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A WEIERSTRASS SEMIGROUP AT A PAIR OF INFLECTION POINTS ON A SMOOTH PLANE CURVE

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Abstract. We classify all semigroups each of which arises as a Weierstrass semigroup at a pair of inflection points of multiplicities d or $d-1$ on a smooth plane curve of degree d.

1. Introduction and preliminaries

Let C be a smooth complex projective curve of genus $g \geq 2$, $\mathcal{M}(C)$ the field of meromorphic functions on C and $\mathbb{N}_0$ the set of all nonnegative integers. For two distinct points P, $Q \in C$, we define the Weierstrass semigroup $H(P) \subset \mathbb{N}_0$ at a point and the Weierstrass semigroup at a pair of points $H(P,Q) \subset \mathbb{N}_0^2$ by

$$\begin{aligned} H(P) &= \{\alpha \in \mathbb{N}_0 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}, \\ H(P,Q) &= \{(\alpha,\beta) \in \mathbb{N}_0^2 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}, \end{aligned}$$

where $(f)_\infty$ means the divisor of poles of the meromorphic function f. Indeed, these sets form sub-semigroups of $\mathbb{N}_0$ and $\mathbb{N}_0^2$, respectively. The cardinality of the set $G(P) = \mathbb{N}_0 \setminus H(P)$ is exactly g. The set $G(P,Q) = \mathbb{N}_0^2 \setminus H(P,Q)$ is also finite, but its cardinality is dependent on the points P and Q. In [3], the upper and lower bound of such sets are given as $\binom{g+2}{2} - 1 \leq \text{card } G(P,Q) \leq \binom{g+2}{2} - 1 - g + g^2$.

We review some basic facts concerning the Weierstrass semigroups at a pair of points on a curve. ([2], [3]).

Lemma 1.1. *For each $\alpha \in G(P)$, let $\beta_\alpha = \min\{\beta \mid (\alpha,\beta) \in H(P,Q)\}$. Then $\alpha = \min\{\gamma \mid (\gamma,\beta_\alpha) \in H(P,Q)\}$. Moreover, we have*

$$\{\beta_\alpha \mid \alpha \in G(P)\} = G(Q).$$

Proof. See [3]. □

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Above lemma implies that the set $H(P,Q)$ defines a bijective mapping $\sigma = \sigma(P,Q)$ from $G(P)$ to $G(Q)$ which is defined by $\alpha \mapsto \beta_\alpha$. Homma [2] obtained the formula for the cardinality of $G(P,Q)$ using the cardinality of the set of pairs (α, α') which are reversed by σ. We use the following notations;

$$\begin{aligned} \Gamma = \Gamma(P,Q) &:= \{(\alpha, \beta_\alpha) \mid \alpha \in G(P)\} \\ &= \{(p_i, q_{\sigma(i)}) \mid i = 1, 2, \ldots, g\}, \\ \widetilde{\Gamma} = \widetilde{\Gamma}(P,Q) &:= \Gamma(P,Q) \cup \big(H(P) \times \{0\}\big) \cup \big(\{0\} \times H(Q)\big). \end{aligned}$$

The above set $\Gamma(P,Q)$ is called the *generating subset* of the Weierstrass semigroup $H(P,Q)$. For given distinct two points P, Q, the set $\Gamma(P,Q)$ determines not only $\widetilde{\Gamma}(P,Q)$ but also the sets $H(P,Q)$ and $G(P,Q)$ completely, as described in the lemma below. To state the lemma we use the natural partial order on the set $\mathbb{N}_0^2$ defined as

$$(\alpha, \beta) \geq (\gamma, \delta) \text{ if and only if } \alpha \geq \gamma \text{ and } \beta \geq \delta,$$

and the least upper bound of two elements (α_1, β_1), (α_2, β_2) is defined as

$$\operatorname{lub}\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} = (\max\{\alpha_1, \alpha_2\},\ \max\{\beta_1, \beta_2\}).$$

