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MODULES OVER THE GENERALIZED CENTROID OF SEMI-PRIME GAMMA RINGS

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ABSTRACT. The aim of this paper is to introduce modules over the generalized centroid of a semi-prime Γ-ring

1. Introduction

Nobusawa studied on Γ-ring for the first time in [3]. After his research, Barnes studied on this Γ-ring in [1]. But Barnes approached to Γ-ring in some different way from that of Nobusawa and he defined the concept of Γ-ring and related definitions. After these two papers were published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa, which are parallel to the results in the ring theory. The concept of "centroid of a prime Γ-ring " was defined and investigated in [4] and [6] by Öztürk and Jun. Furthermore it was shown in [5] and [6] that the extended centroid is a Γ-field and the generalized centroid of a semi-prime Γ-ring is regular Γ-ring respectively by Öztürk and Jun. The aim of this paper is to introduce modules over the generalized centroid of semi-prime Γ-ring.

2. Preliminaries

The gamma ring is defined in [3] as follows: A Γ-ring is a pair (M, Γ) where M and Γ are (additive) abelian groups for which exists a $(.,.,.) : Mx\Gamma xM \to M$ (the image of (a, α, b) being denoted by $a\alpha b$ for $a, b \in M$ and $\alpha \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:

(i) $(a + b)\,\alpha c = a\alpha c + b\alpha c$
(ii) $a\,(\alpha + \beta)\,b = a\alpha b + a\beta b$
(iii) $a\alpha\,(b + c) = a\alpha b + a\alpha c$
(iv) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)$.

Let M be a Γ-ring. A right (resp. left) ideal of M is an additive subgroup U such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and left ideal, then we say that U is an ideal. For each $a \in M$, the smallest right (resp. left)

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ideal containing a is called the principal right (resp. left) ideal generated by a and is denoted by $|a\rangle$ (resp. $\langle a|$). Also, we $\langle a\rangle$ denotes the principal two-side (right and left) ideal generated by a. An ideal Q of M is semi-prime if, for any ideal U of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is said to be semi-prime if the zero ideal is semi-prime.

Remark 1. A Γ-ring M is semi-prime if and only if all of its non-zero ideals have a non-zero multiplication, i.e., for an ideal U the equality $U\Gamma U = \langle 0\rangle$ implies $U = \langle 0\rangle$.

Theorem 1 ([2, Theorem 1]). *If Q is an ideal of a Γ-ring M, then the following conditions are equivalent.*

(i) Q is a semi-prime ideal.
(ii) If $a \in M$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$.
(iii) If $\langle a\rangle$ is a principal ideal in M such that $\langle a\rangle \Gamma \langle a\rangle \subseteq Q$, then $a \in Q$.
(iv) If U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.
(v) If U is a left ideal in M such that $V\Gamma V \subseteq Q$, then $V \subseteq Q$.

Lemma 1 ([2, Theorem 4]). *A Γ-ring M is semi-prime if and only if $a\Gamma M\Gamma a = \langle 0\rangle$ implies $a = 0$.*

Let M be a Γ-ring. For a subset U of M, $Ann_l U = \{a \in M|\ a\Gamma U = \langle 0\rangle\}$ is called the left annihilator of U. A right annihilator $Ann_r U$ can be defined similarly. An ideal of M is said to be essential if it has non-zero intersection with any non-zero ideal of M.

Let M be a semi-prime Γ-ring. We denote F a set of all ideals of M which have zero annihilator in M.

Lemma 2 ([6, Lemma 3.2]). *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then $U \in F$ if and only if U is essential.*

Let M be a semi-prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U, f)|\ f : U \to M \text{is a right } M\text{-module homomorphism for all} U \in F\}.$$

Denote a relation "$\sim$" on $\mathcal{M}$ by $(U, f) \sim (V, g) \Leftrightarrow \exists W \subseteq U \cap V$ such that $f = g$ on $W \in F$. Since the set F is closed under multiplication, it is possible to find such an ideal $W \in F$ and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We denote the equivalence class by $Cl\,(U, f) = \widehat{f}$, where $\widehat{f} := \{g : V \to M|\ (\ U, f\ ) \sim (\ V, g\ )\}$ and denote by Q_r the set of all equivalence classes. We define an addition "+" on Q_r as follows:

