Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 337–350

ON NONSINGULAR EMBRY QUARTIC MOMENT PROBLEM

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Abstract. Given a collection of complex numbers $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i + j \le 2n,\ |i-j| \le n)$ with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, we consider the moment problem for γ in the case of $n = 2$, which is referred to Embry quartic moment problem. In this note we give a partial solution for the nonsingular case of Embry quartic moment problem.

1. Introduction and preliminaries

For $n \in \mathbb{N}$, let $m = m(n) := ([\frac{n}{2}] + 1)([\frac{n+1}{2}] + 1)$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the algebra of $m \times m$ complex matrices), we denote the successive rows and columns according to the following ordering:

$$\underbrace{\mathit{1}}_{(1)}, \underbrace{Z}_{(1)}, \underbrace{Z^2, \bar{Z}Z}_{(2)}, \underbrace{Z^3, \bar{Z}Z^2}_{(2)}, \underbrace{Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2}_{(3)}, \ldots. \tag{1.1}$$

For a collection of complex numbers

$$\gamma \equiv \{\gamma_{ij}\} \ \ (0 \le i + j \le 2n,\ \ |i-j| \le n) \ \text{ with } \gamma_{00} > 0 \text{ and } \gamma_{ji} = \bar{\gamma}_{ij}, \tag{1.2}$$

we define the moment matrix $E(n) \equiv E(n)(\gamma)$ in $\mathcal{M}_m(\mathbb{C})$ as follows:

$$E(n)_{(k,l)(i,j)} := \gamma_{l+i,j+k}.$$

For example, if $n = 2$, i.e.,

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{12}, \gamma_{21}, \gamma_{13}, \gamma_{22}, \gamma_{31},$$

then we obtain the moment matrix

$$E(2) = \begin{array}{c} \begin{array}{cccc} \mathit{1} & Z & Z^2 & \bar{Z}Z \end{array} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}.$$

Received August 30, 2006.

2000 *Mathematics Subject Classification.* Primary 47A57, 44A60; Secondary 15A57, 47A20.

Key words and phrases. truncated complex moment problem, representing measure, nonsingular quartic moment problem, flat extension.

Embry truncated complex moment problem entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z) \qquad (0 \le i+j \le 2n, \, |i-j| \le n);$$

μ is called a *representing measure* for γ as in (1.2). Recall that Embry quadratic moment problem for $n = 1$ was solved in [6], but Embry quartic moment problem for $n = 2$ did not solved completely yet. Recently the first author of this paper solved the singular case in [5], whose case means that $\det E(2) = 0$. In fact, two methods can be used to solve the nonsingular case of such problem: one is characterizing the double flat extension $E(4)$ of $E(2)$ according to [4] and the other is that, first, extending $E(2)$ to $M(2)$ as

$$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}$$

and next, considering the extension problem of singular moment matrix $M(2)$ as in [3] and [8]. In this paper, by using the latter method, we solve the nonsingular Embry quartic moment problem under useful special cases.

The calculations in this article were obtained throughout computer experiments using the software tool *Mathematica* [9].

2. $E(2)$ has positive flat extension $M(2)$

Proposition 2.1. *Assume that $E(2)$ is positive and nonsingular. If*

$$\text{(2.1)} \qquad \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{vmatrix} > 0 \ \text{ and } \ \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{22} \end{vmatrix} > 0,$$

then $E(2)$ has a positive flat extension

$$E(2,1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{pmatrix},$$

for some $\gamma_{03} \in \mathbb{C}$.

Proof. Let

$$\text{(2.2)} \qquad \det E(2,1) = A \, |\gamma_{03}|^2 + 2\mathrm{Re}(C\gamma_{03}) + B = 0.$$

Then a straightforward calculation shows that

$$|C|^2 - AB = \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{vmatrix} \cdot \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{22} \end{vmatrix} \geq 0.$$

So the equation (2.2) has a solution γ_{03} (see [5, Lemma 2.13]). By (2.1), $E(2,1)$ is positive and rank $E(2,1) =$ rank $E(2) = 4$. Therefore, $E(2)$ has a positive flat extension $E(2,1)$. □

Remark. The conditions (2.1) of Proposition 2.1 are essential. In fact, let

$$E(2) = \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 4 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 4 \end{pmatrix}.$$

Then $E(2)$ is positive and nonsingular. But since

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2},$$

the former condition of (2.1) is not satisfied. Let

$$E(2) = \begin{pmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 2 & \frac{1+i}{2} \\ \frac{1}{2} & 0 & \frac{1-i}{2} & 2 \end{pmatrix}.$$

Then $E(2)$ is positive and nonsingular. But since

$$\begin{vmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 2 \end{vmatrix} = -\frac{21}{16},$$

the latter condition of (2.1) is not satisfied.

