

ASYMPTOTIC NORMALITY OF WAVELET ESTIMATOR OF REGRESSION FUNCTION UNDER NA ASSUMPTIONS

HAN-YING LIANG AND YAN-YAN QI

ABSTRACT. Consider the heteroscedastic regression model $Y_i = g(x_i) + \sigma_i \epsilon_i$ ($1 \leq i \leq n$), where $\sigma_i^2 = f(u_i)$, the design points (x_i, u_i) are known and nonrandom, and g and f are unknown functions defined on closed interval $[0, 1]$. Under the random errors ϵ_i form a sequence of NA random variables, we study the asymptotic normality of wavelet estimators of g when f is a known or unknown function.

1. Introduction

Consider the following heteroscedastic regression model:

$$(1) \quad Y_i = g(x_i) + \sigma_i \epsilon_i, \quad 1 \leq i \leq n,$$

where $\sigma_i^2 = f(u_i)$, (x_i, u_i) are nonrandom design points, and assume $0 \leq x_0 \leq x_1 \leq \cdots \leq x_n = 1$ and $0 \leq u_0 \leq u_1 \leq \cdots \leq u_n = 1$, Y_i are the response variables, ϵ_i are random errors, and $f(\cdot)$ and $g(\cdot)$ are unknown functions defined on closed interval $[0, 1]$.

It is well known that regression model has many applications in practical problems, so the model (1) and its special cases have been studied extensively. For instance, when $\sigma_i^2 = \sigma^2$, the model (1) reduces to the usual nonparametric regression model, when the errors ϵ_i are i.i.d. random variables, various estimation methods have been used to obtain estimators of g , see Georgiev [9], Wang [26] and Xue [28]. For the estimator of g under mixing assumptions, Roussas [20] had found the strong consistency and consistency in quadratic mean, and Roussas and Tran [23] considered the asymptotic normality.

A wavelet analysis has been used extensively in engineering and technological fields, such as signal and image processing, objects are frequently inhomogeneous. In order to meet practical demand, since 90's, some authors have considered to use wavelet methods in statistics. For instance, Antoniadis et al. [2] and Donoho et al. [7] estimated regression function and density function by using wavelet technique, respectively. For wavelet approach, it is well known

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that the hypotheses of degrees of smoothness of the underlying function is less restrictive. Due to this ability to adapt to local features of curves, many authors have applied wavelet procedures to estimate nonparametric models, such as Antoniadis et al. [2], Hall and Patil [10], Liang et al. [16], Qian and Cai [18], Zhou and Yao [29], and so on.

Related to wavelet estimation for model (1) under the case $\sigma_i^2 = \sigma^2$ and independent errors, see Antoniadis et al. [2] and Xue [28]. Liang et al. [15] discussed the convergence rates of wavelet estimators of g and f for model (1) under martingale difference errors; Chen et al. [6] investigated the consistency of wavelet estimators of g and f for model (1) under negatively associated errors. But, up to now, there have been no results available related to asymptotic normality on wavelet estimation for model (1) under negatively associated error assumptions.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (NA) if, for every pair of disjoint subsets A and B of $\{1, 2, \dots, n\}$, we have

$$\text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. The definition of NA random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [11]. Because of its wide applications in multivariate statistical analysis and systems reliability, the notion of NA received considerable attention recently. We refer to Joag-Dev and Proschan [11] for fundamental properties, Shao [24] for moment equalities, Shao and Su [25] for the law of the iterated logarithm, Liang [12] as well as Baek, Kim and Liang [3] for complete convergence, Liang and Baek [13] for some strong law, and Roussas [21] for the central limit theorem of random fields. Asymptotic properties of estimates related to NA samples have also been studied by some authors. See e.g., Cai and Roussas [4], established Berry-Esseen bounds for a smooth estimate of the distribution function. Roussas [22] derived asymptotic normality of the kernel estimate of a probability density function. Liang and Jing [14] discussed the asymptotic properties for estimates of nonparametric regression models.

In this paper, the unobservable errors ϵ_i are assumed to be NA random variables, and we shall investigate the asymptotic normality of wavelet estimators of g in the model (1) when f is a known or unknown function.

