Bull. Korean Math. Soc. 44 (2007), No. 2, pp. 291–308

INTUITIONISTIC FUZZY n-NORMED LINEAR SPACE

Srinivasan Vijayabalaji, Natesan Thillaigovindan, and Young Bae Jun

Abstract. The motivation of this paper is to present a new and interesting notion of intuitionistic fuzzy n-normed linear space. Cauchy sequence and convergent sequence in intuitionistic fuzzy n-normed linear space are introduced and we provide some results on it. Furthermore we introduce generalized cartesian product of the intuitionistic fuzzy n-normed linear space and establish some of its properties.

1. Introduction

In [10, 11], S. Gähler introduced an attractive theory of 2-norm and n-norm on a linear space. A systematic development of an n-normed linear space has been extensively made by S. S. Kim and Y. J. Cho [15], R. Malceski [17], A. Misiak [18] and Hendra Gunawan [13]. In [13], Hendra Gunawan and Mashadi gave a simple way to derive an $(n-1)$-norm from the n-norm and realized that any n-normed space is an $(n-1)$-normed space. A detailed theory of fuzzy normed linear space can be found in [5, 6, 7, 8, 9, 14, 16, 21]. In [19], we have extended n-normed linear space to fuzzy n-normed linear space. The origin and development of intuitionistic fuzzy set theory can be found in [1, 2, 3, 4, 12].

The purpose of this paper is to introduce the notion of intuitionistic fuzzy n-normed linear space as a further generalization of fuzzy n-normed linear space [19]. Cauchy sequence and convergent sequence in intuitionistic fuzzy n-normed linear space are introduced. The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is introduced and we provide some results on it.

2. Preliminaries

This section is devoted to the collection of basic definitions and results which will be needed in the sequel.

Definition 2.1 ([10]). Let X be a real linear space of dimension greater than 1 and let $||\bullet, \bullet||$ be a real valued function on $X \times X$ satisfying the following conditions :

Received July 4, 2006.
2000 *Mathematics Subject Classification.* 03E55, 46S40, 03E72.
Key words and phrases. n-norm, fuzzy n-norm, intuitionistic fuzzy n-norm.

(1) $||x, y|| = 0$ if any only if x and y are linearly dependent
(2) $||x, y|| = ||y, x||$
(3) $||\alpha x, y|| = |\alpha|\ ||x, y||$, where $\alpha \in R$ (set of real numbers)
(4) $||x, y + z|| \leq ||x, y|| + ||x, z||$.

$||\bullet, \bullet||$ is called a 2-norm on X and the pair (X,$||\bullet, \bullet||$) is called a 2-normed linear space.

Definition 2.2 ([13]). Let $n \in N$(natural numbers) and X be a real linear space of dimension $d \geq n$.(Here we allow d to be infinite). A real valued function $||\bullet, \dots, \bullet||$ on $\underbrace{X \times \cdots \times X}_{n} = X^n$ satisfying the following four properties:

(1) $||x_1, x_2, \dots, x_n|| = 0$ if any only if $x_1, x_2, \dots, x_n$ are linearly dependent
(2) $||x_1, x_2, \dots, x_n||$ is invariant under any permutation
(3) $||x_1, x_2, \dots, \alpha x_n|| = |\alpha|\ ||x_1, x_2, \dots, x_n||$, for any $\alpha \in R$(set of real numbers)
(4) $||x_1, x_2, , x_{n-1}, y + z|| \leq ||x_1, x_2, \dots, x_{n-1}, y|| + ||x_1, x_2, \dots, x_{n-1}, z||$,

is called an n-norm on X and the pair $(X, ||\bullet, \dots, \bullet||)$ is called an n-normed linear space.

Definition 2.3 ([13]). A sequence $\{x_n\}$ in an n-normed linear space

$$(X, ||\bullet, \dots, \bullet||)$$

is said to converge to an $x \in X$ (in the n-norm) whenever

$$\lim_{n \to \infty} ||x_1, x_2, \dots, x_{n-1}, x_n - x|| = 0.$$

Definition 2.4 ([13]). A sequence $\{x_n\}$ in an n-normed linear space

$$(X, ||\bullet, \dots, \bullet||)$$

is called Cauchy sequence if $\lim\limits_{n,k \to \infty} ||x_1, x_2, \dots, x_{n-1}, x_n - x_k|| = 0$.

Definition 2.5 ([13]). An n-normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition 2.6 ([19]). Let X be a linear space over a field F. A fuzzy subset N of $X^n \times R$ (set of real numbers) is called a fuzzy n-norm on X if and only if :

(N1) For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$.
(N2) For all $t \in R$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, ..., x_n are linearly dependent.
(N3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$.
(N4) For all $t \in R$ with $t > 0$,

$$N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|}), \text{ if } c \neq 0, c \in F(\text{field}).$$

(N5) For all $s, t \in R$,

$$N(x_1, x_2, \ldots, x_n + x_n^{'}, s+t) \geq \min\left\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x_n^{'}, t)\right\}.$$

(N6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$ and

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Definition 2.7 ([22]). A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative
(2) $*$ is continuous
(3) $a * 1 = a$, for all $a \in [0,1]$
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.8 ([22]). A binary operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is continuous t-co-norm if $\diamond$ satisfies the following conditions:

(1) $\diamond$ is commutative and associative
(2) $\diamond$ is continuous
(3) $a \diamond 0 = a$, for all $a \in [0,1]$
(4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Remark 2.9 ([20]). (a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamond r_2$.