Proposition 2.2. *Assume that $E(2)$ is positive and nonsingular. If (2.1) is satisfied, and there exists $\gamma_{03} \in \mathbb{C}$ such that*

$$\begin{vmatrix} \gamma_{00} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{vmatrix} \geq 0, \tag{2.3}$$

then $E(2)$ has a positive flat extension

$$E(2,2) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}$$

for some $\gamma_{04} \in \mathbb{C}$.

Proof. In fact, if

$$\det E(2,2) = A\,|\gamma_{04}|^2 + 2\mathrm{Re}(C\gamma_{04}) + B = 0, \tag{2.4}$$

then a straightforward calculation shows that

$$|C|^2 - AB = \begin{vmatrix} \gamma_{00} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{vmatrix} \cdot \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{vmatrix} \geq 0.$$

Thus equation (2.4) has a solution γ_{04} (see [5, Lemma 2.13]). Since (2.3) is satisfied, we know that $E(2,2)$ is positive, and rank $E(2,2) =$ rank $E(2) = 4$. Therefore, $E(2)$ has a positive flat extension $E(2,2)$. □

By Proposition 2.1 and Proposition 2.2, we have the following theorem.

Theorem 2.3. *Assume that $E(2)$ is positive and nonsingular. If (2.1) is satisfied and (2.3) is also satisfied for some $\gamma_{03} \in \mathbb{C}$, then $E(2)$ can be positively flat extended to $M(2)$.*

3. Main results

Proposition 3.1. *Let $\gamma^{(1)} := \{1,0,0,0,a,0,0,0,0,b,0\}$ with $a, b \in \mathbb{R}$ and let the corresponding moment matrix be*

$$E(2) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & b \end{pmatrix}. \tag{3.1}$$

If $a > 0$ and $b > a^2$, then $E(2)$ as in (3.1) is positive and invertible, and it has a positive double flat extension $E(4)$. Therefore, $\gamma^{(1)}$ admits a 4-atomic representing measure.

Proof. See the proof of [7, Corollary 5.2]. □

The following result is an improvement of Proposition 3.1.

Proposition 3.2. *Let* $\gamma^{(2)} := \{1,0,0,0,a,0,0,0,\gamma_{13},b,\gamma_{31}\}$ *with* $a,b \in \mathbb{R}$, $\gamma_{31} = \bar{\gamma}_{13}$ *and let the corresponding moment matrix be*

$$E(2) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & \gamma_{31} \\ a & 0 & \gamma_{13} & b \end{pmatrix}, \tag{3.2}$$

where $\gamma_{13} \neq 0$. *If* $a > 0, b > a^2$, *and* $|\gamma_{13}|^2 < b\left(b - a^2\right)$, *then* $E(2)$ *as in* (3.2) *is positive and invertible, and it has a positive flat extension* $M(3)$. *Furthermore,* $\gamma^{(2)}$ *admits a 4-atomic representing measure.*

Proof. The conditions of Theorem 2.3 are $a > 0$, $b > a^2$ and

$$|\gamma_{03}|^2 \leq \frac{a\left(b^2 - a^2 b - |\gamma_{13}|^2\right)}{b - a^2}.$$

First, we can extend $E(2)$ to $M(2)$ as following

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & a & 0 \\ 0 & a & 0 & 0 & 0 & \gamma_{30} \\ 0 & 0 & a & \gamma_{03} & 0 & 0 \\ 0 & 0 & \gamma_{30} & b & \gamma_{31} & \gamma_{40} \\ a & 0 & 0 & \gamma_{13} & b & \gamma_{31} \\ 0 & \gamma_{03} & 0 & \gamma_{04} & \gamma_{13} & b \end{pmatrix}, \tag{3.3}$$

where $|\gamma_{03}|^2 = \frac{a\left(b^2 - a^2 b - |\gamma_{13}|^2\right)}{b-a^2}$ $(< ab)$ and $\gamma_{04} = \frac{\gamma_{13}^2}{b-a^2}$. Since