The paper is organized as follows. In Section 2, we introduce estimators and give main results, and some lemmas are stated in Section 3. Proofs of theorems and lemmas will be provided in Sections 4 and 5, respectively.

2. Main results

Let ϕ be father wavelet with compact support and unit integral of multiresolution analysis $\{V_m, m \in Z\}$, where Z is integer set. Since $\{\phi(x - k), k \in Z\}$

is an orthogonal family of $L^2(\mathbb{R})$ and V_0 is the subspace spanned, if we denote

$$\phi_{mk}(x) = 2^{m/2} \phi(2^m x - k), \quad k \in Z,$$

then $\{\phi_{0k}, k \in Z\}$ is an orthogonal basis of V_0 , and $\{\phi_{mk}, k \in Z\}$ is an orthogonal basis of V_m . For the more on wavelet see Watler [27].

By ϕ , we can define the following Meyer wavelet kernel:

$$E_m(x, u) = 2^m E_0(2^m x, 2^m u), \quad E_0(x, u) = \sum_{k \in Z} \phi(x - k) \phi(u - k).$$

We now construct the wavelet estimator of g (also see Antoniadis et al. [2]) when $f(\cdot)$ is known:

$$\tilde{g}_n(x) = \sum_{i=1}^n Y_i \int_{A_i} E_m(x, s) ds,$$

where $A_i = [s_{i-1}, s_i]$ are a partition of interval $[0, 1]$ with $x_i \in A_i$ for $1 \leq i \leq n$.

Under $f(\cdot)$ is unknown, the wavelet estimator of g is defined by

$$\hat{g}_n(x) = \sum_{i=1}^n \tilde{Y}_i \int_{A_i} E_m(x, s) ds,$$

where $\tilde{Y}_i = g(x_i) + \hat{\sigma}_{ni} \epsilon_i$, $\hat{\sigma}_{ni}^2 = \hat{f}_n(u_i)$ and

$$\hat{f}_n(u) = \sum_{i=1}^n (Y_i - \tilde{g}_n(x_i))^2 \int_{B_i} E_m(u, s) ds,$$

here $B_i = [s_{i-1}, s_i]$ are another partition of interval $[0, 1]$ with $u_i \in B_i$ for $1 \leq i \leq n$.

In order to list some restrictions for ϕ and g , we give two definitions here.

Definition 1.1. A father wavelet ϕ is said to be q -regular ($S_q, q \in \mathcal{N}$) if for any $l \leq q$, and for any integer k , one has $\left| \frac{d^l \phi}{dx^l} \right| \leq C_k (1 + |x|)^{-1}$, where C_k is a generic constant depending only on k .

Definition 1.2. A function space H^γ ($\gamma \in \mathbf{R}$) is said to be Sobolev space with order γ , i.e., if $h \in H^\gamma$ then $\int |\hat{h}(w)|^2 (1 + w^2)^\gamma dw < \infty$, where \hat{h} is the Fourier transform of h .

Now we list the following some assumptions.

- (A.1) $g(\cdot) \in H^\gamma, \gamma > 3/2$;
- (A.2) $\phi(\cdot) \in S_q, q \geq \gamma$. Let $\phi(\cdot)$ satisfy Lipschitz condition of order 1 and $|\hat{\phi}(\xi) - 1| = O(\xi)$ as $\xi \rightarrow 0$, where $\hat{\phi}$ is the Fourier transformation of ϕ ;
- (A.3) $\max_{1 \leq i \leq n} (s_i - s_{i-1}) = O(n^{-1})$;
- (A.4) $0 < m_0 \leq \min_{1 \leq i \leq n} f(u_i) \leq \max_{1 \leq i \leq n} f(u_i) \leq M_0 < \infty$;
- (A.5) $H(n) = \max_{1 \leq i \leq n} |s_i - s_{i-1} - n^{-1}| = o(n^{-1})$;

(A.6) The errors $\{\epsilon_i\}$, as modelled in (1.1), satisfy the following conditions:

- (i) $v(q) = \sup_k \sum_{j:|k-j|>q} |\text{cov}(e_k, e_j)| \rightarrow 0$ as $q \rightarrow \infty$;
- (ii) The spectral density function $f(w)$ of $\{\epsilon_i\}$ satisfies $0 < C_1 \leq f(w) \leq C_2 < \infty$ for $w \in (-\pi, \pi]$;

(A.7) $f(\cdot) \in H^\gamma, \gamma > 3/2$.