$$f + g := Cl\,(U, f) + Cl\,(V, g) = \ Cl(U \cap V, f + g),$$

where $f + g : U \cap V \to M$ is a right M-module homomorphism. $(Q, +)$ is an abelian group. Since $M\Gamma M \neq M$ and M is a semi-prime Γ-ring, $M\Gamma M(\neq 0)$ is an ideal of M and so is $M\beta M$ for every $\beta(\neq 0) \in \Gamma$. $0 \neq M\beta M\Gamma U \subset M\beta M \cap U$ where U is a non-zero ideal of M. Therefore $M\beta M$ is essential and so $M\beta M \in F$ for every $(\beta \neq 0) \in \Gamma$ by Lemma 2. We can take the homomorphism $1_{M\beta} : M\beta M \to M$ defined by $1_{M\beta}(m_1\beta m_2) = m_1\beta m_2$ as non-zero M-module homomorphism. Denote $\mathcal{N} := \{(M\beta M, 1_{M\beta}) | 0 \neq \beta \in \Gamma\}$ and define a relation, "$\approx$" on $\mathcal{N}$ by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma}) \Longleftrightarrow \exists W := M\alpha M\,(\in F) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on $W \in F$. "$\approx$" is an equivalence relation on $\mathcal{N}$. Denote by $Cl(M\beta M, 1_{M\beta}) = \widehat{\beta}$, the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\widehat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to "$\approx$", that is, $\widehat{\Gamma} := \left\{ \widehat{\beta} \middle| 0 \neq \beta \in \Gamma \right\}$. Define an addition "+" on $\widehat{\Gamma}$ as follows:

$$\begin{aligned} \widehat{\beta} + \widehat{\gamma} &:= Cl(M\beta M, 1_{M\beta}) + Cl(M\gamma M, 1_{M\gamma}) \\ &= Cl\,(M\beta M \cap M\gamma M, 1_{M\beta} + 1_{M\gamma}) \end{aligned}$$

for every $\beta\,(\neq 0)\,, \gamma\,(\neq 0) \in \Gamma$. Then, $(\widehat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(.,.,.) : Q_r x\Gamma x Q_r \to Q_r$, $(\widehat{f}, \widehat{\beta}, \widehat{g}) \longmapsto \widehat{\widehat{f}\widehat{\beta}\widehat{g}}$ as follows:

$$\begin{aligned} \widehat{\widehat{f}\widehat{\beta}\widehat{g}} &= Cl(U, f).Cl(M\beta M, 1_{M\beta}).Cl(V, g) \\ &= Cl\,(V\Gamma M\beta M\Gamma U, f1_{M\beta}g) \end{aligned}$$

where $V\Gamma M\beta M\Gamma U \in F$ and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \to M$, which is given by

$$(f1_{M\beta}g)\left(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i\right) = f\left(\sum g(\ v_i\ )\gamma_i m_i\beta n_i\alpha_i u_i\right)$$

is a right M-module homomorphism. Notice that the mapping $\varphi : \Gamma \to \widehat{\Gamma}$ defined by $\varphi(\ \beta\ ) = \widehat{\beta}$ for every $0 \neq \beta \in \Gamma$. The mapping φ is an isomorphism, we know that the $\widehat{\Gamma}$-ring Q_r is a Γ-ring. For a fixed element a in M and every element γ in Γ, consider a mapping $\lambda_{a\gamma} : M \to M$ defined by $\lambda_{a\gamma}(\ x\ ) = a\gamma x$ for all $x \in M$. The mapping $\lambda_{a\gamma}$ is a right M-module homomorphism and so $\lambda_{a\gamma}$ is an element of Q_r. Define a mapping $\Psi : M \to Q_r$ by $\Psi(a) = \widehat{a}$=$Cl(M, \lambda_{a\gamma})$ for all $a \in M$ and $\gamma \in \Gamma$. The mapping Ψ is a right M-module injective homomorphism and so M is a subring of Q_r and in this case, we call Q_r the right quotient Γ-ring of M. One can, of course characterize Q_l, the left quotient Γ-ring of M in a similar manner. The ring Q is called a two-sided quotient Γ-ring of M, if Q is both right and left quotient Γ-ring of M, then (see [6]). For purposes of convenience, we use q instead of $\widehat{q} \in Q$.

Definition 1 ([6, Definition 3.4]). Let M be a semi-prime Γ-ring and Q_r the quotient Γ-ring of M. Then the set

$$C_\Gamma := \{g \in Q_r |\ g\gamma f = f\gamma g \text{ for all } f \in Q_r \text{ and } \gamma \in \Gamma\}$$

is called the generalized centroid of M.

Theorem 2 ([6, Theorem 3.5]). *Let M be a semi-prime Γ-ring and Q_r the quotient Γ-ring of M. Then the Γ-ring Q_r satisfies the following properties:*

(i) *For any element* $q \in Q_r$, *there exists an ideal* $U_q \in F$ *which is an essential ideal with a right* M*-module homomorphism* $q : U \to M$, *such that* $q(U_q) \subseteq M$ (*or* $q\gamma U_q \subseteq M$ *for all* $\gamma \in \Gamma$).