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & \gamma_{03} \\ 0 & 0 & \gamma_{30} & b \end{vmatrix} = -a\left(|\gamma_{03}|^2 - ab\right) > 0,$$

we know that $\{\mathit{1}, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$. If we let

$$\bar{Z}Z = A\mathit{1} + BZ + C\bar{Z} + DZ^2,$$

then

$$A = a, \ B = 0, \ C = \frac{\gamma_{31}\gamma_{03}}{|\gamma_{03}|^2 - ab}, \ D = -\frac{a\gamma_{31}}{|\gamma_{03}|^2 - ab},$$

where $D \neq 0$. By [3, Theorem 1.3], we know that $M(2)$ has a flat extension $M(3)$ if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that

$$\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}.$$

For $M(2)$ as in (3.3), this means

$$\bar{\gamma}_{23} = \gamma_{03}\frac{\gamma_{31}^2}{|\gamma_{03}|^2 - ab} - a\gamma_{31}\frac{\gamma_{23}}{|\gamma_{03}|^2 - ab}. \tag{3.4}$$

Claim. There exists $\gamma_{23} \in \mathbb{C}$ satisfying (3.4).

Indeed, if we let $\gamma_{03} = x + yi, \quad \gamma_{13} = s + ti, \quad \gamma_{23} = u + vi$, then (3.4) is equivalent to

$$\begin{array}{c}(as - ab + x^2 + y^2)\,u + atv - 2sty - s^2x + t^2x = 0,\\ -atu + (ab + as - x^2 - y^2)v + 2stx - s^2y + t^2y = 0.\end{array} \tag{3.5}$$

Since

$$\begin{aligned}&\begin{vmatrix} as - ab + x^2 + y^2 & at \\ -at & ab + as - x^2 - y^2 \end{vmatrix}\\ = \ & \left(a\sqrt{s^2+t^2} + (ab - x^2 - y^2)\right)\left(a\sqrt{s^2+t^2} - (ab - x^2 - y^2)\right)\end{aligned}$$

and

$$a\sqrt{s^2+t^2} - (ab - x^2 - y^2) = \frac{a\,|\gamma_{13}|}{b - a^2}\left(b - a^2 - |\gamma_{13}|\right),$$

if $|\gamma_{13}| \neq b - a^2$, then the equation (3.4) admits a unique solution.

For the case $|\gamma_{13}| = b - a^2$, $|\gamma_{03}|^2 = a^3$, and also (3.5) is equivalent to

$$\begin{aligned} u\left(as - ab + a^3\right) + atv &= \left(s^2 - t^2\right)x + 2sty,\\ -atu + v\left(ab + as - a^3\right) &= \left(s^2 - t^2\right)y - 2stx. \end{aligned}$$

Let $x = a\sqrt{a}\cos\theta, y = a\sqrt{a}\sin\theta, s = \left(b - a^2\right)\cos\alpha, t = \left(b - a^2\right)\sin\alpha$. Then

$$\begin{vmatrix} \left(s^2 - t^2\right)x + 2sty & at \\ \left(s^2 - t^2\right)y - 2stx & ab + as - a^3 \end{vmatrix} = 2a^2\sqrt{a}\left(b - a^2\right)^3\cos\left(\theta - \frac{3}{2}\alpha\right)\cos\alpha.$$

Thus, if $\cos\alpha = 0$, i.e., $\mathrm{Re}\gamma_{13} = 0$, then the equation (3.5) admits infinitely many solutions. If $\cos\alpha \neq 0$, then we consider $\cos\left(\theta - \frac{3}{2}\alpha\right) = 0$. So $\theta - \frac{3}{2}\alpha = \frac{\pi}{2}$ or $\theta - \frac{3}{2}\alpha = \frac{3\pi}{2}$. Therefore, in each case, $E(2)$ admits a flat extension $M(3)$. Thus we have our conclusion. □

Example 3.3. Let

$$E(2) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -\frac{i}{2} \\ 1 & 0 & \frac{i}{2} & 2 \end{pmatrix}.$$

Then $E(2)$ has positive flat extension

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{2-\sqrt{3}i}{2} \\ 0 & 0 & 1 & \frac{2+\sqrt{3}i}{2} & 0 & 0 \\ 0 & 0 & \frac{2-\sqrt{3}i}{2} & 2 & -\frac{i}{2} & -\frac{1}{4} \\ 1 & 0 & 0 & \frac{i}{2} & 2 & -\frac{i}{2} \\ 0 & \frac{2+\sqrt{3}i}{2} & 0 & -\frac{1}{4} & \frac{i}{2} & 2 \end{pmatrix}.$$