Remark 2.1. Conditions (A.1)-(A.3) were used by Xue [28] and Qian et al. [19]; Condition (A.1-A.4) also were assumed in Chen et al. [5] and Liang et al. [15]; (A.5) is somewhat weaker than the ‘‘asymptotic equidistance’’ assumption of Gasser and Müller [8] in which $H(n) = O(n^{-\delta})$ for some $\delta > 1$.

Remark 2.2. In (A.6) (i), $v(q) \rightarrow 0$ can easily be achieved. For example:

- (i) If $v(1) < \infty$ (which is usually the case, cf. e.g. Roussas [22]), then $v(q) \rightarrow 0$ as $q \rightarrow \infty$.
- (ii) For a stationary sequence, Cai and Roussas [4] use the covariance coefficient: $v'(n) := \sum_{j=n}^\infty |\text{cov}(e_1, e_{j+1})|^{1/3}$ and $v'(1) < \infty$. In this case, we have $|\text{cov}(e_1, e_{j+1})| = o(j^{-3})$. Hence

$$v(q) := \sum_{j=q}^\infty |\text{cov}(e_1, e_{j+1})| = O\left(\sum_{j=q}^\infty |\text{cov}(e_1, e_{j+1})|^{1/3} j^{-2}\right) = O(q^{-2}).$$

In the sequel, assume that $\{\epsilon_i\}$ are identically distributed, and negatively associated random variables with $E\epsilon_i = 0$. C and c stand for positive constants whose value may change at different places. Denote by $x^{(m)} = [2^m x]/2^m$ for $x \in [0, 1]$, here $[x]$ denotes integer part no more than x ;

$$\Delta_n^2 = \text{Var}\left(\sum_{i=1}^n \sigma_i \epsilon_i \int_{A_i} E_m(x^{(m)}, s) ds\right).$$

Our results are as follows.

Theorem 2.1. *Suppose that (A.1)-(A.6) are satisfied, and that $2^m/n \rightarrow 0$ and $2^{3m}/n \rightarrow \infty$. If $E\epsilon_1^2 < \infty$, then*

$$(\tilde{g}_n(x^{(m)}) - g(x^{(m)}))/\Delta_n \rightarrow_d N(0, 1), \quad x \in [0, 1].$$

Theorem 2.2. *Suppose that (A.1)-(A.7) are satisfied, and that $2^{2m}/n \rightarrow 0$ and $2^{3m}/n \rightarrow \infty$. If $E\epsilon_1^2 = 1$ and $E\epsilon_1^4 < \infty$, then*

$$(\hat{g}_n(x^{(m)}) - g(x^{(m)}))/\Delta_n \rightarrow_d N(0, 1), \quad x \in [0, 1].$$

Remark 2.3. Since the proofs of Theorems 2.1 and 2.2 relate to estimate a bound below of Δ_n , we choose the dyadic points $x^{(m)}$ of order m of x (see Antoniadis et al. [2]) in the conclusions of Theorems 2.1 and 2.2. If we assume a bound below for Δ_n , $x^{(m)}$ can be replaced by x . Moreover, the exact estimation of Δ_n is not necessary in the proofs, so it is not necessary to discuss whether Theorems 2.1 and 2.2 are valid for the boundary points, 0 and 1, here. When

related to the exact estimation of Δ_n , the points in interior and boundary of $[0, 1]$ need to be considered, respectively.

3. Some lemmas

Lemma 3.1 (Walter [27], or Antoniadis et al. [2]). *Under the condition of (A.2), we have*

$$(i) \sup_x \int_0^1 |E_m(x, u)| du \leq C; \quad (ii) \sup_{u \in [0, 1]} |E_m(x, u)| = O(2^m).$$

Lemma 3.2. *Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying $\sum_{i=1}^n a_{ni}^2 = O(1)$ and $\max_{1 \leq i \leq n} |a_{ni}| \rightarrow 0$ as $n \rightarrow \infty$. Assume that (A.6) (i) holds and $E\epsilon_1^2 < \infty$. If $\text{Var}(\sum_{i=1}^n a_{ni}\epsilon_i) \rightarrow 1$, then $\sum_{i=1}^n a_{ni}\epsilon_i \rightarrow_d N(0, 1)$.*

