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ADDITIVITY OF LIE MAPS ON OPERATOR ALGEBRAS

Jia Qian and Pengtong Li

Abstract. Let $\mathscr{A}$ be a standard operator algebra which does not contain the identity operator, acting on a Hilbert space of dimension greater than one. If Φ is a bijective Lie map from $\mathscr{A}$ onto an arbitrary algebra, that is

$$\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)$$

for all $A, B \in \mathscr{A}$, then Φ is additive. Also, if $\mathscr{A}$ contains the identity operator, then there exists a bijective Lie map of $\mathscr{A}$ which is not additive.

1. Introduction

Throughout, for a Hilbert space $\mathscr{H}$, we write $B(\mathscr{H})$ for the algebra of all bounded linear operators on $\mathscr{H}$. Usually, a standard operator algebra on $\mathscr{H}$ will mean a subalgebra of $B(\mathscr{H})$ containing all finite rank operators. Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras or rings. A map $\Phi : \mathscr{A} \to \mathscr{B}$ is called a *Lie map* if it is multiplicative with respect to the Lie product $AB - BA$, that is

$$\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)$$

for all $A, B \in \mathscr{A}$.

Characterizing the interrelation between the multiplicative and the additive structures of a ring is an interesting topic. This question was first studied by Martindale who obtained the surprising result that every bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive [10]. For operator algebras, the same problem was treated in [1, 7, 15]. In the papers [2, 3, 8, 9, 11, 12, 13], the additivity of maps on operator algebras which are multiplicative with respect to other products, such as the Jordan product $AB + BA$ or the Jordan triple product ABA, were investigated. Also, the papers [4, 5, 6, 14] studied the similar questions for elementary maps and Jordan elementary maps on rings or operator algebras.

In this note, we shall study the additivity of Lie maps on operator algebras. More precisely, it will be proved that every bijective Lie map on a standard operator algebra $\mathscr{A}$ which does not contain the identity operator, acting on a Hilbert space $\mathscr{H}$ of dimension greater than one, is automatically additive. In

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particular, if $\dim \mathscr{H} = \infty$ and $\mathscr{A}$ is either the ideal of all finite rank operators or the ideal of all compact operators in $B(\mathscr{H})$, then every bijective Lie map on $\mathscr{A}$ is additive. Furthermore, we show that if $\mathscr{A}$ contains the identity operator, then there must exist a bijective Lie map on $\mathscr{A}$ which is not additive.

It should be mentioned that Lu in [8] proved that a bijective Jordan map on a standard operator algebra which is allowed to contain the identity operator, is additive. Although the basic ideas used in our proof are similar to those in [8], some concrete techniques are new.

2. Result and Proof

Our main result reads as follows.

Theorem 1. *Let $\mathscr{H}$ be a real or complex Hilbert space with $\dim \mathscr{H} > 1$, $\mathscr{A} \subseteq B(\mathscr{H})$ be a standard operator algebra which does not contain the identity operator I and $\mathscr{B}$ be an arbitrary algebra. If $\Phi : \mathscr{A} \to \mathscr{B}$ is a bijective Lie map, then Φ is necessarily additive.*

We shall organize the proof of Theorem 1 in a series of lemmas, in which the notation of the theorem will be kept. Since $\dim \mathscr{H} > 1$, we can take a non-trivial orthogonal projection P_1 which has finite rank. Then $P_1 \in \mathscr{A}$. Put $P_2 = I - P_1$. Note that P_2 is not in $\mathscr{A}$. Let $\mathscr{A}_{ij} = P_i \mathscr{A} P_j$, $i, j = 1, 2$. Then

$$\mathscr{A} = \mathscr{A}_{11} \oplus \mathscr{A}_{12} \oplus \mathscr{A}_{21} \oplus \mathscr{A}_{22}$$

which is the Peirce decomposition of $\mathscr{A}$. This idea is essentially from Martindale [10].

Lemma 1. $\Phi(0) = 0$.

Proof. It is obvious. □

Lemma 2. *If $A, B, S \in \mathscr{A}$ such that $\Phi(S) = \Phi(A) + \Phi(B)$, then for all $T \in \mathscr{A}$, we have*

(1) $\Phi(ST - TS) = \Phi(AT - TA) + \Phi(BT - TB)$,
(2) $\Phi(TS - ST) = \Phi(TA - AT) + \Phi(TB - BT)$.