(ii) *If* $q \in Q_r$ *and* $q(U_q) = \langle 0 \rangle$ *for a certain* $U_q \in F$ (*or* $q\gamma U_q = \langle 0 \rangle$ *for a certain* $U_q \in F$ *and for all* $\gamma \in F$), *then* $q = 0$.

(iii) *If* $U \in F$ *and* $\Psi : U \to M$ *is a right* M*-module homomorphism, then there exists an element* $q \in Q_r$ *such that* $\Psi(u) = q(u)$ *for all* $u \in U$ (*or* $\Psi(u) = q\gamma u$ *for all* $u \in U$ *and* $\gamma \in \Gamma$).

(iv) *Let* W *be a submodule* (*an* (M, M)*-sub-bimodule*) *in* Q_r *and* $\Psi : W \to Q_r$ *a right* M*-module homomorphism. If* W *contains the ideal* U *of the* Γ*-ring* M *such that* $\Psi(U) \subseteq M$ *and* $AnnU = Ann_r W$, *then there is an element* $q \in Q_r$ *such that* $\Psi(b) = q(b)$ *for any* $b \in W$ (*or* $\Psi(b) = q\gamma b$ *for any* $b \in W$ *and* $\gamma \in \Gamma$) *and* $q(a) = 0$ *for any* $a \in Ann_r W$ (*or* $q\gamma a = 0$ *for any* $a \in Ann_r W$ *and* $\gamma \in \Gamma$).

Definition 2 ([6, Definition 3.7]). A Γ-ring M is called regular if for any element $x \in M$, there exists an element $x' \in M$ such that $x'\beta x\gamma x = x$, where $\gamma, \beta \in \Gamma$.

Theorem 3 ([6, Theorem 3.8]). *Let* M *be a semi prime* Γ*-ring and* C_Γ *the generalized centroid of* M. *Then* C_Γ *is a regular* Γ*-ring.*

In the set E of all idempotents of C_Γ, the relation $\leq$ defined by $e_1 \leq e_2 \Leftrightarrow e_2\gamma e_1 = e_1$, $\gamma \in \Gamma$ is a partial order.

Definition 3 ([6, Definition 3.10]). Let M be a semi-prime Γ-ring, Q_r the quotient Γ-ring of M and let $S \subseteq Q_r$. The least of idempotent elements $e(S) = e \in C_\Gamma$ such that $e\gamma s = s$ for all $s \in S$, $\gamma \in \Gamma$ is called the support of the set S.

Lemma 3 ([6, Lemma 3.11]). *Let* M *be a semi-prime* Γ*-ring,* Q_r *the quotient* Γ*-ring of* M *and* $S \subseteq Q_r$. *If* S *has a support* $e(S) = e \in C_\Gamma$, *then the equality* $q\gamma M\Gamma S = 0$ *for an element* $q \in Q$ ($S\Gamma M\gamma q = 0$) *is equivalent to* $q\gamma e(S) = 0$.

Let A be module in Γ-ring M. Let $\Pi : A_2 \to A_1$ be an epimorphism where A_1 and A_2 are any modules in Γ-ring M. If there exists a module homomorphism $\Psi : A \to A_2$ for module homomorphism $\varphi : A_1 \to A$ such that $\Pi\Psi = \varphi$, then A is called projective module.

This property is equivalent to the fact that for any epimorphism $\theta : A' \to A$ there exists a decomposition $A' = \operatorname{Ker}\theta \oplus A''$. The direct sum of modules is projective if and only if all summands are projective.

Let A be module in Γ-ring M. Let $\Pi : A_1 \to A_2$ be a monomorphism, where A_1 and A_2 are any modules in Γ-ring M. If there exists a module homomorphism $\Psi : A_2 \to A$ for module homomorphism $\varphi : A_1 \to A$ such that $\Psi\Pi = \varphi$, then A is called injective module.

This property is equivalent to fact that for any monomorphism $\theta : A \to A'$, there exists a decomposition $A' = \operatorname{Im}\theta \oplus A''$. In this case A is extracted a direct summand of any module containing it. The direct product of modules is injective if and only if all the cofactors are injective.

Let M be Γ-ring. If M is injective over itself as (left) module, then Γ-ring M is called (left) self-injective.

3. Main results

Definition 4. Let M be a semi-prime Γ-ring and C_Γ be its generalized centroid.

(i) Let I be a directed partially ordered set, A be C_Γ-module and $a \in A$. If there exists a direct set of idempotents $\{e_i \mid i \in I\}$ such that $e_i \leq e_j$ for $i \leq j$, $i, j \in I$, $\sup\{e_i\} = 1$ and $e_i \gamma a_i = e_i \gamma a$, $\forall i \in I, \forall \gamma \in \Gamma$, then a is called a limit of the set $\{a_i \in A \mid i \in I\}$. Denote by $\lim_I a_i = a$.