Proof. See the proof of Theorem 2.7 in Liang and Jing [14]. \square

Lemma 3.3 (Shao [24]). *Let $\{X_i, i \geq 1\}$ be a sequence of NA random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $p \geq 1$. Then, there exists constant $C_p > 0$ such that $E \max_{1 \leq k \leq n} |\sum_{i=1}^k X_i|^p \leq C_p \sum_{i=1}^n E|X_i|^p$ for $1 \leq p < 2$.*

Lemma 3.4. *Under the assumptions of Theorem 2.2, we have*

$$\sup_{0 \leq u \leq 1} |\hat{f}_n(u) - f(u)| = O(2^{-m} + n^{-1}) + O_p(2^m/\sqrt{n}).$$

The proof of Lemma 3.4 will be given in Section 5.

4. Proofs of main results

Proof of Theorem 2.1. We write

$$(2) \quad \tilde{g}_n(x) - g(x) = [\tilde{g}_n(x) - E\tilde{g}_n(x)] + [E\tilde{g}_n(x) - g(x)].$$

We first prove that

$$(\tilde{g}_n(x^{(m)}) - E\tilde{g}_n(x^{(m)}))/\Delta_n \rightarrow_d N(0, 1).$$

Note that $\tilde{g}_n(x) - E\tilde{g}_n(x) = \sum_{i=1}^n \sigma_i \epsilon_i \int_{A_i} E_m(x, s) ds$. So, according to Lemma 3.2, we need only to verify that

$$(3) \quad \sup_n \sum_{i=1}^n (\sigma_i \int_{A_i} E_m(x^{(m)}, s) ds / \Delta_n)^2 < \infty;$$

$$(4) \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} |\sigma_i \int_{A_i} E_m(x^{(m)}, s) ds / \Delta_n| \rightarrow 0.$$

From the assumptions of this theorem, it follows

$$\begin{aligned} \Delta_n^2 &= \text{Var}\left(\sum_{k=1}^n \sigma_k \epsilon_k \int_{A_k} E_m(x^{(m)}, s) ds\right) \\ &= \int_{-\pi}^{\pi} f(w) \left| \sum_{k=1}^n \sigma_k \int_{A_k} E_m(x^{(m)}, s) ds e^{-ikw} \right|^2 dw, \end{aligned}$$

which yields from (A.6)(ii) that

$$(5) \quad C_1 \sum_{k=1}^n \sigma_k^2 \left(\int_{A_k} E_m(x^{(m)}, s) ds \right)^2 \leq \Delta_n^2 \leq C_2 \sum_{k=1}^n \sigma_k^2 \left(\int_{A_k} E_m(x^{(m)}, s) ds \right)^2,$$

which yields (3). In addition, it is easy to see that

$$\begin{aligned} & \left| \sum_{i=1}^n \sigma_i^2 \left(\int_{A_i} E_m(x^{(m)}, s) ds \right)^2 - \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \int_{A_i} E_m^2(x^{(m)}, s) ds \right| \\ & \leq \sum_{i=1}^n \sigma_i^2 \left| (s_i - s_{i-1})^2 E_m^2(x^{(m)}, \xi_i^{(1)}) - \frac{1}{n} (s_i - s_{i-1}) E_m^2(x^{(m)}, \xi_i^{(2)}) \right| \\ & \quad (\text{where both } \xi_i^{(1)} \text{ and } \xi_i^{(2)} \text{ belong to } A_i) \\ & = O(n^{-1}) \sum_{i=1}^n \sigma_i^2 \left| \left(s_i - s_{i-1} - \frac{1}{n} \right) E_m^2(x^{(m)}, \xi_i^{(1)}) \right. \\ & \quad \left. - \frac{1}{n} (E_m^2(x^{(m)}, \xi_i^{(2)}) - E_m^2(x^{(m)}, \xi_i^{(1)})) \right|. \end{aligned}$$

