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.6. Since $C^{\pm}_{m,n}$ in Example 3.5 are zero for some cases, we make a choice of $\{a_{f,\sigma} \in \mathbb{C} : 1 \neq f|m, \sigma \in G\}$ which gives nonzero $C^{\pm}_{m,n}$ as [6, Example 4]. Let $m = \prod_{i=1}^s p_i^{r_i}$ and $S = \{1, \ldots, s\}$. For $I \subseteq S$, define $m_I = \prod_{i \in I} p_i^{r_i}$. For $1 \neq f|m$ and $\sigma \in G$, we define

$$a_{f,\sigma} = \begin{cases} \prod_{i \notin I} (-\Phi(p_i^{r_i})^{-1}) & \text{if } f = m_I \text{ for some } I \subseteq S \text{ and } \sigma = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as Kučera [6], we have $C^+_{m,n} = \pm 1$ and $C^-_{m,n} = \pm 1$.

4. Cyclotomic fields of prime power conductor cases

For any prime (or monic irreducible) p in $\mathbb{A}$ and $\ell \geq 2$ ($\ell \geq 3$ if $p = 2$), put $m = p^\ell$ and $n = p^{\ell-1}$. In this section we show that recursion formulas of Girstmair [3, Theorem] and Jung-Ahn [4, Thm. 4.1] are immediately obtained from Theorem 3.3. We also obtain a similar recursion formula for the real class numbers $h(\mathcal{O}_{K_m})$ and $h(\mathcal{O}_{K_n})$.

Assume that $k = \mathbb{Q}$. For $j \in \mathbb{Z}$ with $(p, j) = 1$, let $j^* \in \{0, \ldots, p-1\}$ such that $(j - (j)_n)/n \equiv j^* \mod p$ and $\langle j \rangle_G = j^*/p$ if $(j)_n < n/2$ and 0 otherwise.

Proposition 4.1 ([3], Theorem). *For prime $p \geq 3$ with $\ell \geq 2$ (or $p = 2$ with $\ell \geq 3$), let $m = p^\ell$ and $n = p^{\ell-1}$. Define the matrix*

$$\mathcal{C} = \big(\langle cd \rangle_G - \langle -cd \rangle_G - \langle c\,(d)_n \rangle_G + \langle -c\,(d)_n \rangle_G : c, d \in \mathbb{M}_m \backslash (\mathbb{M}^{**}_{m,n} \cup \mathbb{M}_n)\big).$$

Then we have

$$h^-(\mathcal{O}_{K_m}) = p\,|\det \mathcal{C}|\, h^-(\mathcal{O}_{K_n}).$$

Proof. We note that $Q_{K_m} = Q_{K_n} = 1$, $w_{K_m} = 2m$ and $w_{K_n} = 2n$ (resp. $w_{K_m} = m$ and $w_{K_n} = n$) for odd p (resp. $p = 2$). Put $a_{f,\sigma} = 1$ if $f = m, \sigma = 1$ and 0 otherwise. Then we have $C^-_{m,n} = 1$ and $f^-_m(a) = g^-_m(a) = \frac{1}{2} - \langle \frac{a}{m} \rangle$ for $a \in \mathbb{A}$ with $(a, m) = 1$. Thus by Theorem 3.3 and Remark 3.4 (i), we have

$$\det\big(\varepsilon_{c,d} : c, d \in \mathbb{M}_m \backslash (\mathbb{M}^{**}_{m,n} \cup \mathbb{M}_n)\big) = \pm 2^{[K^+_m : k] - [K^+_n : k]} \frac{h^-(\mathcal{O}_{K_m})}{p\, h^-(\mathcal{O}_{K_n})}, \tag{4.1}$$

where $\varepsilon_{c,d} = f^-_m(cd) - f^-_m(cd/\operatorname{sgn}_n(d)) - f^-_m(c\,(d)_n) + f^-_m(c\,(d)_n/\operatorname{sgn}_n(d))$. We claim that

$$\begin{aligned} & \frac{1}{2}\big(\langle cd/m \rangle - \langle -cd/m \rangle - \langle c\,(d)_n/m \rangle + \langle -c\,(d)_n/m \rangle\big) \\ = \; & \langle cd \rangle_G - \langle -cd \rangle_G - \langle c\,(d)_n \rangle_G + \langle -c\,(d)_n \rangle_G \end{aligned}$$

for any $c, d \in \mathbb{M}_m \backslash (\mathbb{M}^{**}_{m,n} \cup \mathbb{M}_n)$. Since $\langle -j/m \rangle = 1 - \langle j/m \rangle$ if $\langle j/m \rangle \neq 0$, we have

$$\frac{1}{2}\big(\langle cd/m \rangle - \langle -cd/m \rangle - \langle c\,(d)_n/m \rangle + \langle -c\,(d)_n/m \rangle\big) = \langle cd/m \rangle - \langle c\,(d)_n/m \rangle.$$