By straightforward calculation, we obtain the following representing measure

$$\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3},$$

where the atoms and the densities are $z_0 \approx -0.8039+0.846697i$, $z_1 \approx -0.59528-1.49938i$, $z_2 \approx 0.175888 - 0.22674i$, $z_3 \approx 1.46731 + 0.501431i$; $\rho_0 \approx 0.25919$, $\rho_1 \approx 0.113986$, $\rho_2 \approx 0.49831$, $\rho_3 \approx 0.128514$.

Proposition 3.4. *Let* $\gamma^{(3)} := \{1, \gamma_{01}, \gamma_{10}, 0, a, 0, 0, 0, 0, b, 0\}$ *with* $a, b \in \mathbb{R}$, $\gamma_{10} = \bar{\gamma}_{01}$ *and the corresponding moment matrix be*

$$E(2) = \begin{pmatrix} 1 & \gamma_{01} & 0 & a \\ \gamma_{10} & a & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & b \end{pmatrix}, \tag{3.6}$$

where $\gamma_{01} \neq 0$. *If* $a > 0, b > a^2$, *and* $|\gamma_{01}|^2 < \frac{a(b-a^2)}{2b}$, *then* $E(2)$ *as in* (3.6) *is positive and invertible, and it has a positive flat extension* $M(3)$. *Furthermore,* $\gamma^{(3)}$ *admits a 4-atomic representing measure.*

Proof. The conditions of Theorem 2.3 are

$$|\gamma_{01}|^2 < \frac{a(b-a^2)}{2b} \left(< \frac{a}{2}\right) \quad \text{and} \quad |\gamma_{03}|^2 \leq \frac{-b\left(a^3 - ab + b\,|\gamma_{01}|^2\right)}{b-a^2}.$$

We extend $E(2)$ as in (3.6) to the following

$$M(2) = \begin{pmatrix} 1 & \gamma_{01} & \gamma_{10} & 0 & a & 0 \\ \gamma_{10} & a & 0 & 0 & 0 & \gamma_{30} \\ \gamma_{01} & 0 & a & \gamma_{03} & 0 & 0 \\ 0 & 0 & \gamma_{30} & b & 0 & \gamma_{40} \\ a & 0 & 0 & 0 & b & 0 \\ 0 & \gamma_{03} & 0 & \gamma_{04} & 0 & b \end{pmatrix}, \tag{3.7}$$

where

$$|\gamma_{03}|^2 = \frac{ab\left(a^3 - ab + 2b\,|\gamma_{01}|^2\right)}{a^3 - ab + b\,|\gamma_{01}|^2} \left(< \frac{-b\left(a^3 - ab + b\,|\gamma_{01}|^2\right)}{b-a^2}\right)$$

and

$$|\gamma_{04}| = \frac{|\gamma_{01}|^2 b^2}{-\left(a^3 - ab + b\,|\gamma_{01}|^2\right)}.$$

Since

$$\begin{vmatrix} 1 & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & a & 0 \\ \gamma_{01} & 0 & a \end{vmatrix} = a^2 - 2a\,|\gamma_{01}|^2 > 0,$$

$$\Delta := \begin{vmatrix} 1 & \gamma_{01} & \gamma_{10} & 0 \\ \gamma_{10} & a & 0 & 0 \\ \gamma_{01} & 0 & a & \gamma_{03} \\ 0 & 0 & \gamma_{30} & b \end{vmatrix} = -\frac{|\gamma_{01}|^2 ba^4}{a^3 - ab + b\,|\gamma_{01}|^2} > 0,$$

we know that $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$. If we let

$$\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2, \tag{3.8}$$

then

$$A = \frac{a^2\left(ab - |\gamma_{03}|^2\right)}{\Delta}, B = \frac{a\gamma_{10}\left(|\gamma_{03}|^2 - ab\right)}{\Delta}, C = -\frac{a^2 b\gamma_{01}}{\Delta}, D = \frac{a^2\gamma_{01}\gamma_{30}}{\Delta}.$$

Since $|\gamma_{01}|^2 < \frac{a(b-a^2)}{2b}$, $\gamma_{30} \neq 0$, so $D \neq 0$. By [3, Theorem 1.3], we know that $M(2)$ has a flat extension $M(3)$ if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that

$$\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}.$$

For $M(2)$ as in (3.7), this means

$$\Delta\bar{\gamma}_{23} = ab\gamma_{10}\left(|\gamma_{03}|^2 - ab\right) + a^2\gamma_{01}\gamma_{30}\gamma_{23}. \tag{3.9}$$

Claim. There exists $\gamma_{23} \in \mathbb{C}$ satisfying (3.9).

