Bull. Korean Math. Soc. **44** (2007), No. 2, pp. 225–231

GALOIS GROUPS OF MODULES AND INVERSE POLYNOMIAL MODULES

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ABSTRACT. Given an injective envelope E of a left R-module M, there is an associative Galois group $Gal(\phi)$. Let R be a left noetherian ring and E be an injective envelope of M, then there is an injective envelope $E[x^{-1}]$ of an inverse polynomial module $M[x^{-1}]$ as a left $R[x]$-module and we can define an associative Galois group $Gal(\phi[x^{-1}])$. In this paper we describe the relations between $Gal(\phi)$ and $Gal(\phi[x^{-1}])$. Then we extend the Galois group of inverse polynomial module and can get $Gal(\phi[x^{-s}])$, where S is a submonoid of $\mathbb{N}$ (the set of all natural numbers).

1. Introduction

Given an injective envelope $M \subset E$, by the Galois group of this envelope we mean all $f \in Hom_R(E, E)$ such that $f(x) = x$ for all $x \in M$ or equivalently such that

$$\begin{array}{ccc} M & \longrightarrow & E \\ & \searrow & \downarrow f \\ & & E \end{array}$$

is a commutative diagram. Any such f is an automorphism of E and we also see that

$$\begin{array}{ccc} M & \longrightarrow & E \\ & \searrow & \downarrow f^{-1} \\ & & E \end{array}$$

is commutative. So we easily see that the set of f form a group (using the composition of functions as operation). If $\phi : M \longrightarrow E$ denotes the canonical injection then the group is denoted $Gal(\phi)$. Northcott ([4]) defined inverse polynomial modules and used inverse polynomial modules to study the properties of injective modules and he studied $K[x^{-1}]$ as $K[x]$-module on field K. And

Received January 2, 2006.
2000 *Mathematics Subject Classification.* Primary 16E30; Secondary 13C11, 16D80.
Key words and phrases. injective module, injective envelope, Galois group, inverse polynomial module.
This paper was supported by Dong-A University Research fund, in 2005.

McKerraw ([2]) showed that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$ as $R[x]$-module. Inverse polynomial modules were studied in ([5]), ([6]) and recently in ([1]), ([7]), ([8]), ([9]).

Definition 1.1 ([5]). Let R be a ring and M be a left R-module, then $M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1}$$

and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}$$

where $r \in R$. We call $M[x^{-1}]$ as an inverse polynomial module.

If R is left noetherian and if $M \subset E$ is as above, then $M[x^{-1}] \subset E[x^{-1}]$ is an injective envelope over $R[x]$. If $\phi[x^{-1}] : M[x^{-1}] \longrightarrow E[x^{-1}]$ denotes the canonical injection, then the group is denoted $Gal(\phi[x^{-1}])$.

Lemma 1.2 ([5]). *Let M and N be left R-modules, then*

$$Hom_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong Hom_R(M, N)[[x]].$$

Theorem 1.3. *There is a ring isomorphism*

$$Hom_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong Hom_R(M, N)[[x]].$$

Proof. By the Lemma 1.2., we know that two groups are isomorphic. Let $\sigma, \tau \in Hom_{R[x]}(M[x^{-1}], N[x^{-1}])$, then σ corresponds to $f_0 + f_1x + f_2x^2 + \cdots \in Hom_R(M, N)[[x]]$ and τ corresponds to $g_0 + g_1x + g_2x^2 + \cdots \in Hom_R(M, N)[[x]]$. Then $\sigma \circ \tau$ corresponds to

$$\sum_{n=0}^{\infty}(\sum_{i+j=n} f_i \circ g_j)x^n.$$

Hence, $Hom_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong Hom_R(M, N)[[x]]$. □

2. $Gal(\phi)$ and $Gal(\phi[x^{-1}])$

Theorem 2.1. *If R is a left noetherian ring and if $M \subset E$ is an injective envelope of R-module, then $f = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots \in End_R(E)[[x]]$ is in $Gal(\phi[x^{-1}])$ if and only if $f_0 \in Gal(\phi)$ and $f_i(M) = 0$ for all $i \geq 1$.*