Lemma 1.2. (1) *The subset $H(P,Q)$ of $\mathbb{N}_0^2$ is closed under the lub(least upper bound) operation.* (2) *Every element of $H(P,Q)$ is expressed as the* lub *of one or two elements of the set $\widetilde{\Gamma}(P,Q)$.* (3) *The set $G(P,Q) = \mathbb{N}_0^2 \setminus H(P,Q)$ is expressed as*

$$G(P,Q) = \bigcup_{l \in G(P)} \big(\{(l, \beta) | \beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l)) | \alpha = 0, 1, \ldots, l-1\}\big).$$

Proof. See [3] and [4]. □

We can characterize the elements of $\Gamma(P,Q)$ and $H(P,Q)$ using the dimensions of divisors. We denote $\dim(\alpha, \beta)$ the dimension of the complete linear series $|\alpha P + \beta Q|$.

Lemma 1.3. *Let (α, β) be an element in $\mathbb{N}_0^2$ with $\beta \geq 1$* [resp. $\alpha \geq 1$]. Then

$$\dim(\alpha, \beta) = \dim(\alpha, \beta - 1) + 1 [\text{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1]$$

if and only if there exists $(\gamma, \beta) \in \widetilde{\Gamma}$ [resp. $(\alpha, \delta) \in \widetilde{\Gamma}$] with $0 \leq \gamma \leq \alpha$ [resp. $0 \leq \delta \leq \beta$].

Proof. See [3]. □

Lemma 1.4. *For $\alpha \geq 1$ and $\beta \geq 1$, the pair (α, β) is an element of $\Gamma(P,Q)$* [resp. $H(P,Q)$] *if and only if*

$$\begin{aligned} \dim(\alpha, \beta) &= \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1 \\ &= \dim(\alpha - 1, \beta - 1) + 1 \end{aligned}$$

$$[\text{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1].$$

Proof. By Lemma 1.3, since $(\alpha,\beta)\in\Gamma(P,Q)$ implies that there is no element (α',β) [resp. (α,β')] $\in H(P,Q)$ with $0\le\alpha'\le\alpha$ [resp. $0\le\beta'\le\beta$], the lemma holds. □

Theorem 1.5. *Let $m\ge 1$, $m'\ge 0$, $n'\ge n\ge 1$ and $a\ge 0$ be integers. Suppose $\dim(s+m,t-n)=\dim(s,t)+a$ for all $s\ge m'$, $t\ge n'$. Then, for $\alpha\ge m'+1$ and $\beta\ge n'+1$, $(\alpha+m,\beta-n)\in H(P,Q)$ [resp. $(\alpha+m,\beta-n)\in\Gamma(P,Q)$] if and only if $(\alpha,\beta)\in H(P,Q)$ [resp. $(\alpha,\beta)\in\Gamma(P,Q)$].*

Proof. It follows from Lemma 1.4. □

Theorem 1.6. *Suppose that mP is linearly equivalent to mQ. If $(\alpha,\beta),(\alpha+m,\beta')\in\Gamma(P,Q)$, then $\beta'=\beta-m$.*

Proof. It follows from Theorem 1.5. □

When we prove the existence of a smooth plane curve with aligned inflection points of given multiplicities, we use the following theorem. Here $\mathbb{P}_d$ denotes the set of all smooth plane curves of degree d, and $i(T,C;P)$ denotes the intersection multiplicity of two curves T and C at the point P.

Theorem 1.7. [1] *Fix a line L in $\mathbb{P}^2$ and different points $P_0,P_1,\ldots,P_{d-e}$ on L with $\mathbb{N}$, $0\le e\le d$. Fix lines $T_1,\ldots,T_{d-e}$ passing through $P_1,\ldots,P_{d-e}$ different from L. For a sequence $\underline{m}=(m_1,\ldots,m_{d-e})$ with $d\ge m_1\ge\cdots\ge m_{d-e}$, let*

$$\mathcal{P}_{(e,\underline{m})}=\{C\in\mathbb{P}_d\,|C \text{ is smooth, } i(L,C;P_0)=e,\\ i(T_j,C;P_j)=m_j \text{ for } 1\le j\le d-e\}.$$

Then $\mathcal{P}_{(e,\underline{m})}$ is not empty if and only if the following condition holds:
For every j, $1\le j<d-e$, if $m_{j+1}<m_j$, then $m_{j+1}\le d-j$.