(b) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.10 ([2]). Let E be any set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in E\}$, where the functions $\mu_A : E \to [0,1]$ and $\gamma_A : E \to [0,1]$ denote the degree of membership and the non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

Definition 2.11 ([13]). If A and B are any two intuitionistic fuzzy sets of a non-empty set E, then $A \subseteq B$ if and only if for all

$x \in E, \mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$;
A=B if and only if for all $x \in E, \mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x)$;
$\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) | x \in E\}$;
$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))) | x \in E\}$;
$A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))) | x \in E\}$.

Definition 2.12 ([12]). Let A and B be intuitionistic fuzzy sets in E_1 and E_2 respectively. Then the generalized cartesian product

$$A \times_{T,S} B = \{((x,y), T(\mu_A(x), \mu_B(y)), S(\gamma_A(x), \gamma_B(y))) | x \in E_1 \text{ and } y \in E_2\},$$

T denotes the t-norm and S denotes the t-co-norm.

3. Intuitionistic fuzzy n-normed linear space

By generalizing Definition 2.6 we obtain a new and interesting notion of intuitionistic fuzzy n-normed linear space as follows:

Definition 3.1. An intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS is an object of the form

$$A = \{(X, N(x_1, x_2, \dots, x_n, t), M(x_1, x_2, \dots, x_n, t)) | (x_1, x_2, \dots, x_n) \in X^n\},$$

where X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$, N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, \dots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions:

(i) $N(x_1, x_2, \dots, x_n, t) + M(x_1, x_2, \dots, x_n, t) \leq 1$;
(ii) $N(x_1, x_2, \dots, x_n, t) > 0$;
(iii) $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
(iv) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$;
(v) $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field);
(vi) $N(x_1, x_2, \dots, x_n, s) * N(x_1, x_2, \dots, x_n', t) \leq N(x_1, x_2, \dots, x_n + x_n', s + t)$;
(vii) $N(x_1, x_2, \dots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t;
(viii) $M(x_1, x_2, \dots, x_n, t) > 0$;
(ix) $M(x_1, x_2, \dots, x_n, t) = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
(x) $M(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$;
(xi) $M(x_1, x_2, \dots, cx_n, t) = M(x_1, x_2, \dots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field);
(xii) $M(x_1, x_2, \dots, x_n, s) \diamond M(x_1, x_2, \dots, x_n', t) \geq M(x_1, x_2, \dots, x_n + x_n', s + t)$;
(xiii) $M(x_1, x_2, \dots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

To strengthen the above definition, we present the following example.

Example 3.2. Let $(X, ||\bullet, \bullet, \dots, \bullet||)$ be an n-normed linear space. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$, $N(x_1, x_2, \dots, x_n, t) = \frac{t}{t + ||x_1, x_2, \dots, x_n||}$, $M(x_1, x_2, \dots, x_n, t) = \frac{||x_1, x_2, \dots, x_n||}{t + ||x_1, x_2, \dots, x_n||}$. Then

$$A = \{(X, N(x_1, x_2, \dots, x_n, t), M(x_1, x_2, \dots, x_n, t)) | (x_1, x_2, \dots, x_n) \in X^n\}$$

is an i-f-n-NLS.

Proof. (i) Clearly $N(x_1, x_2, \dots, x_n, t) + M(x_1, x_2, \dots, x_n, t) \leq 1$.
(ii) Obviously $N(x_1, x_2, \dots, x_n, t) > 0$.

(iii)

$$\begin{aligned} N(x_1, x_2, \ldots, x_n, t) = 1 &\Leftrightarrow \frac{t}{t + ||x_1, x_2, \ldots, x_n||} = 1 \\ &\Leftrightarrow t = t + ||x_1, x_2, \ldots, x_n|| \\ &\Leftrightarrow ||x_1, x_2, \ldots, x_n|| = 0 \\ &\Leftrightarrow x_1, x_2, \ldots, x_n \text{ are linearly dependent.} \end{aligned}$$

(iv)

$$\begin{aligned} N(x_1, x_2, \ldots, x_n, t) &= \frac{t}{t + ||x_1, x_2, \ldots, x_n||} = \frac{t}{t + ||x_1, x_2, \ldots, x_n, x_{n-1}||} \\ &= N(x_1, x_2, \ldots, x_n, x_{n-1}, t) \\ &= \cdots . \end{aligned}$$

(v)

$$\begin{aligned} N(x_1, x_2, \ldots, x_n, \frac{t}{|c|}) &= \frac{\frac{t}{|c|}}{\frac{t}{|c|} + ||x_1, x_2, \ldots, x_n||} = \frac{\frac{t}{|c|}}{\frac{t + |c|||x_1, x_2, \ldots, x_n||}{|c|}} \\ &= \frac{t}{t + |c|||x_1, x_2, \ldots, x_n||} = \frac{t}{t + ||x_1, x_2, \ldots, cx_n||} \\ &= N(x_1, x_2, \ldots, cx_n, t). \end{aligned}$$

(vi) Without loss of generality assume that,

$$\begin{aligned} &N(x_1, x_2, \ldots, x_n', t) \leq N(x_1, x_2, \ldots, x_n, s). \\ \Rightarrow\ &\frac{t}{t + ||x_1, x_2, \ldots, x_n'||} \leq \frac{s}{s + ||x_1, x_2, \ldots, x_n||} \\ \Rightarrow\ &t(s + ||x_1, x_2, \ldots, x_n||) \leq s(t + ||x_1, x_2, \ldots, x_n'||) \\ \Rightarrow\ &t||x_1, x_2, \ldots, x_n|| \leq s||x_1, x_2, \ldots, x_n'|| \\ \Rightarrow\ &||x_1, x_2, \ldots, x_n|| \leq \frac{s}{t}||x_1, x_2, \ldots, x_n'||. \end{aligned}$$