Proof. Let $m \in M$ and $f \in Gal(\phi[x^{-1}])$, then

$$\begin{aligned} &f(m + 0x^{-1} + 0x^{-2} + \cdots + 0x^{-i}) \\ &= (f_0 + f_1x + f_2x^2 + \cdots)(m + 0x^{-1} + 0x^{-2} + \cdots + 0x^{-i}) \\ &= (f_0 + f_1x + f_2x^2 + \cdots)(m) \\ &= f_0(m) + f_1(m)x + f_2(m)x^2 + \cdots \\ &= m. \end{aligned}$$

Thus $f_0(m) = m$ for all $m \in M$, so that $f_0 \in Gal(\phi)$. And

$$\begin{aligned} & f(m+mx^{-1}) \\ &= (f_0 + f_1x + f_2x^2 + \cdots)(m+mx^{-1}) \\ &= f_0(m) + f_0(m)x^{-1} + f_1(m)x + f_1(m) + f_2(m)x^2 + f_2(m)x + \cdots \\ &= (f_0(m) + f_1(m)) + f_0(m)x^{-1} + (f_1(m) + f_2(m))x + \cdots \\ &= m + mx^{-1}. \end{aligned}$$

Since $f_0(m) = m$, $m + f_1(m) = m$ implies $f_1(m) = 0$. Thus $f_1(m) = 0$. And

$$\begin{aligned} & f(m+mx^{-1}+mx^{-2}) \\ &= (f_0 + f_1x + f_2x^2 + \cdots)(m+mx^{-1}+mx^{-2}) \\ &= f_0(m) + f_0(m)x^{-1} + f_1(m)x + f_1(m) + f_1(m)x^{-1} + f_2(m)x^2 \\ &\quad + f_2(m)x + f_2(m) + \cdots \\ &= (f_0(m) + f_1(m) + f_2(m)) + (f_0(m) + f_1(m))x^{-1} + f_0(m)x^{-2} \\ &\quad + (f_1(m) + f_2(m))x + \cdots \\ &= m + mx^{-1} + mx^{-2}. \end{aligned}$$

Since $f_0(m) = m, f_1(m) = 0$, $f_0(m) + f_1(m) + f_2(m) = m$ implies $f_2(m) = 0$. Thus $f_2(m) = 0$. By the same process we can get $f_i(M) = 0$ for all $i \geq 1$.

Conversely, let $f = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots$ with $f \in Gal(\phi[x^{-1}])$ and $f_i(M) = 0, i \geq 1$. Let $m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i} \in M[x^{-1}]$. We want to show

$$f(m_0+m_1x^{-1}+m_2x^{-2}+\cdots+m_ix^{-i}) = m_0+m_1x^{-1}+m_2x^{-2}+\cdots+m_ix^{-i}.$$

Then

$$\begin{aligned} & f(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}) \\ &= (f_0 + f_1x + f_2x^2 + \cdots)(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}) \\ &= f_0(m_0) + f_0(m_1)x^{-1} + f_0(m_2)x^{-2} + \cdots + f_0(m_i)x^{-i} \\ &\quad + f_1(m_0)x + f_1(m_1) + f_1(m_2)x^{-1} + \cdots + f_1(m_i)x^{-i+1} + f_2(m_0)x^2 \\ &\quad + f_2(m_1)x + f_2(m_2) + f_2(m_3)x^{-1} + \cdots + f_2(m_i)x^{-i+2} + \cdots + f_i(m_i) \\ &= m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}, \end{aligned}$$

since $f_0 \in Gal(\phi)$ and $f_i(M) = 0$ for all $i \geq 1$. Therefore, $f = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots \in Gal(\phi[x^{-1}])$. □

There are natural group homomorphisms $Gal(\phi) \to Gal(\phi[x^{-1}])$ by $g \longmapsto g + 0x + 0x^2 + \cdots$ and $Gal(\phi[x^{-1}]) \to Gal(\phi)$ by $f_0 + f_1x + f_2x^2 + \cdots \longmapsto f_0$. The composition $Gal(\phi) \to Gal(\phi[x^{-1}]) \to Gal(\phi)$ is the identity map on $Gal(\phi)$. The kernel of $Gal(\phi[x^{-1}]) \to Gal(\phi)$ consists of all $id_E + f_1x + f_2x^2 + \cdots$, where $f_i \in Hom_R(E, E)$ and $f_i(M) = 0$, for all $i \geq 1$.