Indeed, if we let $\gamma_{03} = x + yi$, $\gamma_{01} = s + ti$, $\gamma_{23} = u + vi$, then (3.9) is equivalent to

$$\begin{aligned} 0 = & -abs\left(x^2+y^2\right) + a^2b^2s \\ & + u\left(a^2b - 2ab\left(s^2+t^2\right) - a^2\left(sx+ty\right) + \left(s^2+t^2-a\right)\left(x^2+y^2\right)\right) \\ & + va^2\left(tx - sy\right), \\ 0 = & \, abt\left(x^2+y^2\right) - a^2b^2t + ua^2\left(sy - tx\right) \\ & + v\left(\left(a - s^2 - t^2\right)\left(x^2+y^2\right) - a^2b + 2ab\left(s^2+t^2\right) - a^2\left(sx+ty\right)\right). \end{aligned}$$

Let

$$\begin{aligned} e_{11} &= a^2b - 2ab\left(s^2+t^2\right) - a^2\left(sx+ty\right) + \left(s^2+t^2-a\right)\left(x^2+y^2\right), \\ e_{12} &= a^2\left(tx - sy\right), \\ e_{21} &= a^2\left(sy - tx\right), \\ e_{22} &= \left(a - s^2 - t^2\right)\left(x^2+y^2\right) - a^2b + 2ab\left(s^2+t^2\right) - a^2\left(sx+ty\right), \\ e_{10} &= abs\left(x^2+y^2\right) - a^2b^2s, \\ e_{20} &= -abt\left(x^2+y^2\right) + a^2b^2t. \end{aligned}$$

Since the determinant of coefficient matrix is

$$\begin{aligned} & \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \\ = & \frac{\left(a^2\left(b-a^2\right)^2 + 2b^2\left(s^2+t^2\right)^2 + ab\left(2a^2-3b\right)\left(s^2+t^2\right)\right)\left(s^2+t^2\right)a^5b}{\left(a^3 - ab + bs^2 + bt^2\right)^2} \\ = & \frac{\left(a^2\left(b-a^2\right)^2 + 2b^2|\gamma_{01}|^4 + ab\left(2a^2-3b\right)|\gamma_{01}|^2\right)|\gamma_{01}|^2 a^5b}{\left(a^3 - ab + b|\gamma_{01}|^2\right)^2}, \end{aligned}$$

if

$$a^2\left(b-a^2\right)^2+2b^2\left|\gamma_{01}\right|^4+ab\left(2a^2-3b\right)\left|\gamma_{01}\right|^2\neq 0,$$

then the equation (3.9) admits a unique solution. For the case

$$a^2\left(b-a^2\right)^2+2b^2\left|\gamma_{01}\right|^4+ab\left(2a^2-3b\right)\left|\gamma_{01}\right|^2=0, \tag{3.10}$$

we claim that

$$\left|\gamma_{01}\right|^2=\frac{a}{4b}\left(3b-2a^2-\sqrt{b^2+4a^2b-4a^4}\right). \tag{3.11}$$

In fact, the solutions of (3.10) are

$$\begin{aligned}\left|\gamma_{01}\right|^2 &= \frac{1}{2b^2}\left(\frac{3}{2}ab^2-a^3b-\frac{1}{2}\sqrt{a^2b^4+4a^4b^3-4a^6b^2}\right), \text{ or}\\ &\ \frac{1}{2b^2}\left(\frac{3}{2}ab^2-a^3b+\frac{1}{2}\sqrt{a^2b^4+4a^4b^3-4a^6b^2}\right).\end{aligned}$$