(ii) Let T be subset of A. If a is limit of the set $\{a_i \in T \mid i \in I\}$ and $a \in T$, then set T is called closed.

Remark 2. (i) The closure of a set is the least closed set containing the given one. Therefore, the operation of closure determines a certain topology on the C_Γ-module A.

(ii) Let A_1 and A_2 be modules on C_Γ and $\varphi : A_1 \to A_2$ be a function. If $\lim_I a_i = a$ implies $\lim_I \varphi(a_i) = \varphi(a)$, then φ is called a quite continuous function. Indeed, in this case preimage of closed set is closed and hence φ is a continuous function. If φ preserves the operator of multiplication by the idempotents, i.e., $\varphi(a\gamma e) = \varphi(a)\gamma e$, $\gamma \in \Gamma$ and $e \in C_\Gamma$, then φ is a quite continuous function.

Proposition 1. *Let M be semi-prime Γ-ring and Q_r be quotient Γ-ring M. If $\lim_I r_i = r$, $\lim_I s_i = s$ where $r, s, r_i, s_i \in Q_r$, then $\lim_I (r_i \pm s_i) = r \pm s$ and $\lim_I (r_i \alpha s_i) = r\alpha s$ for all $\alpha \in \Gamma$.*

Proof. Let $\lim_I r_i = r$, $\lim_I s_i = s$. Let $\{e_i \gamma f_i \mid i \in I, \gamma \in \Gamma\}$ be directed set of idempotents for sets $\{r_i \pm s_i \mid r_i, s_i \in Q_r\}$ and $\{r_i \alpha s_i \mid r_i, s_i \in Q_r, \alpha \in \Gamma\}$, where $\{e_i\}$ and $\{f_i\}$ are directed sets of idempotents. If $e\gamma e_i \beta f_i = 0$ for all $\gamma, \beta \in \Gamma$, then for an arbitrary $j \in I$, $k \geq i, j$, we get, for $\gamma, \beta, \alpha, \alpha', \beta' \in \Gamma$,

$$\begin{aligned}(e\gamma e_i)\beta f_j &= e\gamma(e_i\alpha e_k)\beta\left(f_j\alpha' e_k\right) = e\gamma(e_i\alpha e_k)\beta\left(f_j\beta' f_k\right)\alpha' e\\ &= \left(e\alpha e_k\beta' f_k\right)\gamma e_i\beta f_j = 0.\end{aligned}$$

Since $\sup\{f_i\} = 1$, $e\gamma e_i = 0$ and $\sup\{e_i\} = 1$, $e = 0$. Further on,

$$\begin{aligned}(e_i\gamma f_i)\beta(r \pm s) &= (e_i\gamma f_i)\beta r \pm (e_i\gamma f_i)\beta s = f_i\beta(e_i\gamma r_i) \pm e_i\gamma(f_i\beta s_i)\\ &= e_i\gamma f_i\beta(r_i \pm s_i).\end{aligned}$$

Consequently from Definition 4(i), we have $\lim_I (r_i \pm s_i) = r \pm s$. Accordingly,

$$\begin{aligned}e_i\gamma f_i\beta(r\alpha s) &= f_i\gamma(e_i\beta r)\alpha s = f_i\gamma(e_i\beta r_i)\alpha s = (e_i\beta r_i)\gamma(f_i\alpha s)\\ &= (e_i\beta r_i)\gamma(f_i\alpha s_i) = e_i\gamma f_i\beta(r_i\alpha s_i),\end{aligned}$$

i.e., $\lim_{I}(r_i\alpha s_i) = r\alpha s$. □

Definition 5. If submodule of A is a union of all the elements having essential annihilators in the C_Γ, then this module is called a singular submodule of A. And, a module is called non-singular if its singular submodule equals zero.

Theorem 4. *The following conditions on the C_Γ-module A are equivalent:*

- (a) *Any directed set of elements of the module A has not more than one limit.*
- (b) *Any one-element set is closed.*
- (c) *The zero submodule is closed.*
- (d) *A is a non-singular module.*

Proof. $(a) \Rightarrow (b)$ Let a be an element of A and $\{a\}$ be one-element set. The set $\{a\}$ is directed. Since $e_i\gamma a_i = e_i\gamma a$, $\gamma \in \Gamma$, where $\{e_i|\ i \in I\}$ is directed set of idempotents and by the Definition (4), $\lim_{i\in I} a_i = a$. Since $a \in \{a\}$, the set $\{a\}$ is closed.

$(b) \Rightarrow (c)$ The submodule $\{0_A\}$ is closed, since $\{0_A\}$ is one-element set.

$(c) \Rightarrow (d)$ Let us prove that closure of a zero submodule is same with the singular ideal.