Therefore,

$$\begin{aligned} &||x_1, x_2, \ldots, x_n|| + ||x_1, x_2, \ldots, x_n'|| \\ \leq\ &\frac{s}{t}||x_1, x_2, \ldots, x_n'|| + ||x_1, x_2, \ldots, x_n'|| \\ \leq\ &(\frac{s}{t} + 1)||x_1, x_2, \ldots, x_n'|| = (\frac{s + t}{t})||x_1, x_2, \ldots, x_n'||. \end{aligned}$$

But,

$$\begin{aligned}||x_1, x_2, \dots, x_n + x_n'|| &\le ||x_1, x_2, \dots, x_n|| + ||x_1, x_2, \dots, x_n'|| \\ &\le (\frac{s+t}{t})||x_1, x_2, \dots, x_n'||.\end{aligned}$$

$$\begin{aligned}&\Rightarrow \frac{||x_1, x_2, \dots, x_n + x_n'||}{s+t} \le \frac{||x_1, x_2, \dots, x_n'||}{t} \\ &\Rightarrow 1 + \frac{||x_1, x_2, \dots, x_n + x_n'||}{s+t} \le 1 + \frac{||x_1, x_2, \dots, x_n'||}{t} \\ &\Rightarrow \frac{s+t+||x_1, x_2, \dots, x_n + x_n'||}{s+t} \le \frac{t+||x_1, x_2, \dots, x_n'||}{t} \\ &\Rightarrow \frac{s+t}{s+t+||x_1, x_2, \dots, x_n + x_n'||} \ge \frac{t}{t+||x_1, x_2, \dots, x_n'||} \\ &\Rightarrow \quad N(x_1, x_2, \dots, x_n + x_n', s+t) \\ &\quad \ge \min\left\{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x_n', t)\right\}.\end{aligned}$$

(vii) Clearly $N(x_1, x_2, \dots, x_n, t)$ is continuous in t.
(viii) $M(x_1, x_2, \dots, x_n, t) > 0$.
(ix)

$$\begin{aligned}&M(x_1, x_2, \dots, x_n, t) = 0 \\ \Leftrightarrow\ &\frac{||x_1, x_2, \dots, x_n||}{t + ||x_1, x_2, \dots, x_n||} = 0 \\ \Leftrightarrow\ &||x_1, x_2, \dots, x_n|| = 0 \\ \Leftrightarrow\ &x_1, x_2, \dots, x_n \text{ are linearly dependent.}\end{aligned}$$

(x)

$$\begin{aligned}M(x_1, x_2, \dots, x_n, t) &= \frac{||x_1, x_2, \dots, x_n||}{t + ||x_1, x_2, \dots, x_n||} \\ &= \frac{||x_1, x_2, \dots, x_n, x_{n-1}||}{t + ||x_1, x_2, \dots, x_n, x_{n-1}||} \\ &= M(x_1, x_2, \dots, x_n, x_{n-1}, t) = \cdots .\end{aligned}$$

(xi)

$$\begin{aligned}M(x_1, x_2, \dots, cx_n, t) &= \frac{||x_1, x_2, \dots, cx_n||}{t + ||x_1, x_2, \dots, cx_n||} = \frac{|c|||x_1, x_2, \dots, x_n||}{t + |c|||x_1, x_2, \dots, x_n||} \\ &= \frac{||x_1, x_2, \dots, x_n||}{\frac{t}{|c|} + ||x_1, x_2, \dots, x_n||} = M(x_1, x_2, \dots, x_n, \frac{t}{|c|}).\end{aligned}$$

(xii) Without loss of generality assume,

$$M(x_1,x_2,\ldots,x_n,s) \le M(x_1,x_2,\ldots,x_n^{'},t).$$

$$\begin{aligned} &\Rightarrow \frac{||x_1,x_2,\ldots,x_n||}{s+||x_1,x_2,\ldots,x_n||} \le \frac{||x_1,x_2,\ldots,x_n^{'}||}{t+||x_1,x_2,\ldots,x_n^{'}||} \\ (1) \quad &\Rightarrow \quad ||x_1,x_2,\ldots,x_n||(t+||x_1,x_2,\ldots,x_n^{'}||) \\ &\le ||x_1,x_2,\ldots,x_n^{'}||(s+||x_1,x_2,\ldots,x_n||) \\ &\Rightarrow t||x_1,x_2,\ldots,x_n|| \le s||x_1,x_2,\ldots,x_n^{'}||. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{||x_1,x_2,\ldots,x_n+x_n^{'}||}{s+t+||x_1,x_2,\ldots,x_n+x_n^{'}||} - \frac{||x_1,x_2,\ldots,x_n^{'}||}{t+||x_1,x_2,\ldots,x_n^{'}||} \\ &\le \frac{||x_1,x_2,\ldots,x_n||+||x_1,x_2,\ldots,x_n^{'}||}{s+t+||x_1,x_2,\ldots,x_n||+||x_1,x_2,\ldots,x_n^{'}||} - \frac{||x_1,x_2,\ldots,x_n^{'}||}{t+||x_1,x_2,\ldots,x_n^{'}||} \\ &= \frac{t||x_1,x_2,\ldots,x_n||-s||x_1,x_2,\ldots,x_n^{'}||}{(s+t+||x_1,x_2,\ldots,x_n||+||x_1,x_2,\ldots,x_n^{'}||)(t+||x_1,x_2,\ldots,x_n^{'}||)}. \end{aligned}$$