Note that they are all positive. But

$$\begin{aligned}\frac{1}{2b^2}\left(\frac{3}{2}ab^2-a^3b-\frac{1}{2}\sqrt{a^2b^4+4a^4b^3-4a^6b^2}\right) &< \frac{a(b-a^2)}{2b},\\ \frac{1}{2b^2}\left(\frac{3}{2}ab^2-a^3b+\frac{1}{2}\sqrt{a^2b^4+4a^4b^3-4a^6b^2}\right) &> \frac{a(b-a^2)}{2b},\end{aligned}$$

which proves our claim. Hence

$$\left|\gamma_{03}\right|^2=\frac{2ab\left(\sqrt{b^2-4a^4+4a^2b}-b\right)}{2a^2-3b+4b^2-4a^2b+\sqrt{b^2-4a^4+4a^2b}} \tag{3.12}$$

and

$$\begin{vmatrix} e_{11} & e_{10}\\ e_{21} & e_{20}\end{vmatrix}=\frac{ba^3\left(x^2-ab+y^2\right)}{\left(a^3-ab+bs^2+bt^2\right)}\Omega,$$

where

$$\begin{aligned}\Omega=&\ a^2bt\left(s^2+t^2\right)+2stx\left(a\left(a^2-b\right)+b\left(s^2+t^2\right)\right)\\ &+y\left(-b\left(s^2+t^2\right)+a\left(b-a^2\right)\right)\left(s^2-t^2\right).\end{aligned}$$

Since $ba^3\left(x^2-ab+y^2\right)\left(a^3-ab+bs^2+bt^2\right)\neq 0$, we consider the case of

$$\Omega=0. \tag{3.13}$$

We now claim that there exists γ_{03} satisfying (3.12) and (3.13). In fact,

$$
\begin{aligned}
& \left(\frac{a^2bt}{a^3-ab+bs^2+bt^2}\right)^2-\frac{ab\left(a^3-ab+2b\left(s^2+t^2\right)\right)}{a^3-ab+b\left(s^2+t^2\right)} \\
\leq\ & \left(\frac{a^2b}{a^3-ab+b\left(s^2+t^2\right)}\right)^2\left(s^2+t^2\right)-\frac{ab\left(a^3-ab+2b\left(s^2+t^2\right)\right)}{a^3-ab+b\left(s^2+t^2\right)} \\
=\ & -\frac{\left(a^2\left(b-a^2\right)^2+2b^2\left|\gamma_{01}\right|^4+ab\left(2a^2-3b\right)\left|\gamma_{01}\right|^2\right)ab}{\left(a^3-ab+b\left|\gamma_{01}\right|^2\right)^2} \\
=\ & 0
\end{aligned}
$$

because of (3.10). So $M(2)$ admits a flat extension $M(3)$. Thus we have our conclusion. $\square$

Example 3.5. Let

$$
E(2)=\begin{pmatrix}
1 & \frac{i}{\sqrt{10}} & 0 & 2 \\
-\frac{i}{\sqrt{10}} & 2 & 0 & 0 \\
0 & 0 & 5 & 0 \\
2 & 0 & 0 & 5
\end{pmatrix}.
$$

Then $E(2)$ has positive flat extension

$$
M(2)=\begin{pmatrix}
1 & \frac{i}{\sqrt{10}} & -\frac{i}{\sqrt{10}} & 0 & 2 & 0 \\
-\frac{i}{\sqrt{10}} & 2 & 0 & 0 & 0 & \frac{2-4i}{\sqrt{3}} \\
\frac{i}{\sqrt{10}} & 0 & 2 & \frac{2+4i}{\sqrt{3}} & 0 & 0 \\
0 & 0 & \frac{2-4i}{\sqrt{3}} & 5 & 0 & 1+\frac{4i}{3} \\
2 & 0 & 0 & 0 & 5 & 0 \\
0 & \frac{2+4i}{\sqrt{3}} & 0 & 1-\frac{4i}{3} & 0 & 5
\end{pmatrix}.
$$

By straightforward calculation, we obtain the following representing measure

$$
\mu=\rho_0\delta_{z_0}+\rho_1\delta_{z_1}+\rho_2\delta_{z_2}+\rho_3\delta_{z_3},
$$

where the atoms and the densities are $z_0 \approx -1.13072+0.84609i$, $z_1 \approx -0.31061-1.79556i$, $z_2 \approx 0.71198+0.73166i$, $z_3 \approx 1.9995+0.22833i$; $\rho_0 \approx 0.334864$, $\rho_1 \approx 0.18872$, $\rho_2 \approx 0.396101$, $\rho_3 \approx 0.0803147$.