In Section 2, we find all candidates of the Weierstrass semigroup at a pair of inflection points of multiplicities $d,d-1$ on a smooth plane curve. Moreover, we prove the existence of curves and points having such semigroups as their Weierstrass semigroups.

2. A Weierstrass semigroup at a pair of inflection points of multiplicities $d,d-1$

Let C be a smooth plane curve of degree $d\ge 4$. Recall that an *inflection point* of a curve is a simple point at which the tangent line intersects the curve three or more times. We consider a pair of inflection points whose multiplicities are d or $d-1$. In this section, T_PC denotes the tangent line to C at a point $P\in C$ and $T_PC\centerdot C$ denotes the divisor on C cut out by T_PC.

There are six possibilities according to multiplicities of points and their location, which we consider case by case.

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
$1+d$	$2+d$	$3+d$	$\cdots$	$\cdots$	$d-3+d$	
$1+2d$	$2+2d$	$3+2d$	$\cdots$	$\cdot$		
$\vdots$	$\vdots$	$\vdots$	$\cdot$			
$\vdots$	$\vdots$	$\cdot$				
$1+(d-4)d$	$2+(d-4)d$					
$1+(d-3)d$						

TABLE 1. $G(P)$ when $T_PC \centerdot C = dP$

2.1. $T_PC \centerdot C = dP$ and $T_QC \centerdot C = dQ$

In this case, we arrange all elements of the set $G(P) = G(Q)$ as Table 1 with $d-2$ columns and rows.

Note that the last element $1 + (d-3)d$ in Table 1 is equal to $2g - 1$, where $g = \frac{(d-1)(d-2)}{2}$ is the genus of C. Since $dP \sim dQ$, by Theorem 1.6, we have the theorem.

Theorem 2.1. *$\Gamma(P, Q)$ is the set of all elements appeared in the following* Table 2.

$(1, 1+(d-3)d)$	$(2, 2+(d-4)d)$	$\cdots$	$(d-3, (d-3)+d)$	$(d-2, d-2)$
$(1+d, 1+(d-4)d)$	$(2+d, 2+(d-5)d)$	$\cdots$	$(d-3+d, d-3)$	
$(1+2d, 1+(d-5)d)$	$(2+2d, 2+(d-6)d)$	$\cdot$		
$\vdots$	$\vdots$			
$(1+(d-4)d, 1+d)$	$(2+(d-4)d, 2)$			
$(1+(d-3)d, 1)$				

TABLE 2. $\Gamma(P, Q)$ when $T_PC \centerdot C = dP$ and $T_QC \centerdot C = dQ$

Furthermore, there exist a smooth plane curve C of degree d and two inflection points P, Q of multiplicity d on it satisfying the condition that $\Gamma(P, Q)$ is equal to the set of elements appeared in Table 2.

Proof. Note that the lengths of columns in the array in Table 1 are all different. So in view of Theorem 1.6, if $(\alpha, \beta) \in \Gamma(P, Q)$, then α and β should belong to the same column in Table 1, hence $\Gamma(P, Q)$ is determined like as Table 2.

In Theorem 1.7, if we let $e = 0$ and $\underline{m} = (d, d, \ldots, d)$, then $\mathcal{P}_{(0,\underline{m})} \neq \emptyset$. Thus, for any collinear points $P_1, \ldots, P_d$, there exists a smooth plane curve having $P_1, \ldots, P_d$ as inflection points of multiplicity d. Hence $\Gamma(P_i, P_j)$ $(i \neq j)$ is equal to the given set in Table 2. □

2.2. $T_PC \centerdot C = dP$ and $T_QC \centerdot C = P + (d-1)Q$

Since $dP \sim P + (d-1)Q$, we have $(d-1)P \sim (d-1)Q$. To use Theorem 1.6, we had better arrange the elements of $G(P)$ and $G(Q)$ as in Table 3 and

Table 4, respectively. Note that the sets of elements appeared in Table 1 and Table 3 are same, of course. Also note that the sequence in each column in both of two tables is increasing by $d-1$ each time.