By (1),

$$\frac{||x_1,x_2,\ldots,x_n+x_n^{'}||}{s+t+||x_1,x_2,\ldots,x_n+x_n^{'}||} \le \frac{||x_1,x_2,\ldots,x_n^{'}||}{t+||x_1,x_2,\ldots,x_n^{'}||}.$$

Similarly,

$$\begin{aligned} &\frac{||x_1,x_2,\ldots,x_n+x_n^{'}||}{s+t+||x_1,x_2,\ldots,x_n+x_n^{'}||} \le \frac{||x_1,x_2,\ldots,x_n||}{s+||x_1,x_2,\ldots,x_n||}. \\ \Rightarrow \quad &M(x_1,x_2,\ldots,x_n+x_n^{'},s+t) \\ &\le \max\left\{M(x_1,x_2,\ldots,x_n,s), M(x_1,x_2,\ldots,x_n^{'},t)\right\}. \end{aligned}$$

(xiii) Clearly $M(x_1,x_2,\ldots,x_n,t)$ is continuous in t. Thus A is an i-f-n-NLS. □

Definition 3.3. A sequence $\{x_n\}$ in an i-f-n-NLS A is said to converge to x if given $r>0$, $t>0$, $0<r<1$ there exists an integer $n_0 \in N$ such that $N(x_1,x_2,\ldots,x_{n-1},x_n-x,t) > 1-r$ and $M(x_1,x_2,\ldots,x_{n-1},x_n-x,t) < r$, for all $n \ge n_0$.

Theorem 3.4. *In an i-f-n-NLS A, a sequence converges to x if and only if $N(x_1,x_2,\ldots,x_{n-1},x_n-x,t) \to 1$ and $M(x_1,x_2,\ldots,x_{n-1},x_n-x,t) \to 0$, as $n\to\infty$.*

Proof. Fix $t>0$. Suppose $\{x_n\}$ converges to x in A. Then for a given r, $0<r<1$, there exists an integer $n_0 \in N$ such that $N(x_1,x_2,\ldots,x_{n-1},x_n-x,t) > 1-r$ and $M(x_1,x_2,\ldots,x_{n-1},x_n-x,t) < r$. Thus $1-N(x_1,x_2,\ldots,x_{n-1},x_n-x,t) < r$ and $M(x_1,x_2,\ldots,x_{n-1},x_n-x,t) < r$, and hence $N(x_1,x_2,\ldots,x_{n-1},x_n-x,t) \to 1$ and $M(x_1,x_2,\ldots,x_{n-1},x_n-x,t) \to 0$, as $n\to\infty$. Conversely, if for each $t > 0$, $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 0$, as $n \to \infty$, then for every r, $0 < r < 1$, there exists an integer n_0 such that $1 - N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r$ for all $n \geq n_0$. Thus $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r$, for all $n \geq n_0$. Hence $\{x_n\}$ converges to x in A. □

Definition 3.5. A sequence $\{x_n\}$ in an i-f-n-NLS A is said to be Cauchy sequence if given $\epsilon > 0$, with $0 < \epsilon < 1$, $t > 0$ there exists an integer $n_0 \in N$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1 - \epsilon$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) < \epsilon$, for all $n, k \geq n_0$.

Theorem 3.6. *In an i-f-n-NLS A, every convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence in A. Suppose $\{x_n\}$ converges to x. Let $t > 0$ and $\epsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Since $\{x_n\}$ converges to x, we have an integer n_0 such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) > 1 - r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) < r$. Now,

$$\begin{aligned}
&N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \\
&= N(x_1, x_2, \ldots, x_{n-1}, x_n - x + x - x_k, \frac{t}{2} + \frac{t}{2}) \\
&\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) * N(x_1, x_2, \ldots, x_{n-1}, x - x_k, \frac{t}{2}) \\
&> (1 - r) * (1 - r), \text{ for all } n, k \geq n_0 \\
&> 1 - \epsilon, \text{ for all } n, k \geq n_0
\end{aligned}$$

and

$$\begin{aligned}
&M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \\
&= M(x_1, x_2, \ldots, x_{n-1}, x_n - x + x - x_k, \frac{t}{2} + \frac{t}{2}) \\
&\leq M(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) \diamond M(x_1, x_2, \ldots, x_{n-1}, x - x_k, \frac{t}{2}) \\
&< r \diamond r \\
&< \epsilon, \text{ for all } n, k \geq n_0.
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in A. □

Definition 3.7. An i-f-n-NLS A is said to be complete if every Cauchy sequence in A is convergent.

The following example shows that there may exist Cauchy sequence in an i-f-n-NLS which is not convergent.

Example 3.8. Let $(X, ||\bullet, \bullet, \ldots, \bullet||)$ be an n-normed linear space and define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1], t > 0$.

$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t+||x_1,x_2,\ldots,x_n||}$, $M(x_1, x_2, \ldots, x_n, t) = \frac{||x_1,x_2,\ldots,x_n||}{t+||x_1,x_2,\ldots,x_n||}$. Then

$$A = \{(X, N(x, x_2, \ldots, x_n, t), M(x_1, x_2, \ldots, x_n, t))|(x_1, x_2, \ldots, x_n) \in X^n\}$$

is an i-f-n-NLS by Example 3.2. Let $\{x_n\}$ be a sequence in A. Then

(a) $\{x_n\}$ is a Cauchy sequence in $(X, ||\bullet, \bullet, \ldots, \bullet||)$ if and only if $\{x_n\}$ is a Cauchy sequence in A.