If U is an essential ideal in ring C_Γ, then $e(U) = 1$. Let U be an essential and $e(U) \neq 1$, i.e., $f = 1 - e(U) \neq 0$. In this case, $U \cap C_\Gamma\gamma f = (0), \gamma \in \Gamma$, since $C_\Gamma\gamma f \neq (0)$ is an ideal of C_Γ and U is an essential ideal. If $U \cap C_\Gamma\gamma f \neq (0)$, then there exists a $x \in U \cap C_\Gamma\gamma f$ such that $x \in U$ and $x \in C_\Gamma\gamma f$. Hence $x = c\gamma f$, $c \in C_\Gamma$. $x = c\gamma f = c\gamma(1 - e(U)) = c - c\gamma e(U)$. Left multiplying by e, we have

$$\begin{aligned} e\beta x &= e\beta c - e\beta\left(c\gamma e\left(U\right)\right) = e\beta c - \left(e\beta c\right)\gamma e \\ &= e\beta c - \left(e\gamma e\right)\beta c = e\beta c - e\beta c = 0, \end{aligned}$$

i.e., $e = 0$. This is a contradiction, since $e \neq 0$. Thus $U \cap C_\Gamma\gamma f = (0)$ and consequently $f = 0$. In this case our acceptance is wrong, i.e., $e(U) = 1$. Inversely, if $e(U) = 1$ and $U \cap V = (0)$, then choosing an arbitrary element f in V, we get $f\gamma U \subseteq U \cap V = (0)$, i.e., $f = e(U)f = 0$ and hence $V = (0)$. Thus U is an essential ideal.

Let $a \in Z(A)$ which is center of A. In this case $Ann_{C_\Gamma}\langle a\rangle$ is an essential ideal. Let J be a set of all idempotents of this ideal. It is a directed set. In this case, $i\alpha a = i\alpha 0$, for all $i \in J$ and $\alpha \in \Gamma$, so that $\lim_{i\in J} a_i = a$, where $a_i = 0$. Consequently, $Z(A) \subseteq \overline{(0)}$. Let us now show that $Z(A)$ is closed submodule. Let $\lim_{i\in J} a_i = a$, where $a_i \in Z(A)$ and let $a \notin Z(A)$. Let $f = 1 - (Ann_{C_\Gamma}\langle a\rangle) \neq 0$. $\sup_{i\in J} e_i = 1$ from definition of limit for set of idempotents $\{e_i|\ i \in J\}$. There is an element $j \in J$, such that $f\gamma e_j \neq 0$, $\gamma \in \Gamma$. Since $a_j \in Z(A)$ then $\sup(Ann_{C_\Gamma}\langle a_j\rangle) = 1$ and there exists an idempotent $u_j \in Ann_{C_\Gamma}\langle a_j\rangle$ such that $u_j\beta f\gamma e_j \neq 0$. Therefore $u_j\beta f\gamma e_j\alpha a = u_j\beta f\gamma e_j\alpha a_j = f\gamma e_j\alpha(u_j\beta a_j) = 0$. It means that $u_j\beta f\gamma e_j \in Ann_{C_\Gamma}\langle a_j\rangle$. Since idempotent f

annihilates $Ann_{C_\Gamma}\langle a_j\rangle$, $f\alpha(u_j\beta f\gamma e_j) = u_j\beta(f\alpha f)\gamma e_j = u_j\beta f\gamma e_j = 0$. This is a contradiction. Thus $a \in Z(A)$.
$(d) \Rightarrow (a)$ Let us assume that a certain directed set of idempotents $\{a_i\}$ has two limits, $a^{(1)} \neq a^{(2)}$. Let sets of idempotents $\{e_i\}$, $\{f_i\}$. In this case

$$\begin{aligned}
&(e_i\beta f_i)\gamma(a^{(1)} - a^{(2)}) \\
&= (e_i\beta f_i)\gamma a^{(1)} - (e_i\beta f_i)\gamma a^{(2)} = f_i\beta(e_i\gamma a^{(1)}) - e_i\beta(f_i\gamma a^{(2)}) \\
&= f_i\beta(e_i\gamma a_i) - e_i\beta(f_i\gamma a_i) = 0.
\end{aligned}$$

Since $\sup_i\{e_i\gamma f_i\} = 1$ from Proposition 1, $a^{(1)} - a^{(2)} = 0$, i.e., $a^{(1)} = a^{(2)}$. Thus set $\{a_i|\ i \in I\}$ has one limit. □

Remark 3. Let Q be Γ-ring and $E(Q,\Gamma)$ be set of endomorphism of additive group of Q. It can be easily show that $E(Q,\Gamma)$ is a Γ-ring. The center of Q is right module over C_Γ.