Lemma 2.2. *Let $\psi : Gal(\phi) \longrightarrow Gal(\phi[x^{-1}])$ be defined by $\psi(f) = f + 0x + 0x^2 + \cdots$. If $End(E)$ is a commutative ring, then $Im(\psi)$ is a normal subgroup of $Gal(\phi[x^{-1}])$.*

Proof. Let $f_0+0x+0x^2+\cdots \in Im(\psi)$, and $g_0+g_1x+g_2x^2+\cdots \in Gal(\phi[x^{-1}])$. Let $(g_0 + g_1x + g_2x^2 + \cdots)^{-1} = h_0 + h_1x + h_2x^2 + \cdots$. Then

$$(g_0 + g_1x + g_2x^2 + \cdots) \circ (h_0 + h_1x + h_2x^2 + \cdots) = id_E + 0x + 0x^2 + \cdots,$$

implies $g_0 \circ h_0 = id_E$ so that $h_0 = g_0^{-1}$ and $\sum_{i+j=n} g_i \circ h_j = 0, n \geq 1$. Thus

$$\begin{aligned}
&(g_0 + g_1x + \cdots) \circ (f_0 + 0x + \cdots) \circ (h_0 + h_1x + \cdots) \\
&= ((g_0 \circ f_0) + (g_1 \circ f_0)x + (g_2 \circ f_0)x^2 + \cdots) \circ (h_0 + h_1x + h_2x^2 + \cdots) \\
&= (g_0 \circ f_0 \circ h_0) + (g_0 \circ f_0 \circ h_1 + g_1 \circ f_0 \circ h_0)x \\
&\quad + (g_0 \circ f_0 \circ h_2 + g_1 \circ f_0 \circ h_1 + g_2 \circ f_0 \circ h_0)x^2 + \cdots = f_0,
\end{aligned}$$

since $End(E)$ is a commutative ring. Hence, $Im(\psi)$ is a normal subgroup of $Gal(\phi[x^{-1}])$. □

We note that $Im(\psi)$ is not a normal subgroup of $Gal(\phi[x^{-1}])$, in general. So $Gal(\phi[x^{-1}])$ is the semidirect product of $Gal(\phi)$ and $K = ker(Gal(\phi[x^{-1}]) \to Gal(\phi))$.

Lemma 2.3. *$Gal(\phi)$ is commutative if and only if $g \circ g' = g' \circ g$ for all $g, g' \in Hom_R(E, E)$ with $g(M) = 0, g'(M) = 0$.*

Proof. If $f \in Gal(\phi)$, then $g = f - id_E \in Hom_R(E, E)$ with $g(M) = 0$. And given $g \in Gal(\phi[x^{-1}])$ with $g(M) = 0$, $f = g + id_E \in Gal(\phi)$. Therefore, there is one to one correspondence between $Gal(\phi)$ and the set of $g \in Hom_R(E, E)$ with $g(M) = 0$. So, given $f, f' \in Gal(\phi)$ choose $g = f - id_E, g' = f' - id_E \in Hom_R(E, E)$ with $g(M) = 0, g'(M) = 0$. Then $g \circ g' = g' \circ g$.

Conversely, given $g, g' \in Hom_R(E, E)$ with $g(M) = 0, g'(M) = 0$ choose $f = g + id_E, f' = g' + id_E \in Gal(\phi)$. Then $f \circ f' = f' \circ f$. Thus, $Gal(\phi)$ is commutative. □

Theorem 2.4. *$Gal(\phi[x^{-1}])$ is commutative if and only if $Gal(\phi)$ is commutative.*