Proof. Let $T \in \mathscr{A}$. Then

$$\begin{aligned} \Phi(ST - TS) &= \Phi(S)\Phi(T) - \Phi(T)\Phi(S) \\ &= (\Phi(A) + \Phi(B))\Phi(T) - \Phi(T)(\Phi(A) + \Phi(B)) \\ &= \Phi(A)\Phi(T) - \Phi(T)\Phi(A) + \Phi(B)\Phi(T) - \Phi(T)\Phi(B) \\ &= \Phi(AT - TA) + \Phi(BT - TB). \end{aligned}$$

So (1) holds. Similarly, we can prove (2). □

In the following, the notation A_{ij} will denote an arbitrary element in $\mathscr{A}_{ij}$.

Lemma 3. *Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$.*

(1) *If* $T_{ij}S_{jk} = 0$ *for all* $T_{ij}, 1 \leq i, j, k \leq 2$, *then* $S_{jk} = 0$. *If* $S_{ki}T_{ij} = 0$ *for all* $T_{ij}, 1 \leq i, j, k \leq 2$, *then* $S_{ki} = 0$;
(2) *If* $ST_{ij} - T_{ij}S \in \mathscr{A}_{ij}$ *for all* $T_{ij}, 1 \leq i \neq j \leq 2$, *then* $S_{ji} = 0$;
(3) *If* $ST_{jj} - T_{jj}S \in \mathscr{A}_{ij}$ *for all* $T_{jj}, 1 \leq i \neq j \leq 2$, *then* $S_{ji} = 0$ *and* $S_{jj} = \lambda P_j$ *for some scalar* λ;
(4) *If* $ST_{jj} - T_{jj}S \in \mathscr{A}_{ji}$ *for all* $T_{jj}, 1 \leq i \neq j \leq 2$, *then* $S_{ij} = 0$ *and* $S_{jj} = \lambda P_j$ *for some scalar* λ;
(5) *If* $ST_{jj} - T_{jj}S \in \mathscr{A}_{jj}$ *for all* $T_{jj}, j = 1, 2$, *then* $S_{ji} = S_{ij} = 0$ *for* $1 \leq i \neq j \leq 2$.

Proof. (1) It is [8, Lemma 2(ii)].

(2) By the hypothesis, we have obviously $S_{ji}T_{ij} = P_j(ST_{ij} - T_{ij}S) = 0$ for all T_{ij} with $i \neq j$. Hence $S_{ji} = 0$ by (1).

(3) Similar to (2), we can easily obtain that $S_{ji} = 0$. Also, for every T_{jj}, we have $P_j(ST_{jj} - T_{jj}S)P_j = 0$. Hence $T_{jj}S_{jj} = S_{jj}T_{jj}$, which implies that S_{jj} commutes all operators in $B(P_j\mathscr{H})$. It is well known that $S_{jj} = \lambda P_j$ for some scalar λ.

Similarly, we can prove (4) and (5). □

Lemma 4. *For* $1 \leq i \neq j \leq 2$, *we have*
(1) $\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij})$;
(2) $\Phi(A_{ii} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ji})$.

Proof. (1) We only give the proof of (1), and for (2) the proof goes similarly. Since Φ is surjective, there is $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}).$$

For any T_{jj}, by Lemma 2 and noticing that $i \neq j$, we have

$$\begin{aligned} \Phi(ST_{jj} - T_{jj}S) &= \Phi(A_{ii}T_{jj} - T_{jj}A_{ii}) + \Phi(A_{ij}T_{jj} - T_{jj}A_{ij}) \\ &= \Phi(0) + \Phi(A_{ij}T_{jj}) = \Phi(A_{ij}T_{jj}). \end{aligned}$$

It follows from the injectivity of Φ that

$$ST_{jj} - T_{jj}S = A_{ij}T_{jj} \in \mathscr{A}_{ij}. \tag{1}$$

So $S_{ji} = 0$ and $S_{jj} = \lambda P_j$ for some scalar λ by Lemma 3(3). Also, from (1) we get that $S_{ij}T_{jj} = A_{ij}T_{jj}$ for all T_{jj}, and hence $S_{ij} = A_{ij}$ by Lemma 3(1).