Theorem 5. *The modules Q_r and $E(Q_r,\Gamma)$ are non-singular.*

Proof. Let us make use of condition (a) of Theorem 4. Let r, s be the limits of set $\{r_i|\ i \in I\}$ and let $\{e_i\}$ and $\{f_i\}$ be sets of idempotents for limits r, s, respectively. In this case, $e_i\beta(f_i\gamma r) = f_i\beta(e_i\gamma r_i) = e_i\beta(f_i\gamma r_i) = e_i\beta(f_i\gamma s)$. Thus $(e_i\beta f_i)\gamma(r - s) = 0$. If $r, s \in Q_r$, then $r - s = \sup\{e_i\beta f_i\}\gamma(r-s) = 0$, i.e., $r = s$. If $r, s \in E(Q_r,\Gamma)$, then for any $x \in Q_r$, the equalities $e_i\beta f_i\gamma(r-s)(x) = 0$ yield $r(x) = s(x)$. The theorem is proved. □

Theorem 6. *If T is a subring of Q_r, then its closure $\widehat{T}$ is also a subring.*

Proof. Let us denote by $k(T)$ a set of all the limiting points of T. Let j is a limit point and $T_1 = T$, $T_{i+1} = k(T_i)$, $T_j = \bigcup_{i\leq j} T_i$.

In this case the union of all T_i is a closed set and it equals $\widehat{T}$. Therefore, according to the transfinite induction, it suffices to prove that $k(T)$ is a subring. Let $\lim_I r_i = r$, $\lim_J s_j = s$ for $r_i, s_j \in T$ and $\{e_i\}, \{e_j\}$ be sets of idempotents for these limits respectively. Let us consider the set $\{e_i\gamma e_j|\ (i,j) \in IxJ, \gamma \in \Gamma\}$. Let us assume that $(i,j) \leq (i_1, j_1) \Leftrightarrow i \leq i_1, j \leq j_1$.

In this case, we get $\lim_{IxJ}(r_i \pm s_j) = r \pm s$, $\lim_{IxJ}(r_i\gamma s_j) = r\gamma s \in k(T)$. Thus $k(T)$ is a subring. □

Theorem 7. *Let U be an ideal of the subring $T \subseteq Q$. Then the closure $\widehat{U}$ is an ideal of $\widehat{T}$.*

Proof. Similar in the Theorem 6, it suffices to prove that $k(U)$ is an ideal of $k(T)$. Let $\lim_I r_i = r$, $\lim_J s_j = s$ for $r_i \in T, s_j \in U$ and $\{e_i\}, \{e_j\}$ be sets of idempotents for these limits respectively. Let us consider the set

$$\{e_i\gamma e_j|\ (i,j) \in IxJ, \gamma \in \Gamma\}.$$

Let us assume that $(i,j) \leq (i_1, j_1) \Leftrightarrow i \leq i_1, j \leq j_1$.

In this case, we get $\lim\limits_{IxJ}(r_i \pm s_j) = r \pm s \in k(U)$, $\lim\limits_{IxJ}(r_i\gamma s_j) = r\gamma s \in k(U)$ and $s\gamma r \in k(U)$, i.e., $k(U)$ is an ideal of $k(T)$. □

Definition 6. Let A be a C_Γ-module and $\{a_i|\ i \in I\}$ be any set of elements of A. The module A is called complete if set $\{a_i|\ i \in I\}$, for which there exists a directed set of idempotents $\{e_i\}$, such that $\sup\{e_i\} = 1$ and at $i \geq j$ the relations $e_j\gamma a_i = e_j\gamma a_j$, $\gamma \in \Gamma$, $e_i \geq e_j$ are valid, has a limit.

Theorem 8. *The modules Q_r and $E(Q_r, \Gamma)$ are complete.*

Proof. Let us assume that $r_i \in Q_r$ and $\{e_i\}$ be a set of idempotents such that $e_j\gamma r_i = e_j\gamma r_j$, $e_i \geq e_j$ at $i \geq j$ and $\sup\{e_i\} = 1$. Let

$$N_i = \{x \in M|\ e_i\gamma x \in M,\ e_i\alpha r_i\beta x \in M,\ \gamma, \alpha, \beta \in \Gamma\}.$$

Then N_i is a right ideal of the M. Indeed, $e_i\gamma x$, $e_i\alpha r_i\beta x \in M$ and $e_i\gamma y$, $e_i\alpha r_i\beta y \in M$ for all $x, y \in N_i$. In this case $e_i\gamma x - e_i\gamma y = e_i\gamma(x-y) \in M$ and $e_i\alpha r_i\beta x - e_i\alpha r_i\beta y = e_i\alpha r_i\beta(x-y) \in M$, i.e., $x - y \in N_i$. If $e_i\gamma x$ and $e_i\alpha r_i\beta x \in M$, for all $x \in N_i$, then $(e_i\gamma x)\alpha' m = e_i\gamma\left(x\alpha' m\right) \in M$ and $(e_i\alpha r_i\beta x)\alpha' m = e_i\alpha r_i\beta\left(x\alpha' m\right)$, for all $m \in M$, i.e., $x\alpha' m \in N_i$. Thus N_i is a right ideal of M.