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
	$2+(d-1)$	$3+(d-1)$	$\cdots$	$\cdots$	$d-3+(d-1)$	$d-2+(d-1)$
		$3+2(d-1)$	$\cdots$	$\cdot$	$d-3+2(d-1)$	$d-2+2(d-1)$
			$\cdot$	$\vdots$	$\vdots$	$\vdots$
				$\cdot$	$\vdots$	$\vdots$
					$d-3+(d-4)(d-1)$	$d-2+(d-4)(d-1)$
						$d-2+(d-3)(d-1)$

TABLE 3. $G(P)$ when $T_PC \centerdot C = dP$

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
$1+(d-1)$	$2+(d-1)$	$3+(d-1)$	$\cdots$	$\cdots$	$d-3+(d-1)$	
$1+2(d-1)$	$2+2(d-1)$	$3+2(d-1)$	$\cdots$	$\cdot$		
$\vdots$	$\vdots$	$\vdots$	$\cdot$			
$\vdots$	$\vdots$	$\cdot$				
$1+(d-4)(d-1)$	$2+(d-4)(d-1)$					
$1+(d-3)(d-1)$						

TABLE 4. $G(Q)$ when $T_QC \centerdot C = (d-1)Q + P$

Theorem 2.2. *For P, Q as above, we have $\Gamma(P, Q)$ as in* Table 5.

$(1, d-2)$	$(2, d-3+(d-1))$	$\cdots$	$(d-3, 2+(d-4)(d-1)$	$(d-2, 1+(d-3)(d-1))$
	$(2+(d-1), d-3)$	$\cdots$	$(d-3+(d-1), 2+(d-5)(d-1))$	$(d-2+(d-1), 1+(d-4)(d-1))$
		$\cdots$	$(d-3+2(d-1), 2+(d-6)(d-1))$	$(d-2+2(d-1), 1+(d-5)(d-1))$
		$\cdot$	$\vdots$	$\vdots$
			$(d-3+(d-4)(d-1), 2)$	$(d-2+(d-4)(d-1), 1+(d-1))$
				$(d-2+(d-3)(d-1), 1)$

TABLE 5. $\Gamma(P, Q)$ when $T_PC \centerdot C = dP$ and $T_QC \centerdot C = (d-1)Q + P$

Furthermore, there exist a smooth plane curve C of degree d and two inflection points P, Q with the above conditions satisfying the condition that $\Gamma(P, Q)$ is equal to the set of elements appeared in Table 5.

Proof. Using Theorem 1.6, we have the desired $\Gamma(P, Q)$.

If we let $e = d-1$ and $\underline{m} = (d)$, then Theorem 1.7 implies $\mathcal{P}_{(d-1,(d))} \neq \emptyset$. Thus if we let $P_0 = Q$ and $P_1 = P$, then $\Gamma(P, Q)$ is equal to Table 5. □

2.3. $T_PC \cdot C = dP$ and $T_QC \cdot C = R + (d-1)Q$ with $R \neq P, Q$

In this case, we need the following theorem at first.

Theorem 2.3. *For $\alpha \geq 0$ and $\beta \geq d-1$, we have* $\dim((\alpha+d)P + (\beta-(d-1))Q) = \dim(\alpha P + \beta Q) + 1$.