We note that $j^*/p = (j - (j)_n)/m$ if $j \in \mathbb{M}_m$. Suppose that $(cd)_n < n/2$. Since $(-cd)_n > n/2$ and $(c\,(d)_n)_n = (cd)_n < n/2$, we have

$$\begin{aligned} & \langle cd \rangle_G - \langle -cd \rangle_G - \langle c\,(d)_n \rangle_G + \langle -c\,(d)_n \rangle_G \\ = \; & \frac{(cd)_m - ((cd)_m)_n}{m} - \frac{(c\,(d)_n)_m - ((c\,(d)_n)_m)_n}{m} \\ = \; & \frac{(cd)_m}{m} - \frac{(c\,(d)_n)_m}{m} = \langle cd/m \rangle - \langle c\,(d)_n/m \rangle \end{aligned}$$

because $((cd)_m)_n = ((c\,(d)_n)_m)_n = (cd)_n$. Thus the claim holds when $(cd)_n < n/2$. Similarly, we show that the claim holds when $(cd)_n > n/2$. Thus we get the result from (4.1). □

Assume that $k=\mathbb{F}_q(T)$. For $a\in\mathbb{A}$ with $(a,m)=1$, we define $\langle a\rangle_{JA}=1$ if $\operatorname{sgn}_m(a)=1$ and 0 otherwise.

Proposition 4.2 ([4], Theorem 4.1). *For a monic irreducible p and $\ell\geq 2$, let $m=p^\ell$ and $n=p^{\ell-1}$. Define the matrix*

$$\begin{aligned}\mathcal{D} \;=\; & \big(\langle cd\rangle_{JA}-\langle cd/\operatorname{sgn}_n(d)\rangle_{JA}-\langle c\,(d)_n\rangle_{JA}+\langle c\,(d)_n/\operatorname{sgn}_n(d)\rangle_{JA}:\\ & c,d\in\mathbb{M}_m\backslash(\mathbb{M}^*_{m,n}\cup\mathbb{M}_n)\big).\end{aligned}$$

Then we have

$$h^-(K_m)=|\det\mathcal{D}|\,h^-(K_n).$$

Proof. In this case, we have that $Q_{K_m}=Q_{K_n}=1$ and $w_{K_m}=w_{K_n}=q-1$. Put $a_{f,\sigma}=1$ if $f=m,\sigma=1$ and 0 otherwise. Then $C^-_{m,n}=1$ and $f^-_m(a)=Z_m(0,a)=\langle a\rangle_{JA}-\frac{1}{q-1}$ for $a\in\mathbb{A}$ with $(a,m)=1$ ([4, Lemma 3.1]). Thus the result immediately follows from Theorem 3.3 and Remark 3.4 (ii). $\square$

Finally we obtain a recursion formula for the real class numbers $h(\mathcal{O}^+_{K_m})$ and $h(\mathcal{O}^+_{K_n})$ from Theorem 3.3.

Proposition 4.3. *Let $p\in\mathbb{A}$ be a prime (or monic irreducible) and $\ell\geq 2$ integer (if $p=2$, we assume $\ell\geq 3$).*

(i) *When $k=\mathbb{Q}$,*

$$\begin{aligned}&\det\big(\log|\sin(\pi ab/m)/\sin(\pi a\,(b)_n/m)|:a,b\in\mathbb{M}^*_{m,n}\backslash\mathbb{M}^+_n\big)\\ =\;&\pm\frac{h(\mathcal{O}_{K_m^+})R(K_m^+)}{h(\mathcal{O}_{K_n^+})R(K_n^+)}.\end{aligned}$$

(ii) *When $k=\mathbb{F}_q(T)$,*

$$\begin{aligned}&\det\big(\deg_m(ab)-\deg_m(a\,(b)_n):a,b\in\mathbb{M}^*_{m,n}\backslash\mathbb{M}^+_n\big)\\ =\;&\pm\frac{h(\mathcal{O}_{K_m^+})R(K_m^+)}{h(\mathcal{O}_{K_n^+})R(K_n^+)}=\pm\frac{h(K_m^+)}{h(K_n^+)}.\end{aligned}$$

Proof. Put $a_{f,\sigma}=1$ if $f=m,\sigma=1$ and 0 otherwise. Then we have $C^+_{m,n}=1$. When $k=\mathbb{Q}$, we have $f^+_m(a)=g^+_m(a)=\log|1-\zeta^a_m|=\log|\sin(\pi a/m)|$. When $k=\mathbb{F}_q(T)$, we have $f^+_m(a)=g^+_m(a)=\deg(m)-1-\deg_m(a)-\frac{1}{q-1}$ and so $f^+_m(ab)-f^+_m(a\,(b)_n)=-(\deg_m(ab)-\deg_m(a\,(b)_n))$. Thus the result immediately follows from Theorem 3.3 and Remark 3.4. $\square$

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