Since $r_i : U \to M$, $x \mapsto r_i(x) = r_i\gamma x$, $\gamma \in \Gamma$ is a right M-module homomorphism. Hence $e_i\gamma x$ and $e_i\alpha r_i\beta x \in M$. Thus there exists an $U_i \in F(M)$ such that $U_i \subseteq N_i$. The union $N = \bigcup\limits_i e_i\gamma N_i$, $\gamma \in \Gamma$ is also a right ideal. Indeed, if $e_i\gamma a_i$, $e_j\gamma a_j \in N$ and let us find an element $k \geq i, j$, then

$$\begin{aligned} &e_k\alpha r_k\beta(e_i\gamma a_i + e_j\gamma a_j) \\ &= e_k\alpha r_k\beta e_i\gamma a_i + e_k\alpha r_k\beta e_j\gamma a_j \\ &= e_k\beta e_i\alpha r_k\gamma a_i + e_k\beta e_j\alpha r_k\gamma a_j \\ &= e_i\alpha r_k\gamma a_i + e_j\alpha r_k\gamma a_j = e_i\alpha r_i\gamma a_i + e_j\alpha r_j\gamma a_j \in M. \end{aligned}$$

Analogously, $e_k\alpha(e_i\gamma a_i + e_j\gamma a_j) = e_k\alpha e_i\gamma a_i + e_k\alpha e_j\gamma a_j = e_i\gamma a_i + e_j\gamma a_j \in M$. Thus $e_i\gamma a_i + e_j\gamma a_j \in e_k\gamma N_k$. Since $e_k\alpha r_k\beta(e_i\gamma a_i)\alpha' m = e_k\alpha r_k\beta e_i\gamma\left(a_i\alpha' m\right) = e_i\alpha r_k\gamma\left(a_i\alpha' m\right) = e_i\alpha r_i\gamma\left(a_i\alpha' m\right) \in M$ for any $m \in M$, $(e_i\gamma a_i)\alpha' m \in e_k\gamma N_k$. In this case N is an ideal. As $e_k\gamma N_k \supseteq e_i\gamma U_i$, then $N \supseteq \sum e_i\gamma U_i = U$ is an ideal of M. Besides, the annihilator of U equals zero. Indeed, if $x\beta U = 0$, $\beta \in \Gamma$, then $x\beta e_i\gamma U_i = 0$, i.e., $x\beta e_i = 0$, since U_i is an essential ideal and hence $x = 0$, since $\sup\{e_i\} = 1$. Let us consider mapping $\xi : N \to M$ defined by $\xi(e_i\gamma a_i) = r_i\beta e_i\gamma a_i$. The mapping ξ is a right M-module homomorphism. If $e_i\gamma a_i = e_j\gamma a_j$ and $k \geq i, j$ then $r_i\beta e_i\gamma a_i = r_k\beta e_i\gamma a_i = r_k\beta e_j\gamma a_j = r_j\beta e_j\gamma a_j$, i.e., ξ is well-defined. Since $\xi \in Q$, then from Theorem 2 there is an element $r \in Q$, such that $r\beta e_i\gamma a_i = r_i\beta e_i\gamma a_i$. In this case $(r\beta e_i - r_i\beta e_i)\gamma a_i = 0$, i.e., $(r\beta e_i - r_i\beta e_i)\gamma U_i = 0$. Therefore $r\beta e_i - r_i\beta e_i = 0$, since U_i is an essential ideal. Thus $r\beta e_i = r_i\beta e_i$. Hence $\lim\limits_I r_i = r$.

If $r_i \in E(Q,\Gamma)$, then for any $x \in Q_r$ there exists a limit of $r_i(x)$ and we can set $r(x) = r_i(x)$. In this case $r = \lim_I r_i$. The theorem is proved. □

Theorem 9. *The subrings Q and C_Γ are closed in Q. Hence they are complete modules over C_Γ.*

Proof. Let $r = \lim_I r_i$, $r_i \in Q$. From Theorem 2 there exists an $U_i \in F$ such that $U_i\beta r_i\alpha e_i \subseteq M$, $U_i\alpha e_i \subseteq M$. Then $V = \sum U_i\alpha e_i \in F$, in which case $U_i\alpha e_i\beta r = U_i\alpha e_i\beta r_i$ and hence $r\gamma V \subseteq M$, i.e., $r \in Q$. Therefore Q is closed. The submodule C_Γ is closed as an intersection of all the kernels of quite continuous mappings $ad\ a : x \mapsto a\gamma x - x\gamma a$, $\gamma \in \Gamma$. Indeed, let $r_i \in C_\Gamma$ and $\lim_I r_i = r$, $ad\ a(r) = \lim_I ad\ a(r_i)$, since $e_i\beta(ad\ a(r)) = e_i\beta(a\gamma r - r\gamma a) = e_i\beta(ad\ a(r_i))$. The theorem is proved. □

Theorem 10. *Any complete non-singular module over C_Γ is injective and, vice versa, any injective module over C_Γ is complete.*

Proof. Let T be a complete non-singular submodule of A. Let us show that T is extracted from A by a direct summand.