For every T_{ij}, applying Lemma 2 we can similarly get that $\Phi(ST_{ij} - T_{ij}S) = \Phi(A_{ii}T_{ij})$, which implies $ST_{ij} - T_{ij}S = A_{ii}T_{ij}$. Therefore

$$A_{ii}T_{ij} = P_i(ST_{ij} - T_{ij}S)P_j = S_{ii}T_{ij} - T_{ij}S_{jj} = S_{ii}T_{ij} - \lambda T_{ij}$$

and so $S_{ii} = A_{ii} + \lambda P_i$. Thus

$$\begin{aligned} S &= S_{ii} + S_{ij} + S_{ji} + S_{jj} \\ &= (A_{ii} + \lambda P_i) + A_{ij} + 0 + \lambda P_j \\ &= A_{ii} + A_{ij} + \lambda I. \end{aligned}$$

Since $I \notin \mathscr{A}$, we have $\lambda = 0$. This proves $S = A_{ii} + A_{ij}$, as required. □

Lemma 5. $\Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = \Phi(T_{ii}A_{ij}) + \Phi(B_{ij}S_{jj})$ *for* $1 \leq i \neq j \leq 2$.

Proof. Making use of Lemma 4, we have

$$\begin{aligned}
&\Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = \Phi((T_{ii} + B_{ij})(A_{ij} + S_{jj}) - (A_{ij} + S_{jj})(T_{ii} + B_{ij}))\\
&= \Phi(T_{ii} + B_{ij})\Phi(A_{ij} + S_{jj}) - \Phi(A_{ij} + S_{jj})\Phi(T_{ii} + B_{ij})\\
&= (\Phi(T_{ii}) + \Phi(B_{ij}))(\Phi(A_{ij}) + \Phi(S_{jj})) - (\Phi(A_{ij}) + \Phi(S_{jj}))(\Phi(T_{ii}) + \Phi(B_{ij}))\\
&= (\Phi(T_{ii})\Phi(A_{ij}) - \Phi(A_{ij})\Phi(T_{ii})) + (\Phi(T_{ii})\Phi(S_{jj}) - \Phi(S_{jj})\Phi(T_{ii}))\\
&\quad + (\Phi(B_{ij})\Phi(A_{ij}) - \Phi(A_{ij})\Phi(B_{ij})) + (\Phi(B_{ij})\Phi(S_{jj}) - \Phi(S_{jj})\Phi(B_{ij}))\\
&= \Phi(T_{ii}A_{ij} - A_{ij}T_{ii}) + \Phi(T_{ii}S_{jj} - S_{jj}T_{ii})\\
&\quad + \Phi(B_{ij}A_{ij} - A_{ij}B_{ij}) + \Phi(B_{ij}S_{jj} - S_{jj}B_{ij})\\
&= \Phi(T_{ii}A_{ij}) + \Phi(B_{ij}S_{jj}),
\end{aligned}$$

completing the proof. □

Lemma 6. $\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})$ *for* $1 \leq i \neq j \leq 2$.

Proof. Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{ij}) + \Phi(B_{ij}). \tag{2}$$

For any T_{ii}, T_{jj}, by Lemmas 2, 5, we have

$$\Phi(ST_{jj} - T_{jj}S) = \Phi(A_{ij}T_{jj}) + \Phi(B_{ij}T_{jj}), \tag{3}$$

and

$$\begin{aligned}
&\Phi(T_{ii}ST_{jj} + T_{jj}ST_{ii})\\
= \;&\Phi(T_{ii}(ST_{jj} - T_{jj}S) - (ST_{jj} - T_{jj}S)T_{ii})\\
= \;&\Phi(T_{ii}A_{ij}T_{jj} - A_{ij}T_{jj}T_{ii}) + \Phi(T_{ii}B_{ij}T_{jj} - B_{ij}T_{jj}T_{ii})\\
= \;&\Phi(T_{ii}A_{ij}T_{jj}) + \Phi(T_{ii}B_{ij}T_{jj})\\
= \;&\Phi(T_{ii}(A_{ij} + B_{ij})T_{jj}).
\end{aligned}$$

Thus $T_{ii}ST_{jj} + T_{jj}ST_{ii} = T_{ii}(A_{ij} + B_{ij})T_{jj}$. Multiplying this equality by P_i from the left, we get $T_{ii}ST_{jj} = T_{ii}(A_{ij} + B_{ij})T_{jj}$. It follows from Lemma 3(1) that $S_{ij} = A_{ij} + B_{ij}$.