Proof. Since

$$\begin{aligned} \alpha P + \beta Q + R &= \alpha P + (\beta - (d-1))Q + (d-1)Q + R \\ &\sim \alpha P + (\beta - (d-1))Q + dP \\ &= (\alpha + d)P + (\beta - (d-1))Q, \end{aligned}$$

the point R is not a base point of the linear series $|\alpha P + \beta Q + R|$. Hence

$$(1) \qquad \begin{aligned} \dim((\alpha+d)P + (\beta-(d-1))Q) &= \dim(\alpha P + \beta Q + R) \\ &= \dim(\alpha P + \beta Q) + 1. \end{aligned}$$

□

Theorem 2.4. *For $\alpha \geq 1$ and $\beta \geq d$, (α, β) is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$] *if and only if $(\alpha + d, \beta - (d-1))$ is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$].

Proof. It follows from Theorem 1.5 and Theorem 2.3. □

To apply Theorem 2.4 efficiently, we use the arrays of elements in $G(P)$ and $G(Q)$ as in Table 1 and Table 4. Note that both tables have the same number of columns and corresponding columns in both tables have same lengths. Also note that the sequence in each column in Table 1 is increasing by d and the one in Table 4 is increasing by $d-1$.

Theorem 2.5. *We have Table 6 as $\Gamma(P,Q)$ for given P, Q as above.*

$(1, 1+(d-3)(d-1))$	$(2, 2+(d-4)(d-1))$	$\cdots$	$(d-3, (d-3)+(d-1))$	$(d-2, d-2)$
$(1+d, 1+(d-4)(d-1))$	$(2+d, 2+(d-5)(d-1))$	$\cdots$	$(d-3+d, d-3)$	
$(1+2d, 1+(d-5)(d-1))$	$(2+2d, 2+(d-6)(d-1))$	$\cdot$		
$\vdots$	$\vdots$			
$(1+(d-4)d, 1+(d-1))$	$(2+(d-4)d, 2)$			
$(1+(d-3)d, 1)$				

TABLE 6. $\Gamma(P,Q)$ when $T_PC \cdot C = dP$ and $T_QC \cdot C = (d-1)Q + R$ with $R \neq P, Q$

Furthermore, there exist a smooth plane curve of degree d and two inflection points P, Q with the above conditions satisfying the condition that $\Gamma(P,Q)$ is equal to the set of elements appeared in Table 6.

Proof. Using Theorem 2.4, we have the desired $\Gamma(P,Q)$.

In Theorem 1.7, if we let $e = 0$ and $\underline{m} = (d, d-1, \ldots, d-1)$, then $\mathcal{P}_{(0,\underline{m})} \neq \emptyset$. For an element C in $\mathcal{P}_{(0,\underline{m})}$, if we let $P = P_1$ and $Q = P_2$, then $\Gamma(P,Q)$ is equal to Table 6. □

2.4. $T_PC \,.\, C = (d-1)P + R$ and $T_QC \,.\, C = (d-1)Q + R$ with $R \neq P, Q$

Since $(d-1)P \sim (d-1)Q$, we may apply Theorem 1.6. Consider the array of elements in $G(P) = G(Q)$ as in Table 4.

Theorem 2.6. *We have Table 7 as $\Gamma(P, Q)$ for given P, Q as above.*

$(1, 1+(d-3)(d-1))$	$(2, 2+(d-4)(d-1))$	$\cdots$	$(d-3, d-3+(d-1))$	$(d-2, d-2)$
$(1+(d-1), 1+(d-4)(d-1))$	$(2+(d-1), 2+(d-5)(d-1))$	$\cdots$	$(d-3+(d-1), d-3)$	
$(1+2(d-1), 1+(d-5)(d-1))$	$(2+2(d-1), 2+(d-6)(d-1))$	$\cdot$		
$\vdots$	$\vdots$			
$(1+(d-4)(d-1), 1+(d-1))$	$(2+(d-4)(d-1), 2)$			
$(1+(d-3)(d-1), 1)$				

TABLE 7. $\Gamma(P, Q)$ when $T_PC.C = (d-1)P+R$ and $T_QC.C = (d-1)Q+R$ with $R \neq P, Q$

Furthermore, there exist a smooth plane curve of degree d and two inflection points P, Q with the above conditions satisfying the condition that $\Gamma(P, Q)$ is equal to the set of elements appeared in Table 7.