Proposition 3.6. *Let $\gamma^{(4)} := \{1,0,0,t,a,t,0,0,f,b,f\}$ with $a,b,t,f \in \mathbb{R}$ and the corresponding moment matrix be*

$$
E(2)=\begin{pmatrix}
1 & 0 & t & a \\
0 & a & 0 & 0 \\
t & 0 & b & f \\
a & 0 & f & b
\end{pmatrix}, \tag{3.14}
$$

where t, f are nonzero real numbers. If $a > 0, b > t^2$, and $\left(b - a^2\right)\left(b - t^2\right) > \left(f - at\right)^2$, then $E(2)$ is positive and invertible. Furthermore, $\gamma^{(4)}$ admits a 4-atomic representing measure in the following two cases:
(i) $f \neq at$, $f \neq b + at - a^2$ *and* $a^2 > t^2$;
(ii) $f = at$ *and* $a^2 > t^2$.

Proof. Let $\gamma_{03} = x, \gamma_{23} = y, \gamma_{04} = z$. The conditions of Theorem 2.3 are

$$\begin{aligned} 0 &< b - a^2, \quad 0 < a^2 - t^2 \quad \text{and} \\ x^2 &\leq \frac{a\left(\left(b-a^2\right)\left(b-t^2\right) - \left(f - at\right)^2\right)}{b - a^2}. \end{aligned}$$

We extend $E(2)$ as in (3.14) to the following

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & t & a & t \\ 0 & a & t & 0 & 0 & x \\ 0 & t & a & x & 0 & 0 \\ t & 0 & x & b & f & z \\ a & 0 & 0 & f & b & f \\ t & x & 0 & z & f & b \end{pmatrix}.$$

It follows from

$$\begin{vmatrix} 1 & 0 & 0 & t & a \\ 0 & a & t & 0 & 0 \\ 0 & t & a & x & 0 \\ t & 0 & x & b & f \\ a & 0 & 0 & f & b \end{vmatrix} = 0$$

that

$$\begin{aligned} x^2 &= \frac{\left(a^2 - t^2\right)\left(\left(b-a^2\right)\left(b-t^2\right) - \left(f - at\right)^2\right)}{a\left(b - a^2\right)} \\ &< \frac{a\left(\left(b-a^2\right)\left(b-t^2\right) - \left(f - at\right)^2\right)}{b - a^2}. \end{aligned}$$

Since

$$\begin{vmatrix} 1 & 0 & 0 & t \\ 0 & a & t & 0 \\ 0 & t & a & x \\ t & 0 & x & b \end{vmatrix} = \frac{\left(a^2 - t^2\right)\left(f - at\right)^2}{b - a^2} \geq 0,$$

we may consider the following two cases.

Case 1. If $f \neq at$ and $a^2 > t^2$, then $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$. By direct calculation, we have

$$\begin{aligned} A &= \frac{abt^2 - ft^3 - a^3b + a^2ft + a^2x^2}{bt^2 - a^2b - t^4 + ax^2 + a^2t^2}, \\ B &= \frac{(at-f)\,tx}{bt^2 - a^2b - t^4 + ax^2 + a^2t^2}, \\ C &= \frac{(at-f)\,ax}{bt^2 - a^2b - t^4 + ax^2 + a^2t^2}, \\ D &= \frac{(at-f)\left(a^2-t^2\right)}{bt^2 - a^2b - t^4 + ax^2 + a^2t^2}. \end{aligned}$$

Observe that

$$\begin{aligned} 0 &= y - (Bb + Cf + Dy) \\ &= \frac{\left(b-a^2\right)x\,(af-bt) + y\left(f-b-at+a^2\right)\left(a^2-t^2\right)}{\left(a^2-t^2\right)(f-at)}, \end{aligned}$$

which implies that

(i) if $f = b + at - a^2$ and $af - bt = 0$, then any y is the solution of the above equality. But under this condition, $f = b + \frac{a^2 f}{b} - a^2$, and so $f = b$, which gives a contradiction because of the positivity of $E(2)$; and if $f = b + at - a^2$ and $af - bt \neq 0$, then $x = 0$ and any y is the solution of the above equality. But under this condition $x^2 = \frac{\left(a^2-t^2\right)\left(\left(b-a^2\right)\left(b-t^2\right)-(f-at)^2\right)}{a\left(b-a^2\right)} = 0$, we obtain $t^2 = a^2$. Hence this is a contradiction since $a^2 > t^2$. So, $M(2)$ has no flat extension $M(3)$.