(b) $\{x_n\}$ is a convergent sequence in $(X, ||\bullet, \bullet, \ldots, \bullet||)$ if and only if $\{x_n\}$ is convergent in A.

Proof. (a) $\{x_n\}$ is a Cauchy sequence in $(X, ||\bullet, \bullet, \ldots, \bullet||)$
$\Leftrightarrow \lim_{n,k\to\infty} ||x_1, x_2, \ldots, x_{n-1}, x_n - x_k|| = 0.$
$\Leftrightarrow \lim_{n,k\to\infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) = \lim_{n,k\to\infty} \frac{t}{t+||x_1,x_2,\ldots,x_{n-1},x_n-x_k||} = 1$
and $\lim_{n,k\to\infty} M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) = \lim_{n,k\to\infty} \frac{||x_1,x_2,\ldots,x_n-x_k||}{t+||x_1,x_2,\ldots,x_n-x_k||} = 0.$
$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \to 1$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \to 0$, as $n, k \to \infty$
$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1 - r$ and
$M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) < r$, $r \in (0, 1)$, for all $n, k \geq n_0$.
$\Leftrightarrow \{x_n\}$ is a Cauchy sequence in A.

(b) $\{x_n\}$ is a convergent sequence in $(X, ||\bullet, \bullet, \ldots, \bullet||)$
$\Leftrightarrow \lim_{n\to\infty} ||x_1, x_2, \ldots, x_{n-1}, x_n - x|| = 0.$
$\Leftrightarrow \lim_{n\to\infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) = \lim_{n\to\infty} \frac{t}{t+||x_1,x_2,\ldots,x_{n-1},x_n-x||} = 1$ and
$\lim_{n\to\infty} M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) = \lim_{n\to\infty} \frac{||x_1,x_2,\ldots,x_n-x||}{t+||x_1,x_2,\ldots,x_n-x||} = 0.$
$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 0$, as $n \to \infty$.
$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r$, $r \in (0, 1)$, for all $n \geq n_0$.
$\Leftrightarrow \{x_n\}$ is a convergent sequence in A.
Thus if there exists an n-normed linear space $(X, ||\bullet, \bullet, \ldots, \bullet||)$ which is not complete, then the intuitionistic fuzzy n-norm induced by such a crisp n-norm $||\bullet, \bullet, \ldots, \bullet||$ on an incomplete n-normed linear space X is an incomplete intuitionistic fuzzy n-normed linear space. □

Theorem 3.9. *Let A be an i-f-n-NLS, such that every Cauchy sequence in A has a convergent subsequence. Then A is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in A and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $\{x_n\}$ converges to x. Let $t > 0$ and $\epsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, \frac{t}{2}) > 1 - r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, \frac{t}{2}) < r$, for all $n, k \geq n_0$. Since $\{x_{n_k}\}$ converges to x, there is a positive $i_k > n_0$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) > 1-r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) < r$. Now,

$$\begin{aligned}
&N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\
&= N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}) \\
&\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, \frac{t}{2}) * N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) \\
&> (1-r) * (1-r) > 1 - \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\
&= M(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}) \\
&\leq M(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, \frac{t}{2}) \diamond M(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) \\
&< r \diamond r < \epsilon.
\end{aligned}$$

Therefore $\{x_n\}$ converges to x in A and hence it is complete. □

4. Generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces

We now proceed to our new notion of generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces in the following theorem.

Theorem 4.1. *Let*

$$A = \{(X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) | (x_1, x_2, \ldots, x_n) \in X^n\}$$

and

$$B = \{(Y, N_2(y_1, y_2, \ldots, y_n, t), M_2(y_1, y_2, \ldots, y_n, t)) | (y_1, y_2, \ldots, y_n) \in Y^n\}$$

be two intuitionistic fuzzy n-normed linear spaces. Then

$$\begin{aligned}
A \times_{*,\diamond} B = \{&(X \times Y, N(z_1, z_2, \ldots, z_n, t), \\
&M(z_1, z_2, \ldots, z_n, t)) | (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}
\end{aligned}$$

is an i-f-n-NLS with

$$N(z_1, z_2, \ldots, z_n, t) = N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

and

$$M(z_1, z_2, \ldots, z_n, t) = M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t),$$

where $z_i = (x_i, y_i), i = 1, 2, \ldots, n$.

Proof. As

$$\begin{aligned}
&N_1(x_1, x_2, \ldots, x_n, t) + M_1(x_1, x_2, \ldots, x_n, t) \leq 1 \text{ and} \\
&N_2(y_1, y_2, \ldots, y_n, t) + M_2(y_1, y_2, \ldots, y_n, t) \leq 1,
\end{aligned} \tag{1}$$

it follows that

$$M_1(x_1, x_2, \dots, x_n, t) \leq 1 - N_1(x_1, x_2, \dots, x_n, t)$$

and

$$M_2(y_1, y_2, \dots, y_n, t) \leq 1 - N_2(y_1, y_2, \dots, y_n, t).$$

By Definition 2.8,

$$\begin{aligned}&(1 - N_1(x_1, x_2, \dots, x_n, t)) \diamond (1 - N_2(y_1, y_2, \dots, y_n, t))\\ &\geq M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t).\end{aligned}$$