Proof. Using Theorem 1.6, we have the desired $\Gamma(P, Q)$.

Modifying the idea in [1], we construct a desired polynomial of degree d. Consider a linear system $\{ay^{d-1}z + b\prod_{n=1}^{d}(x - nz) \mid (a, b) \in \mathbb{P}^1\}$. By Bezout's theorem, a general element in this system is smooth. In fact, easy calculation shows that $C := y^{d-1}z + \prod_{n=1}^{d}(x - nz)$ is smooth and $T_{P_n}C.C = (x - nz).C = (d-1)P_n + P_\infty$, $n = 1, \ldots, d$, where $P_n = (n, 0, 1)$ and $P_\infty = (0, 1, 0)$. Now we have Table 7 as $\Gamma(P_i, P_j)$ $(i \neq j)$. □

2.5. $T_PC \,.\, C = (d-1)P + R_1$ and $T_QC \,.\, C = (d-1)Q + R_2$ for 4 distinct points P, Q, R_1, R_2

We need the following theorem computing the dimensions.

Theorem 2.7. *For $\alpha \geq 0$ and $\beta \geq d-1$, we have* $\dim(\alpha, \beta) = \dim(\alpha + (d-1), \beta - (d-1))$.

Proof. Since

$$\begin{aligned}
&(\alpha + (d-1))P + (\beta - (d-1))Q + R_1 \\
&= \alpha P + (\beta - (d-1))Q + (d-1)P + R_1 \\
&\sim \alpha P + (\beta - (d-1))Q + (d-1)Q + R_2 \\
&= \alpha P + \beta Q + R_2,
\end{aligned}$$

neither R_1 nor R_2 is a base point of the linear series

$$|(\alpha + (d-1))P + (\beta - (d-1))Q + R_1| = |\alpha P + \beta Q + R_2|.$$

Hence

$$\begin{aligned}&\dim(\alpha+(d-1),\beta-(d-1))\\ =\ &\dim|(\alpha+(d-1))P+(\beta-(d-1))Q+R_1|-1\\ =\ &\dim|\alpha P+\beta Q+R_2|-1\\ =\ &\dim|\alpha P+\beta Q|.\end{aligned}$$

□

Theorem 2.8. *For $\alpha \geq 1$ and $\beta \geq d$, (α,β) is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$] *if and only if $(\alpha+(d-1),\beta-(d-1))$ is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$].

Proof. We can prove this theorem similarly as Theorem 2.4. □

In view of this theorem, it is better to use the array of elements of $G(P)=G(Q)$ as in Table 4. We have the same $\Gamma(P,Q)$ as the former case, however $(d-1)P \not\sim (d-1)Q$ in this case.

Theorem 2.9. *For P,Q as above, $\Gamma(P,Q)$ is the same as Table 7 in Theorem 2.6. Furthermore, there exist a smooth plane curve of degree d and two inflection points P,Q with the above conditions whose $\Gamma(P,Q)$ is equal to the set of elements appeared in Table 7.*

Proof. Using Theorem 1.6, we have the desired $\Gamma(P,Q)$.

In Theorem 1.7, if we let $e=0$, $\underline{m}=(d-1,d-1,\ldots,d-1)$ and T_1,T_2,T_3 three lines not concurrent, then $\mathcal{P}_{(0,\underline{m})} \neq \emptyset$. Then, for an element C in $\mathcal{P}_{(0,\underline{m})}$, at least one of three points $T_i \cap T_j$ for $1 \leq i,j \leq 3$ is not contained in C, say T_1,T_2. Then if we let $P=P_1$ and $Q=P_2$, then $(d-1)P \not\sim (d-1)Q$ and $\Gamma(P,Q)$ is equal to Table 7. □

2.6. $T_PC\,.\,C=(d-1)P+Q$ and $T_QC\,.\,C=(d-1)Q+R$ with $R \neq P,Q$

Since $(d-1)P+Q \sim (d-1)Q+R$, we have $(d-1)P \sim (d-2)Q+R$.