For every T_{ij}, applying Lemma 2 to (2) and (3) respectively, we get

$$ST_{ij} - T_{ij}S = 0,$$

$$(ST_{jj} - T_{jj}S)T_{ij} - T_{ij}(ST_{jj} - T_{jj}S) = 0.$$

It follows easily that $S_{ji} = 0$ and $T_{ij}(ST_{jj} - T_{jj}S) = 0$. Hence $S_{jj}T_{jj} = T_{jj}S_{jj}$ by Lemma 3(1), and so there exists a scalar λ such that $S_{jj} = \lambda P_j$. Also,

$$S_{ii}T_{ij} - \lambda T_{ij} = S_{ii}T_{ij} - T_{ij}S_{jj} = P_i(ST_{ij} - T_{ij}S)P_j = 0.$$

By Lemma 3(1) again, we have $S_{ii} = \lambda P_i$. Therefore,

$$S = \lambda P_i + (A_{ij} + B_{ij}) + 0 + \lambda P_j = A_{ij} + B_{ij} + \lambda I.$$

Recalling that $I \notin \mathscr{A}$, we have $\lambda = 0$ and hence $S = A_{ij} + B_{ij}$. This completes the proof. □

Lemma 7. $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$ *for* $i = 1, 2$.

Proof. Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(B_{ii}). \tag{4}$$

Take $j \neq i$. For any T_{jj}, applying Lemma 2 to (4) we obtain $ST_{jj} - T_{jj}S = 0$. Thus, by Lemma 3(3)-(4) we have

$$S_{ij} = S_{ji} = 0 \quad \text{and} \quad S_{jj} = \lambda P_j$$

for some scalar λ. Further, for any T_{ij}, applying Lemmas 2, 6, it follows from (4) that

$$\Phi(ST_{ij} - T_{ij}S) = \Phi(A_{ii}T_{ij}) + \Phi(B_{ii}T_{ij}) = \Phi(A_{ii}T_{ij} + B_{ii}T_{ij}).$$

Thus $ST_{ij} - T_{ij}S = A_{ii}T_{ij} + B_{ii}T_{ij}$. Hence

$$S_{ii}T_{ij} - \lambda T_{ij} = S_{ii}T_{ij} - T_{ij}S_{jj} = P_i(ST_{ij} - T_{ij}S)P_j = (A_{ii} + B_{ii})T_{ij},$$

from which we get $S_{ii} = A_{ii} + B_{ii} + \lambda P_i$ by Lemma 3(1). So

$$S = (A_{ii} + B_{ii} + \lambda P_i) + 0 + 0 + \lambda P_j = A_{ii} + B_{ii} + \lambda I.$$

Since $I \notin \mathscr{A}$, we have $\lambda = 0$ and $S = A_{ii} + B_{ii}$, as desired. □

Lemma 8. $\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22})$.

Proof. Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{22}). \tag{5}$$

For any T_{11}, by Lemma 2 we get $ST_{11} - T_{11}S = A_{11}T_{11} - T_{11}A_{11}$, which implies $T_{11}S_{12} = S_{21}T_{11} = 0$. So $S_{12} = S_{21} = 0$. Also, we have

$$T_{11}(S_{11} - A_{11}) = (S_{11} - A_{11})T_{11}$$

and hence there exists a scalar λ such that $S_{11} = A_{11} + \lambda P_1$.