Note that the number of terms contributing to the above sum is of order $O(n2^{-m})$, $\sup_{x,s} E_m^2(x, s) = O(2^{2m})$ and

$$\begin{aligned} & |E_m^2(x, \xi_i^{(2)}) - E_m^2(x, \xi_i^{(1)})| \\ & = 2^{2m} |E_0^2(2^m x, 2^m \xi_i^{(2)}) - E_0^2(2^m x, 2^m \xi_i^{(1)})| = O(2^{3m} n^{-1}). \end{aligned}$$

Therefore, from (A.4)-(A.5) we have

$$\begin{aligned} & \left| \sum_{i=1}^n \sigma_i^2 \left(\int_{A_i} E_m(x^{(m)}, s) ds \right)^2 - \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \int_{A_i} E_m^2(x^{(m)}, s) ds \right| \\ (6) \quad & = O(n^{-1}) n 2^{-m} \left(H(n) 2^{2m} + O(2^{3m} n^{-2}) \right) \\ & = o(2^m n^{-1}) + O(2^{2m} n^{-2}). \end{aligned}$$

In view of (5) and (6), from (A.4) we find

$$\begin{aligned} \Delta_n^2 & \geq C n^{-1} \int_0^1 E_m^2(x^{(m)}, s) ds + o(n^{-1} 2^m) \\ (7) \quad & \geq C n^{-1} 2^{2m} \int_0^1 E_0^2(2^m x^{(m)}, 2^m s) ds + o(n^{-1} 2^m) = C n^{-1} 2^m. \end{aligned}$$

According to Lemma 3.1, from (7) we get

$$\frac{\max_{1 \leq i \leq n} |\sigma_i \int_{A_i} E_m(x^{(m)}, s) ds|}{\Delta_n} \leq C \sqrt{\frac{2^m}{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e., (4) holds.

Next, according to $2^m/n \rightarrow 0$ and $2^{3m}/n \rightarrow \infty$ we need only to verify from (7) that

$$(8) \quad \sup_x |E \tilde{g}_n(x) - g(x)| = O(2^{-m} + n^{-1}).$$

In fact,

$$\begin{aligned} E \tilde{g}_n(x) - g(x) &= \left[\sum_{i=1}^n \int_{A_i} E_m(x, u) g(x_i) du - \int_0^1 E_m(x, u) g(u) du \right] \\ &\quad + \left[\int_0^1 E_m(x, u) g(u) du - g(x) \right] := I_{n1}(x) + I_{n2}(x). \end{aligned}$$

Since condition (A.1) follows that $g(\cdot)$ is continuous and differentiable from Antoniadis et al. [2], $g(\cdot)$ satisfies Lipschitz condition of order 1 on $[0, 1]$. Hence, by Lemma 3.1, we obtain that

$$\sup_x |I_{n1}(x)| = \sup_x \left| \sum_{i=1}^n \int_{A_i} E_m(x, u) [g(x_i) - g(u)] du \right| = O(n^{-1}).$$

From the proof of Theorem 3.2 in Antoniadis et al. [2], we have

$$\sup_x |I_{n2}(x)| = O(2^{-m}).$$

Thus, (8) is verified. Therefore, Theorem 2.1 is proved. \square

Proof of Theorem 2.2. Obviously

$$(9) \quad \frac{\hat{g}_n(x) - g(x)}{\Delta_n} = \frac{\hat{g}_n(x) - \tilde{g}_n(x)}{\Delta_n} + \frac{\tilde{g}_n(x) - g(x)}{\Delta_n}.$$

From Theorem 2.1, it suffices to show that

$$(10) \quad [\hat{g}_n(x) - \tilde{g}_n(x)]/\Delta_n = o_p(1).$$

In view of (A.3) and Lemmas 3.1 and 3.3 we have

$$\begin{aligned} &P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k \epsilon_i \int_{A_i} E_m(x, s) ds \right| \geq K \sqrt{2^m/n}\right) \\ &\leq \frac{cn}{K^2 2^m} \sum_{i=1}^n \left(\int_{A_i} E_m(x, s) ds \right)^2 E \epsilon_i^2 \\ &\leq \frac{cn}{K^2 2^m} \max_{1 \leq i \leq n} \int_{A_i} |E_m(x, s)| ds \int_0^1 |E_m(x, s)| ds = O\left(\frac{1}{K^2}\right) = o(1) \text{ as } K \rightarrow \infty, \end{aligned}$$

i.e.,

$$(11) \quad \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \epsilon_i \int_{A_i} E_m(x, s) ds \right| = O_p(\sqrt{2^m/n}).$$