Then,

$$\begin{aligned}&N_1(x_1, x_2, \dots, x_n, t) \hat{\diamond} N_2(y_1, y_2, \dots, y_n, t))\\ &= 1 - ((1 - N_1(x_1, x_2, \dots, x_n, t)) \diamond (1 - N_2(y_1, y_2, \dots, y_n, t)))\\ (2)\quad &\leq 1 - (M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t)).\\ \Rightarrow\quad &N_1(x_1, x_2, \dots, x_n, t) \hat{\diamond} N_2(y_1, y_2, \dots, y_n, t)\\ &\leq 1 - (M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t)),\end{aligned}$$

where $a \hat{\diamond} b = 1 - ((1-a) \diamond (1-b))$ is defined as the dual t-norm with respect to $\diamond$. So, if

$$N_1(x_1, x_2, \dots, x_n, t) * N_2(y_1, y_2, \dots, y_n, t) \leq N_1(x_1, x_2, \dots, x_n, t) \hat{\diamond} N_2(y_1, y_2, \dots, y_n, t),$$

then by (2), we have,

$$\begin{aligned}&N_1(x_1, x_2, \dots, x_n, t) * N_2(y_1, y_2, \dots, y_n, t)\\ &\leq 1 - (M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t)).\\ \Rightarrow\quad &N_1(x_1, x_2, \dots, x_n, t) * N_2(y_1, y_2, \dots, y_n, t)\\ &+ M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t) \leq 1.\\ &\Rightarrow N(z_1, z_2, \dots, z_n, t) + M(z_1, z_2, \dots, z_n, t) \leq 1.\end{aligned}$$

Similarly we can verify the other conditions. Thus, $A \times_{*,\diamond} B$ is an i-f-n-NLS. □

Theorem 4.2. *The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is commutative. In other words if A and B are two intuitionistic fuzzy n-normed linear spaces. Then*

$$A = B \Rightarrow A \times_{*,\diamond} B = B \times_{*,\diamond} A.$$

Proof. Assume $A = B$; $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$. Then,

$$N_1(x_1, x_2, \dots, x_n, t) * N_2(y_1, y_2, \dots, y_n, t) = N_2(x_1, x_2, \dots, x_n, t) * N_1(y_1, y_2, \dots, y_n, t)$$

and

$$M_1(x_1, x_2, \dots, x_n, t) \diamond M_2(y_1, y_2, \dots, y_n, t) = M_2(x_1, x_2, \dots, x_n, t) \diamond M_1(y_1, y_2, \dots, y_n, t)$$

Thus, $A \times_{*,\diamond} B = B \times_{*,\diamond} A$. However the converse is not true. For example, let

$$\begin{aligned}A = \{(X, N_1(x_1, x_2, \dots, x_n, t), M_1(x_1, x_2, \dots, x_n, t)) | N_1(x_1, x_2, \dots, x_n, t) = a,\\ M_1(x_1, x_2, \dots, x_n, t) = b, (x_1, x_2, \dots, x_n) \in X^n\}\end{aligned}$$

and

$$B = \{(X, N_2(x_1, x_2, \ldots, x_n, t), M_2(x_1, x_2, \ldots, x_n, t)) | N_2(x_1, x_2, \ldots, x_n, t) = c, \\ M_2(x_1, x_2, \ldots, x_n, t) = d, (x_1, x_2, \ldots, x_n) \in X^n\}, a, b, c, d \in [0, 1].$$

Then

$$N_1(x_1, x_2, \ldots, x_n, t) * N_2(x_1, x_2, \ldots, x_n, t) = a * c = c * a \\ = N_2(x_1, x_2, \ldots, x_n, t) * N_1(x_1, x_2, \ldots, x_n, t)$$

and

$$M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(x_1, x_2, \ldots, x_n, t) = b \diamond d = d \diamond b \\ = M_2(x_1, x_2, \ldots, x_n, t) \diamond M_1(x_1, x_2, \ldots, x_n, t).$$

So, we obtain $A \times_{*,\diamond} B = B \times_{*,\diamond} A$, but $A \neq B$ if $a \neq c$ or $b \neq d$. □

Theorem 4.3. *The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is distributive with respect to union and intersections. In other words if*

$$A = \{(X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) | (x_1, x_2, \ldots, x_n) \in X^n\}$$

and

$$B = \{(Y, N_2(y_1, y_2, \ldots, y_n, t), M_2(y_1, y_2, \ldots, y_n, t)) | (y_1, y_2, \ldots, y_n) \in Y^n\}$$

$$C = \{(Y, N_3(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)) | (y_1, y_2, \ldots, y_n) \in Y^n\}$$

are the intuitionistic fuzzy n-normed linear spaces, then

$$A \times_{*,\diamond} (B \cap C) = (A \times_{*,\diamond} B) \cap (A \times_{*,\diamond} C)$$

and

$$A \times_{*,\diamond} (B \cup C) = (A \times_{*,\diamond} B) \cup (A \times_{*,\diamond} C).$$

Proof. We have,

$$A \times_{*,\diamond} (B \cap C) \\ = \{(X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\}, \\ M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}) | \\ (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}$$

and

$$\begin{aligned}
&(A \times_{*,\diamond} B) \cap (A \times_{*,\diamond} C) \\
&= \{(X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\
&M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t)) | (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\} \cap \\
&\{(X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t), \\
&M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)) | (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\} \\
&= \{(X \times Y, \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\
&N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\}, \\
&\max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), \\
&M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}) | (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}.
\end{aligned}$$

So it is enough to prove that,

$$\begin{aligned}
&N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\} \\
(3) \quad &= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\
&\quad N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\}
\end{aligned}$$

and

$$\begin{aligned}
&M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\
(4) \quad &= \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), \\
&\quad M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}.
\end{aligned}$$

Let

$$(5) \qquad N_2(y_1, y_2, \ldots, y_n, t) \leq N_3(y_1, y_2, \ldots, y_n, t).$$

Then by Definition 2.7,

$$\begin{aligned}
(6) \qquad &N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\
&\leq N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t).
\end{aligned}$$

Therefore by (5) and (6),

$$\begin{aligned}
&\text{LHS of (3)} \\
&= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\
&= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\
&\quad N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= \text{RHS of (3)}.
\end{aligned}$$

Let

$$(7) \qquad N_2(y_1, y_2, \ldots, y_n, t) > N_3(y_1, y_2, \ldots, y_n, t).$$

By Definition 2.7,

$$
\begin{aligned}
&N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\
&> N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t).
\end{aligned} \tag{8}
$$

Therefore by (7) and (8),

$$
\begin{aligned}
&\text{LHS of (3)} \\
&= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t) \\
&= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\
&\quad N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= \text{RHS of (3)}.
\end{aligned}
$$

Thus equality holds in (3).