Proof. Since $Gal(\phi)$ is a subgroup of $Gal(\phi[x^{-1}])$, $Gal(\phi)$ is commutative. Conversely, let $f_0 + f_1x + f_2x^2 + \cdots, g_0 + g_1x + g_2x^2 + \cdots \in Gal(\phi[x^{-1}])$. Then by the Theorem 2.1., $f_0, g_0 \in Gal(\phi), f_i(M) = 0, g_j(M) = 0$, for all $i, j \geq 1$. And by the Lemma 2.3., $f_i \circ g_j = g_j \circ f_i, i, j \geq 1$. Given $f_i \in Gal(\phi)$ choose $g_i = f_i - id_E \in Hom(E, E)$ with $g_i(M) = 0$. Then

$$\begin{aligned}
f_0 \circ g_i &= f_0 \circ (f_i - id_E) = f_0 \circ f_i - f_0 = f_i \circ f_0 - f_0 \\
&= (f_i - id_E) \circ f_0 = g_i \circ f_0.
\end{aligned}$$

So

$$\begin{aligned}&(f_0 + f_1x + f_2x^2 + \cdots) \circ (g_0 + g_1x + g_2x^2 + \cdots)\\ &= (f_0 \circ g_0) + (f_0 \circ g_1 + f_1 \circ g_0)x + (f_0 \circ g_2 + f_1 \circ g_1 + f_2 \circ g_0)x^2 + \cdots\\ &= (g_0 \circ f_0) + (g_1 \circ f_0 + g_0 \circ f_1)x + (g_2 \circ f_0 + g_1 \circ f_1 + g_0 \circ f_2)x^2 + \cdots\\ &= (g_0 + g_1x + g_2x^2 + \cdots) \circ (f_0 + f_1x + f_2x^2 + \cdots).\end{aligned}$$

Therefore, $Gal(\phi[x^{-1}])$ is commutative. □

Theorem 2.5. *Let $\varphi : Gal(\phi[x^{-1}]) \longrightarrow Gal(\phi)$ be defined by $\varphi(f_0 + f_1x + f_2x^2 + \cdots) = f_0$. Then $Gal(\phi[x^{-1}])$ is the direct product of K and $Gal(\phi)$ if and only if $Gal(\phi)$ is commutative, where $K = ker(\varphi)$.*

Proof. Let $g, g' \in Gal(\phi)$. Then $id_E + g \in Gal(\phi)$ and $(id_E + g'x)^{-1} \circ (id_E + g) \circ (id_E + g'x) \in Gal(\phi)$. So let $(id_E + g'x)^{-1} = id_E - g'x +$ etc., then

$$\begin{aligned}&(id_E + g'x)^{-1} \circ (id_E + g) \circ (id_E + g'x)\\ &= (id_E - g'x + \text{etc}) \circ (id_E + g) \circ (id_E + g'x)\\ &= id_E + (-g' \circ g + g \circ g')x + \text{etc.} \in Gal(\phi)\end{aligned}$$

implies $-g' \circ g + g \circ g' = 0$ so that $g' \circ g = g \circ g'$.
Therefore, $Gal(\phi)$ is commutative.

Conversely, by the Theorem 2.4., if $Gal(\phi)$ is commutative then $Gal(\phi[x^{-1}])$ is commutative. Therefore, $Gal(\phi[x^{-1}])$ is the direct product of K and $Gal(\phi)$. □

3. Generalization of Galois group

Definition 3.1 ([8]). Let R be a ring and M be a left R-module, and $S = \{0, k_1, k_2, \ldots\}$ be a submonoid of $\mathbb{N}$ (the set of all natural numbers). Then $M[x^{-s}]$ is a left $R[x^s]$-module such that

$$\begin{aligned}&x^{k_i}(m_0 + m_1x^{-k_1} + m_2x^{-k_2} + \cdots + m_nx^{-k_n})\\ &= m_1^{-k_1+k_i} + m_2x^{-k_2+k_i} + \cdots + m_nx^{-k_n+k_i}\end{aligned}$$

where

$$x^{-k_j+k_i} = \begin{cases} x^{-k_j+k_i} & \text{if } k_j - k_i \in S\\ 0 & \text{if } k_j - k_i \notin S.\end{cases}$$

For example, if $S = \{0, 2, 3, \ldots\}$, then $m_0 + m_2x^{-2} + m_3x^{-3} + \cdots + m_ix^{-i} \in M[x^{-s}]$ and if $S = \{0, 1, 2, 3, \ldots\}$, then $M[x^{-s}] = M[x^{-1}]$.