For any T_{12}, applying Lemmas 2, 6, we obtain from (5) that

$$\begin{aligned} &\Phi(ST_{12} - T_{12}S) = \Phi(A_{11}T_{12} - T_{12}A_{11}) + \Phi(A_{22}T_{12} - T_{12}A_{22}) \\ = \;& \Phi(A_{11}T_{12}) + \Phi(-T_{12}A_{22}) = \Phi(A_{11}T_{12} - T_{12}A_{22}). \end{aligned}$$

It follows that $ST_{12} - T_{12}S = A_{11}T_{12} - T_{12}A_{22}$. Hence

$$S_{11}T_{12} - T_{12}S_{22} = A_{11}T_{12} - T_{12}A_{22},$$

in which putting $A_{11} + \lambda P_1$ for S_{11}, we have $S_{22} = A_{22} + \lambda P_2$ by Lemma 3(1). So

$$S = (A_{11} + \lambda P_1) + 0 + 0 + (A_{22} + \lambda P_2) = A_{11} + A_{22} + \lambda I.$$

Then $S = A_{11} + A_{22}$ since $I \notin \mathscr{A}$, completing the proof. □

Lemma 9. $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.

Proof. Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}). \tag{6}$$

For any T_{12}, by Lemma 2 one has

$$ST_{12} - T_{12}S = A_{21}T_{12} - T_{12}A_{21}. \tag{7}$$

Multiplying this equality by P_1 from the right, we get $T_{12}S_{21} = T_{12}A_{21}$, which implies $S_{21} = A_{21}$. With the same discussion for T_{21}, we can get $S_{12} = A_{12}$. For any T_{11}, by Lemma 2 we get from (6)

$$\Phi(ST_{11} - T_{11}S) = \Phi(-T_{11}A_{12}) + \Phi(A_{21}T_{11}).$$

Moreover, for any T_{21}, applying Lemma 2 to the above equality, we have

$$T_{11}ST_{21} + T_{21}(ST_{11} - T_{11}S) = T_{11}A_{12}T_{21} - T_{21}T_{11}A_{12}.$$

Multiplying this equality by P_1 from the right and noting that $S_{12} = A_{12}$, we get $T_{21}(ST_{11} - T_{11}S)P_1 = 0$. Then $S_{11}T_{11} = T_{11}S_{11}$, and so $S_{11} = \lambda P_1$ for some scalar λ. Also, observing that $S_{11}T_{12} - T_{12}S_{22} = 0$ from (7), we have $S_{22} = \lambda P_2$. So $S = A_{12} + A_{21} + \lambda I$. Hence $S = A_{12} + A_{21}$ since $I \notin \mathscr{A}$. □

Lemma 10. $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Proof. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).$$

Then by Lemmas 4, 9, we have that

$$\Phi(S) = \Phi(A_{11} + A_{12}) + \Phi(A_{21}), \tag{8}$$

$$\Phi(S) = \Phi(A_{11} + A_{21}) + \Phi(A_{12}), \tag{9}$$

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12} + A_{21}). \tag{10}$$

For any T_{21}, by Lemma 2 we get

$$ST_{21} - T_{21}S = A_{12}T_{21} - T_{21}(A_{11} + A_{12}) \tag{11}$$

from (8). Multiplying this equality by P_1 from the left, we get $S_{12}T_{21} = A_{12}T_{21}$ and so $S_{12} = A_{12}$. Similarly, one has $S_{21} = A_{21}$ from (9). Further, for any T_{22}, applying Lemma 2 to (10), we obtain

$$ST_{22} - T_{22}S = A_{12}T_{22} - T_{22}A_{21}.$$

Multiplying this equality by P_2 from both sides, we see that $S_{22}T_{22} - T_{22}S_{22} = 0$. It follows that there exists a scalar λ such that $S_{22} = \lambda P_2$. Also, multiplying (11) by P_2 from the left and by P_1 from the right respectively, we get

$$S_{22}T_{21} - T_{21}S_{11} = -T_{21}A_{11}.$$

So $S_{11} = A_{11} + \lambda P_1$. Thus $S = A_{11} + A_{12} + A_{21} + \lambda I$ and consequently, $S = A_{11} + A_{12} + A_{21}$. The proof is complete. □

Lemma 11. $\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

Proof. Suppose that $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ are such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Then by the Lemma 10, we can write

$$\Phi(S) = \Phi(A_{11} + A_{12} + A_{21}) + \Phi(A_{22}). \tag{12}$$

For any T_{11}, by Lemma 2 we see that

$$ST_{11} - T_{11}S = (A_{11} + A_{21})T_{11} - T_{11}(A_{11} + A_{12}).$$

By multiplying this equality by P_2 from the left and the right respectively, it is easily seen that $S_{21} = A_{21}$ and $S_{12} = A_{12}$. Also, multiplying this equality by P_1 from both sides, we can get $(S_{11} - A_{11})T_{11} = T_{11}(S_{11} - A_{11})$. It follows that there exists a scalar λ such that $S_{11} = A_{11} + \lambda P_1$.