Let

$$
M_2(y_1, y_2, \ldots, y_n, t) \leq M_3(y_1, y_2, \ldots, y_n, t). \tag{9}
$$

Then by Definition 2.8,

$$
\begin{aligned}
&M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t) \\
&\leq M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t).
\end{aligned} \tag{10}
$$

Therefore by (9) and (10),

$$
\begin{aligned}
&\text{LHS of (4)} \\
&= M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\
&= M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) \\
&= \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), \\
&\quad M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \\
&= \text{RHS of (4)}.
\end{aligned}
$$

Let

$$
M_2(y_1, y_2, \ldots, y_n, t) > M_3(y_1, y_2, \ldots, y_n, t). \tag{11}
$$

By Definition 2.8,

$$
\begin{aligned}
&M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t) \\
&> M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t).
\end{aligned} \tag{12}
$$

Therefore by (11) and (12),

$$\begin{aligned}
&\text{LHS of (4)}\\
&= M_1(x_1,x_2,\dots,x_n,t)\diamond \max\{M_2(y_1,y_2,\dots,y_n,t), M_3(y_1,y_2,\dots,y_n,t)\}\\
&= M_1(x_1,x_2,\dots,x_n,t)\diamond M_2(y_1,y_2,\dots,y_n,t)\\
&= \max\{M_1(x_1,x_2,\dots,x_n,t)\diamond M_2(y_1,y_2,\dots,y_n,t),\\
&\quad M_1(x_1,x_2,\dots,x_n,t)\diamond M_3(y_1,y_2,\dots,y_n,t)\}\\
&= \text{RHS of (4)}.
\end{aligned}$$

Thus equality holds in (4). Finally from (3) and (4), we have $A\times_{*,\diamond}(B\cap C) = (A\times_{*,\diamond} B)\cap(A\times_{*,\diamond} C)$. Similarly, we can prove that $A\times_{*,\diamond}(B\cup C) = (A\times_{*,\diamond} B)\cup(A\times_{*,\diamond} C)$. □

Theorem 4.4. *The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is distributive with respect to difference. In other words if*

$$A = \{(X, N_1(x_1,x_2,\dots,x_n,t), M_1(x_1,x_2,\dots,x_n,t))|(x_1,x_2,\dots,x_n)\in X^n\}$$

and

$$B = \{(Y, N_2(y_1,y_2,\dots,y_n,t), M_2(y_1,y_2,\dots,y_n,t))|(y_1,y_2,\dots,y_n)\in Y^n\}$$

$$C = \{(Y, N_3(y_1,y_2,\dots,y_n,t), M_3(y_1,y_2,\dots,y_n,t))|(y_1,y_2,\dots,y_n)\in Y^n\}$$

are the intuitionistic fuzzy n-normed linear spaces, then

$$A\times_{*,\diamond}(B\backslash C)\subseteq (A\times_{*,\diamond} B)\backslash(A\times_{*,\diamond} C).$$

If, $B=\{(Y, N_2(y_1,y_2,\dots,y_n,t)=1,\ M_2(y_1,y_2,\dots,y_n,t)=0)\,|\,(y_1,y_2,\dots,y_n)\in Y^n\}$, $C\subseteq A, * = \min, \diamond = \max$, *then equality holds.*

Proof. We need to prove, $A\times_{*,\diamond}(B\cap\overline{C})\subseteq(A\times_{*,\diamond}B)\cap\overline{(A\times_{*,\diamond} C)}$. It is enough to prove,

$$\begin{aligned}
&N_1(x_1,x_2,\dots,x_n,t) * \min\{N_2(y_1,y_2,\dots,y_n,t), M_3(y_1,y_2,\dots,y_n,t)\}\\
(13)\quad &\le \min\{N_1(x_1,x_2,\dots,x_n,t) * N_2(y_1,y_2,\dots,y_n,t),\\
&\quad M_1(x_1,x_2,\dots,x_n,t)\diamond M_3(y_1,y_2,\dots,y_n,t)\}
\end{aligned}$$

and

$$\begin{aligned}
&M_1(x_1,x_2,\dots,x_n,t)\diamond \max\{M_2(y_1,y_2,\dots,y_n,t), N_3(y_1,y_2,\dots,y_n,t)\}\\
(14)\quad &\ge \max\{M_1(x_1,x_2,\dots,x_n,t)\diamond M_2(y_1,y_2,\dots,y_n,t),\\
&\quad N_1(x_1,x_2,\dots,x_n,t) * N_3(y_1,y_2,\dots,y_n,t)\}.
\end{aligned}$$

Case (i) Let

$$(15)\qquad N_2(y_1,y_2,\dots,y_n,t) < M_3(y_1,y_2,\dots,y_n,t)$$

and using the fact

$$(16)\qquad a*b\le\min\{a,b\}\le a\le\max\{a,c\}\le a\diamond c.$$

Then by Definition 2.7 and (16),

$$\begin{aligned} & N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\ & < N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \leq M_3(y_1, y_2, \ldots, y_n, t) \\ & \leq N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t). \quad (17) \\ \Rightarrow \quad & N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\ & < N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t). \end{aligned}$$

Therefore by (15) and (17),

$$\begin{aligned} & \text{LHS of (13)} \\ & = N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ & = N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\ & = \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\ & \quad N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \\ & = \text{RHS of (13)}. \end{aligned}$$

Thus equality holds in (13).