Theorem 2.10. *For $\alpha \geq 0$ and $\beta \geq d-2$,* $\dim(\alpha+(d-1),\beta-(d-2)) = \dim(\alpha,\beta)+1$.

Proof. Since

$$\begin{aligned}\alpha P+\beta Q+R &= \alpha P+(\beta-(d-2))Q+(d-2)Q+R\\ &\sim (\alpha+(d-1))P+(\beta-(d-2))Q,\end{aligned}$$

the point R is not a base point of the linear series $|\alpha P+\beta Q+R|$. Hence

$$\begin{aligned}\dim(\alpha+(d-1),\beta-(d-2)) &= \dim(\alpha P+\beta Q+R)\\ &= \dim(\alpha,\beta)+1.\end{aligned}$$

□

Theorem 2.11. *For $\alpha \geq 1$ and $\beta \geq d-1$, (α, β) is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$] *if and only if $(\alpha+(d-1), \beta-(d-2))$ is an element of $H(P,Q)$* [resp. $\Gamma(P,Q)$].

Proof. We can prove this theorem similarly as Theorem 2.4. □

In view of Theorem 2.11, we had better use the array of elements in $G(P)$ [resp. $G(Q)$] same as that in Table 4 [resp. Table 8]. Note that both tables have the same columns and the same rows. Also note that the sequence in each column in Table 4 is increasing by $d-1$ and the one in Table 8 is increasing by $d-2$.

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
	$2+(d-2)$	$3+(d-2)$	$\cdots$	$\cdots$	$d-3+(d-2)$	$d-2+(d-2)$
		$3+2(d-2)$	$\cdots$	$\cdot$	$d-3+2(d-2)$	$d-2+2(d-2)$
			$\cdot$	$\vdots$	$\vdots$	$\vdots$
				$\cdot$	$\vdots$	$\vdots$
					$d-3+(d-4)(d-2)$	$d-2+(d-4)(d-2)$
						$d-2+(d-3)(d-2)$

TABLE 8. $G(Q)$ when $T_PC \cdot C = (d-1)Q + R$

Thus we have the following theorem.

Theorem 2.12. *We have Table 9 as $\Gamma(P,Q)$ for given P, Q as above.*

$(1, d-2+(d-3)(d-2))$	$(2, d-3+(d-4)(d-2))$	$\cdots$	$(d-3, 2+(d-2))$	$(d-2, 1)$
$(1+(d-1), d-2+(d-4)(d-2))$	$(2+(d-1), d-3+(d-5)(d-2))$	$\cdots$	$(d-3+(d-1), 2)$	
$(1+2(d-1), d-2+(d-5)(d-2))$	$(2+2(d-1), d-3+(d-6)(d-2))$	$\cdot$		
$\vdots$	$\vdots$			
$(1+(d-4)(d-1), d-2+(d-2))$	$(2+(d-4)(d-1), d-3)$			
$(1+(d-3)(d-1), d-2)$				

TABLE 9. $\Gamma(P,Q)$ when $T_PC \cdot C = (d-1)P+Q$ and $T_QC \cdot C = (d-1)Q + R$ with $R \neq P, Q$

Furthermore, there exist a smooth plane curve of degree d and two inflection points P, Q with the above conditions whose $\Gamma(P,Q)$ is equal to the set of elements appeared in Table 9.

Proof. Using Theorem 1.6, we have the desired $\Gamma(P,Q)$.

In Theorem 1.7, if we let $e = d-1$ and $\underline{m} = (d-1)$, then $\mathcal{P}_{(d-1,\underline{m})} \neq \emptyset$. For an element C in $\mathcal{P}_{(d-1,\underline{m})}$, if we let $P = P_0$ and $Q = P_1$, then $\Gamma(P,Q)$ is equal to Table 9. □

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