Similarly, we define $M[[x^{-s}]]$, $M[x^s, x^{-s}]$, $M[[x^s, x^{-s}]]$, $M[x^s, x^{-s}]]$ and $M[[x^s, x^{-s}]$ as left $R[x^s]$-modules.

Definition 3.2. Given any module M and $f \in End(E)$ we say f is locally nilpotent on M if for every $x \in M$, there exist $n \geq 1$ such that $f^n(x) = 0$.

Theorem 3.3 (Matlis and Gabriel). *If R is a left noetherian ring and E is an injective left R-module and $f \in End(E)$ is such that E is an essential extension of $ker(f)$ then f is locally nilpotent on E.*

Theorem 3.4. *Let R be a commutative noetherian ring and S be a submonoid, and E be an injective left R-module. Then $E[x^{-s}]$ is an injective left $R[x^s]$-module.*

Proof. Let $S = \{0, k_1, k_2, \ldots\}$ be a submonoid. Then $Hom_R(R[x^s], E) \cong E[[x^{-s}]]$ is an injective left $R[x^s]$-module. Define $\phi : E[[x^{-s}]] \longrightarrow E[[x^{-s}]]$ by $\phi(f) = x^{k_1} f$ for $f \in E[[x^{-s}]]$. Then ϕ is not locally nilpotent on $E[[x^{-s}]]$. So $E[[x^{-s}]]$ is not an essential extension of $ker(\phi)$. Let $\bar{E}$ be an injective envelope of $ker(\phi)$. Then

$$ker(\phi) \subset \bar{E} \subset E[[x^{-s}]].$$

Then $\phi : \bar{E} \longrightarrow \bar{E}$ defined by

$$\phi(f) = x^{k_1} f,$$

for $f \in \bar{E}$ is locally nilpotent on $\bar{E}$. So $\bar{E} \subset E[x^{-s}]$. But $E[x^{-s}]$ is an essential extension of $ker(\phi)$, so that $E[x^{-s}]$ is an essential extension of $\bar{E}$. Therefore, $\bar{E} = E[x^{-s}]$. Hence, $E[x^{-s}]$ is an injective left $R[x^s]$-module. □

We can generalize the Theorem 1.3. and get

$$Hom_{R[x^s]}(M[x^{-s}], N[x^{-s}]) \cong Hom_R(M, N)[[x^s]].$$

If $\phi[x^{-s}] : M[x^{-s}] \longrightarrow E[x^{-s}]$ denotes the canonical injection, then the group is denoted $Gal(\phi[x^{-s}])$.

Theorem 2.1. can be extended to the following remark.

Remark 1. If R is a left noetherian ring and if $M \subset E$ is an injective envelope of R-module, then $f = f_{k_0} + f_{k_1} x^{k_1} + f_{k_2} x^{k_2} + f_{k_3} x^{k_3} + \cdots \in End_R(E)[[x^s]]$ is in $Gal(\phi[x^{-s}])$ if and only if $f_{k_0} \in Gal(\phi)$ and $f_{k_i}(M) = 0, k_i \in S, k_i \neq k_0$.

Lemma 2.2. can be extended to the following remark.

Remark 2. Let $\psi : Gal(\phi) \longrightarrow Gal(\phi[x^{-s}])$ be defined by $\psi(f) = f + 0x^{k_1} + 0x^{k_2} + \cdots$. If $End(E)$ is a commutative ring, then $Im(\psi)$ is a normal subgroup of $Gal(\phi[x^{-s}])$.

Theorem 2.4. can be extended to the following remark.

Remark 3. $Gal(\phi[x^{-s}])$ is commutative if and only if $Gal(\phi)$ is commutative.

Theorem 2.5. can be extended to the following remark.

Remark 4. Let $\varphi : Gal(\phi[x^{-s}]) \longrightarrow Gal(\phi)$ be defined by $\varphi(f_{k_0} + f_{k_1} x^{k_1} + f_{k_2} x^{k_2} + \cdots) = f_{k_0}$. Then $Gal(\phi[x^{-s}])$ is the direct product of K and $Gal(\phi)$ if and only if $Gal(\phi)$ is commutative, where $K = ker(Gal(\phi[x^{-s}]) \rightarrow Gal(\phi))$.

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