Case (ii) Let

$$N_2(y_1, y_2, \ldots, y_n, t) \geq M_3(y_1, y_2, \ldots, y_n, t). \quad (18)$$

By Definition 2.7,

$$\begin{aligned} & N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\ & \geq N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t). \end{aligned} \quad (19)$$

Therefore by (18) and (19),

(20)

$$\begin{aligned} & N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ & = N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \\ & \leq N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t). \\ \Rightarrow \quad & N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ & \leq N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)\}. \end{aligned}$$

By (16) and (18),

$$\begin{aligned} & N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ & = N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \quad (21) \\ & \leq M_3(y_1, y_2, \ldots, y_n, t) \leq M_1(y_1, y_2, \ldots, y_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t). \end{aligned}$$

From (20) and (21) we have,

$$\begin{aligned} & N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ & \leq \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\ & \quad M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}. \end{aligned}$$

Thus we have proved (13) and (14) can be proved similarly. So,

$$A \times_{*,\diamond} (B \backslash C) \subseteq (A \times_{*,\diamond} B) \backslash (A \times_{*,\diamond} C).$$

Let $B=\{(Y, N_2(y_1, y_2, \ldots, y_n, t) = 1,\ M_2(y_1, y_2, \ldots, y_n, t) = 0) \mid (y_1, y_2, \ldots, y_n) \in Y^n\}$, $C \subseteq A$, $* = \min$, $\diamond = \max$.

$$\begin{aligned} &\text{LHS of (13)} \\ &= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ &= N_1(x_1, x_2, \ldots, x_n, t) * \min\{1, M_3(y_1, y_2, \ldots, y_n, t)\} \\ &= N_1(y_1, y_2, \ldots, y_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \\ &= \min\{N_1(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}. \end{aligned}$$

$$\begin{aligned} &\text{RHS of (13)} \\ &= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\ &\quad M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \\ &= \min\{\min\{N_1(x_1, x_2, \ldots, x_n, t), 1\}, \\ &\quad \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}\} \\ &= \min\{N_1(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}. \end{aligned}$$

Thus equality holds in (13).

Similarly we can prove the equality in (14). □

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets VII ITKR's session*, Sofia, June 1983(Deposed in Central Sci-Techn.Library of Bulg. Acad. of. Sci., 1967/84) (in Bulgarian).

[2] ———, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), no. 1, 87–96.

[3] ———, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems **33** (1989), no. 1, 37–45.

[4] ———, *Intuitionistic fuzzy sets*, Theory and applications. Studies in Fuzziness and Soft Computing, 35. Physica-Verlag, Heidelberg, 1999.

[5] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), no. 3, 687–705.

[6] S. C. Chang and J. N. Mordesen, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), no. 5, 429–436.

[7] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems **48** (1992), no. 2, 239–248.

[8] ———, *The completion of a fuzzy normed linear space*, J. Math. Anal. Appl. **174** (1993), no. 2, 428–440.

[9] ———, *Finite dimensional fuzzy normed linear space. II.*, J. Anal. **7** (1999), 117–131.

[10] S. Gähler, *Lineare 2-normierte Raume*, Math. Nachr. **28** (1964), 1–43.

[11] ———, *Untersuchungen uber verallgemeinerte m-metrische Raume, I, II, III.*, Math. Nachr. **40** (1969), 165–189.

[12] G. Deschrijver and E. Kerre, *On the Cartesian product of intuitionistic fuzzy sets*, J. Fuzzy Math. **11** (2003), no. 3, 537–547.

[13] H. Gunawan and M. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (2001), no. 10, 631–639.

[14] A. K. Katsaras, *Fuzzy topological vector spaces. II.*, Fuzzy Sets and Systems **12** (1984), no. 2, 143–154.

[15] S. S. Kim and Y. J. Cho, *Strict convexity in linear n-normed spaces*, Demonstratio Math. **29** (1996), no. 4, 739–744.

[16] S. V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), no. 2, 207–217.

[17] R. Malceski, *Strong n-convex n-normed spaces*, Mat. Bilten No. **21** (1997), 81–102.

[18] A. Misiak, *n-inner product spaces*, Math. Nachr. **140** (1989), 299–319.

[19] Al. Narayanan and S. Vijayabalaji, *Fuzzy n-normed linear space*, Int. J. Math. Math. Sci. (2005), no. 24, 3963–3977.

[20] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals **22** (2004), no. 5, 1039–1046.

[21] G. S. Rhie, B. M. Choi, and D. S. Kim, *On the completeness of fuzzy normed linear spaces*, Math. Japon. **45** (1997), no. 1, 33–37.

[22] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313–334.

Srinivasan Vijayabalaji
Department of Mathematics
Annamalai University
Annamalainagar-608002, Tamilnadu, India
E-mail address: balaji_nandini@rediffmail.com

Natesan Thillaigovindan
Department of Mathematics
Annamalai University
Annamalainagar-608002, Tamilnadu, India
E-mail address: thillai_n@sify.com

Young Bae Jun
Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: ybjun@gsnu.ac.kr