For any T_{12}, applying Lemma 2 to (12), we get

$$\Phi(ST_{12} - T_{12}S) = \Phi(A_{11}T_{12} + A_{21}T_{12} - T_{12}A_{21}) + \Phi(-T_{12}A_{22}).$$

Again, for any T_{11}, by Lemmas 2, 4 and 6, we obtain from the above equality that

$$\begin{aligned}
&\Phi(-T_{12}ST_{11} - T_{11}ST_{12} + T_{11}T_{12}S) \\
&= \Phi(-T_{12}A_{21}T_{11} - T_{11}A_{11}T_{12} + T_{11}T_{12}A_{21}) + \Phi(T_{11}T_{12}A_{22}) \\
&= \Phi(T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11}) + \Phi(-T_{11}A_{11}T_{12}) + \Phi(T_{11}T_{12}A_{22}) \\
&= \Phi(T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11}) + \Phi(T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}) \\
&= \Phi(T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11} + T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&T_{11}T_{12}S - T_{12}ST_{11} - T_{11}ST_{12} \\
= \;&T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11} + T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12},
\end{aligned}$$

in which multiplying by P_2 from the right and making use of $S_{11} = A_{11} + \lambda P_1$, we get $S_{22} = A_{22} + \lambda P_2$. Hence $S = A_{11} + A_{12} + A_{21} + A_{22} + \lambda I$. So $S = A_{11} + A_{12} + A_{21} + A_{22}$ because of $I \notin \mathscr{A}$, completing the proof. □

Proof of Theorem 1. Let $A, B \in \mathscr{A}$. By writing $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, then it is easily seen that $\Phi(A+B) = \Phi(A) + \Phi(B)$ making use of Lemmas 6, 7 and 11. We are done. □

Theorem 1 has the following obvious corollary.

Corollary 1. *Let $\mathscr{H}$ be a real or complex Hilbert space with $\dim \mathscr{H} = \infty$, and $\mathscr{A} \subseteq B(\mathscr{H})$ be either the ideal of all finite rank operators or the ideal of all compact operators. Then every bijective Lie map from $\mathscr{A}$ onto an arbitrary algebra is additive.*

We shall conclude by considering the case that $\mathscr{A}$ contains the identity operator I in Theorem 1. In this case, define the map $\Phi : \mathscr{A} \to \mathscr{A}$ by

$$\Phi(A) = \begin{cases} 2A, & \text{if } A \in \mathbb{F}I, \\ A, & \text{otherwise}, \end{cases}$$

where $\mathbb{F}$ denotes the real or complex field. Then

(1) Φ is bijective;

(2) Φ is not additive;

(3) Φ is a Lie map.

In fact, (1) and (2) are obvious. Let us prove that (3). Suppose $A, B \in \mathscr{A}$. We distinguish two cases.

Case 1. At least one of A, B is in $\mathbb{F}I$. Then clearly, $AB - BA = 0$ and

$$\Phi(AB - BA) = 0 = \Phi(A)\Phi(B) - \Phi(B)\Phi(A).$$

Case 2. Both A and B are not in $\mathbb{F}I$. If $AB - BA \notin \mathbb{F}I$, then

$$\Phi(A)\Phi(B) - \Phi(B)\Phi(A) = AB - BA = \Phi(AB - BA).$$

Suppose now that $AB - BA = \lambda I$ for some $\lambda \in \mathbb{F}$. We then have $\sigma(AB) = \lambda + \sigma(BA)$, where $\sigma(\cdot)$ denotes the spectrum of an operator. It is well known that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$. This leads to $\lambda = 0$ and hence

$$\Phi(AB - BA) = 0 = AB - BA = \Phi(A)\Phi(B) - \Phi(B)\Phi(A).$$

So Φ is a Lie map.

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Jia Qian
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, P. R. China
E-mail address: qianjia10001408@163.com

Pengtong Li
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, P. R. China
E-mail address: pengtongli@nuaa.edu.cn