(ii) if $f \neq b + at - a^2$, then $M(2)$ has a unique flat extension $M(3)$.

Case 2. If $f = at$ and $a^2 > t^2$, then $E(2)$ has a positive flat extension

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & t & a & t \\ 0 & a & t & 0 & 0 & x \\ 0 & t & a & x & 0 & 0 \\ t & 0 & x & b & at & z \\ a & 0 & 0 & at & b & at \\ t & x & 0 & z & at & b \end{pmatrix},$$

where

$$x = \sqrt{\frac{\left(b-t^2\right)\left(a^2-t^2\right)}{a}}, \quad z = \frac{t\left(at-b+t^2\right)}{a}.$$

Since $1, Z, Z^2$, and $\bar{Z}Z$ are linearly independent, we have

$$\begin{aligned} \bar{Z} &= p_0 1 + p_1 Z + p_2 Z^2 + p_3 \bar{Z}Z, \\ \bar{Z}^2 &= q_0 1 + q_1 Z + q_2 Z^2 + q_3 \bar{Z}Z. \end{aligned}$$

By a straightforward calculation, we have

$$p_2 = \frac{\sqrt{\frac{1}{a}\left(b-t^2\right)\left(a^2-t^2\right)}}{b-t^2} \neq 0, \quad p_3 = q_3 = 0,$$

and so $\gamma^{(4)}$ admits a 4-atomic representing measure for $E(2)$. □

Example 3.7. Let

$$E(2) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 5 & \frac{5}{2} \\ 2 & 0 & \frac{5}{2} & 5 \end{pmatrix},$$

that is, $f = \frac{5}{2}, a = 2, b = 5, t = 1$. Then the condition (i) of Proposition 3.6 is satisfied. In this case, $E(2)$ has a positive flat extension

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & \frac{3\sqrt{10}}{4} \\ 0 & 1 & 2 & \frac{3\sqrt{10}}{4} & 0 & 0 \\ 1 & 0 & \frac{3\sqrt{10}}{4} & 5 & \frac{5}{2} & -\frac{5}{8} \\ 2 & 0 & 0 & \frac{5}{2} & 5 & \frac{5}{2} \\ 1 & \frac{3\sqrt{10}}{4} & 0 & -\frac{5}{8} & \frac{5}{2} & 5 \end{pmatrix}.$$

By a straightforward calculation, we obtain the following representing measure

$$\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3},$$

where the atoms and the densities are $z_0 \approx -1.18585-1.04582i$, $z_1 \approx -1.18585+1.04582i$, $z_2 \approx 0$, $z_3 \approx 1.58114$; $\rho_0 \approx 0.228571$, $\rho_1 \approx 0.228571$, $\rho_2 \approx 0.2$, $\rho_3 \approx 0.342857$.

Example 3.8. Let

$$E(2) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 5 & 2 \\ 2 & 0 & 2 & 5 \end{pmatrix},$$

that is, $f = 2, a = 2, b = 5, t = 1$. Then the condition (ii) of Proposition 3.6 is satisfied. In this case, $E(2)$ has a positive flat extension

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & \sqrt{6} \\ 0 & 1 & 2 & \sqrt{6} & 0 & 0 \\ 1 & 0 & \sqrt{6} & 5 & 2 & -1 \\ 2 & 0 & 0 & 2 & 5 & 2 \\ 1 & \sqrt{6} & 0 & -1 & 2 & 5 \end{pmatrix}.$$

By a straightforward calculation, we obtain

$$\begin{aligned}\bar{Z} &= -\frac{\sqrt{6}}{4}\mathit{1} + \frac{1}{2}Z + \frac{\sqrt{6}}{4}Z^2, \\ \bar{Z}^2 &= \frac{3}{2}\mathit{1} + \frac{\sqrt{6}}{2}Z - \frac{1}{2}Z^2.\end{aligned}$$

The four atoms are the roots of $3z^4 + 2\sqrt{6}z^3 - 6\sqrt{6}z - 9 = 0$, that is, $z_0 = -0.671\,88, z_1 = 1.488\,4, z_2 = -1.224\,7 - 1.224\,7i, z_3 = -1.224\,7 + 1.224\,7i$, and the densities are $\rho_0 \approx 0.16667, \rho_1 \approx 0.16667, \rho_2 \approx 0.27033, \rho_3 \approx 0.39633$. So we obtain a representing measure as following

$$\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3}.$$

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