When n is large enough, we know from Lemma 3.4 and Condition (A.4) that

$$(12) \quad 0 < m'_0 \leq \min_{1 \leq i \leq n} \hat{f}_n(u_i) \leq \max_{1 \leq i \leq n} \hat{f}_n(u_i) \leq M'_0 < \infty, \text{ in Probability.}$$

In order to estimate $|\hat{g}_n(x) - g(x)|$ we need the following the **Abel Inequality** (see P.32, Theorem 1 of Mitrinovic [17]): Let $A_1, A_2, \dots, A_n; B_1, B_2, \dots, B_n$ ($B_1 \geq B_2 \geq \dots \geq B_n \geq 0$) be two sequences of real numbers, and let $S_k = \sum_{i=1}^k A_i$, $M_1 = \min_{1 \leq k \leq n} S_k$ and $M_2 = \max_{1 \leq k \leq n} S_k$. Then

$$(13) \quad B_1 M_1 \leq \sum_{k=1}^n A_k B_k \leq B_1 M_2.$$

Hence, for any real number G_k, H_k ($1 \leq k \leq n$), without loss of generality, assume $H_1 \geq H_2 \geq \dots \geq H_n$. Let $Q_s = H_s - H_n$, $1 \leq s \leq n - 1$, $Q_n = 0$. Applying (13) we have

$$(14) \quad \left| \sum_{k=1}^n G_k H_k \right| \leq \left| \sum_{k=1}^n G_k Q_k \right| + \left| \sum_{k=1}^n G_k H_n \right| \leq 5 \max_{1 \leq i \leq n} |H_i| \max_{1 \leq m \leq n} \left| \sum_{k=1}^m G_k \right|.$$

Hence, on applying (14), and from Lemma 3.4, (A.4), (7) and (11)-(12) we obtain that

$$\begin{aligned} \frac{|\hat{g}_n(x) - \tilde{g}_n(x)|}{\Delta_n} &\leq \frac{C}{\Delta_n} \max_{1 \leq i \leq n} |\hat{\sigma}_{ni} - \sigma_i| \cdot \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \epsilon_i \int_{A_i} E_m(x, s) ds \right| \\ &\leq \frac{C \sup_{1 \leq u \leq n} |\hat{f}_n(u) - f(u)|}{\Delta_n (\sqrt{\hat{f}_n(u)} + \sqrt{f(u)})} \cdot \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \epsilon_i \int_{A_i} E_m(x, s) ds \right| \\ &= O_p(2^{-m} + n^{-1} + 2^m/\sqrt{n}) = o_p(1). \end{aligned}$$

Therefore, (10) is verified. □

5. Proof of Lemma 3.4

We write

$$\begin{aligned} \hat{f}_n(u) - f(u) &= \sum_{i=1}^n (\sqrt{f(u_i)} \epsilon_i + g(x_i) - \tilde{g}_n(x_i))^2 \int_{B_i} E_m(u, s) ds - f(u) \\ &= \left[\sum_{i=1}^n f(u_i) \epsilon_i^2 \int_{B_i} E_m(u, s) ds - f(u) \right] \\ (15) \quad &+ 2 \sum_{i=1}^n \sqrt{f(u_i)} \epsilon_i (g(x_i) - \tilde{g}_n(x_i)) \int_{B_i} E_m(u, s) ds \\ &+ \sum_{i=1}^n (g(x_i) - \tilde{g}_n(x_i))^2 \int_{B_i} E_m(u, s) ds \\ &:= L_{n1}(u) + 2L_{n2}(u) + L_{n3}(u). \end{aligned}$$

By $Ee_i^2 = 1$, we have

$$\begin{aligned} L_{n1}(u) &= \sum_{i=1}^n f(u_i) e_i^2 \int_{B_i} E_m(u, s) ds - f(u) \\ &= \sum_{i=1}^n f(u_i) (e_i^2 - Ee_i^2) \int_{B_i} E_m(u, s) ds + \left[\sum_{i=1}^n f(u_i) \int_{B_i} E_m(u, s) ds - f(u) \right] \\ &:= L_{n11}(u) + L_{n12}(u). \end{aligned}$$

Similarly to the estimate for (8), we can obtain that $\sup_{0 \leq u \leq 1} |L_{n12}(u)| = O(2^{-m} + n^{-1})$. We write

$$\begin{aligned} L_{n11}(u) &= \sum_{i=1}^n f(u_i) [(e_i^+)^2 - E(e_i^+)^2] \int_{B_i} E_m(u, s) ds \\ &\quad + \sum_{i=1}^n f(u_i) [(e_i^-)^2 - E(e_i^-)^2] \int_{B_i} E_m(u, s) ds \\ &:= \sum_{i=1}^n f(u_i) \xi_i \int_{B_i} E_m(u, s) ds + \sum_{i=1}^n f(u_i) \eta_i \int_{B_i} E_m(u, s) ds \\ &:= L_{n11}^{(1)}(u) + L_{n11}^{(2)}(u), \end{aligned}$$

where $a_i^+ = \max(a, 0)$, $a_i^- = \max((-a), 0)$. Then, according to the NA property, both $\{\xi_i, i \geq 1\}$ and $\{\eta_i, i \geq 1\}$ are sequences of NA random variables, and

$$E(\xi_1^2) < \infty, E(\eta_1^2) < \infty \quad \text{by} \quad E(\epsilon_1^4) < \infty.$$

Hence, in view of (14), (A.3)-(A.4) and Lemmas 3.1 and 3.3 we can obtain that

$$\sup_{0 \leq u \leq 1} |L_{n11}^{(1)}(u)| \leq C \sup_{1 \leq u \leq 1} \max_{1 \leq i \leq n} \int_{B_i} |E_m(u, s)| ds \cdot \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \xi_i \right| = O_p(2^m / \sqrt{n}).$$

Similarly $\sup_{0 \leq u \leq 1} |L_{n11}^{(2)}(u)| = O_p(2^m / \sqrt{n})$, and we also have

$$\begin{aligned} &\sup_{0 \leq x \leq 1} |g(x) - \tilde{g}_n(x)| \\ &\leq \sup_{0 \leq x \leq 1} |\tilde{g}_n(x) - E \tilde{g}_n(x)| + \sup_{0 \leq x \leq 1} |E \tilde{g}_n(x) - g(x)| \\ (16) \quad &= \sup_{0 \leq x \leq 1} \left| \sum_{i=1}^n \sigma_i \epsilon_i \int_{A_i} E_m(x, s) ds \right| + O(2^{-m} + n^{-1}) \\ &\leq C \sup_{0 \leq x \leq 1} \max_{1 \leq i \leq n} \int_{A_i} |E_m(x, s)| ds \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \sigma_i \epsilon_i \right| + O(2^{-m} + n^{-1}) \\ &= O_p(2^m / \sqrt{n}) + O(2^{-m} + n^{-1}). \end{aligned}$$

Therefore, by using (16) we can obtain that

$$\begin{aligned} \sup_{0 \leq u \leq 1} |L_{n2}(u)| &\leq C \sup_{0 \leq x \leq 1} |g(x) - \tilde{g}_n(x)| \sup_{0 \leq u \leq 1} \left| \int_{B_i} E_m(u, s) ds \right| \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \sigma_i \epsilon_i \right| \\ &= O(2^{-m} + n^{-1}) + O_p(2^m / \sqrt{n}); \\ \sup_{0 \leq u \leq 1} |L_{n3}(u)| &\leq C \sup_{0 \leq x \leq 1} [g(x) - \tilde{g}_n(x)]^2 \sup_{0 \leq u \leq 1} \sum_{i=1}^n \left| \int_{B_i} E_m(u, s) ds \right| \\ &= O(2^{-m} + n^{-1}) + O_p(2^m / \sqrt{n}). \end{aligned}$$

Put the estimations for $L_{n1}(u)$, $L_{n2}(u)$, $L_{n3}(u)$ into (15), the conclusion is proved.

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HAN-YING LIANG
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092, CHINA
E-mail address: hyliang83@yahoo.com

YAN-YAN QI
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092, CHINA
E-mail address: irisqiy@126.com