Let us consider a set of all submodules having zero intersection with T in A. This set is directed by inclusion relation. In this case, according Zorn Lemma, this set has at least one maximal element A'. Let us show that A' is closed submodule in A. Let $\lim_I a_i = a$, $a_i \in A'$ and let $\{e_i \mid i \in I\}$ be set of idempotents. If $a \notin A'$ then $\left(A' + C_\Gamma\gamma a\right) \cap T \neq (0)$. Let $t = a' + c\gamma a \neq 0$, $\gamma \in \Gamma$ for $a' \in A'$ and $c \in C_\Gamma$. For any $i \in I$, $e_i\alpha t = e_i\alpha a' + e_i\alpha c\gamma a \in A' \cap T$ and $A' \cap T = (0)$, i.e., due to non-singularity of the module T the element t is equal zero. This is a contradiction. $a \in A'$.

Let a be an arbitrary element of A. Let us show that $a \in A' + T$. Let us assume that $a \in A'$. Let us consider a set I of idempotents $i \in C_\Gamma$, such that $i\gamma a \in A' + T$, $\gamma \in \Gamma$. This set is directed and if $i_1, i_2 \in I$, then $= i_1 + i_2 - i_1 i_2 \in I$ such that $(i_1 \vee i_2)\, a = i_1 a + (1 - i_1)\, i_2 a$.

Let us show that $\sup I = 1$. Let, on the contrary, $f = 1 - \sup I \neq 0$. If $f \in I$, then $f^2 = f$. $f(1-f) = f.\sup I = 0$. If $f \sup I = 0$, then $f = 0$. This is a contradiction. It must be $f \notin I$. Thus $f\gamma a \notin A' + T$ and since $f\gamma a \notin A'$, $\left(A' + C_\Gamma\alpha f\gamma a\right) \cap T \neq (0)$. Let $0 \neq a' + c\alpha f\gamma a \in T$, for $c \in C_\Gamma$ and $a' \in A'$. In this case $0 \neq c\alpha f\gamma a$ is element of $A' + T$. Since C_Γ is regular ring, there exists an element c' such that $e_1 = c'\beta c$ is an idempotent and $c\alpha' e_1 = c$. Therefore $e_1\beta' f \neq 0$. On the other hand since $e_1\beta' f\gamma a = \left(c'\beta c\right)\beta' f\gamma a$ in $A' + T$, $e_1\beta f \in I$. However $I\gamma' f = I\gamma'(1 - \sup I) = I - I\gamma' \sup I = I - I = 0$ and so that $e_1\beta' f = \left(e_1\beta' f\right)^2 = \left(e_1\beta' f\right)\gamma'\left(e_1\beta' f\right) = \left(e_1\beta' f\right)\gamma' f = 0$. This is a contradiction. It must be $\sup I = 1$. Let $i\gamma a = a_i + t_i$ where $a_i \in A'$, $t_i \in T$.

If $j \leq i$, then $j\beta a = j\beta(i\gamma a) = j\beta a_i + j\beta t_i$. As the sum $A' + T$, is direct, $a_j = j\beta a_i$, $t_j = j\beta t_i$. Since the module T is complete, there exists a limit $\lim_{i \in I} t_i = t$. Since $i\gamma a = a_i + t_i$, $i\gamma a - t_i = a_i \in A'$. Since $\sup I = 1$, $a - t = \lim_I a_i$ and A', i.e., $a \in A' + T$.

Inversely, let A be injective C_Γ-module. $C_\Gamma \alpha b \cong C_\Gamma$ where $C_\Gamma \alpha b$ is a free one-generated module. Indeed, the mapping $\varphi : C_\Gamma \to C_\Gamma \alpha b$, $x \mapsto x\alpha b$ is an isomorphism. If $x = y$, then $x\alpha b = y\alpha b$, i.e., $\varphi(x) = \varphi(y)$. $\varphi(x + y) = (x + y)\alpha b = x\alpha b + y\alpha b = \varphi(x) + \varphi(y)$. $\varphi(x\beta c) = (x\beta c)\alpha b = (x\alpha b)\beta c = \varphi(x)\beta c$, i.e., φ is a module homomorphism. If $x\alpha y = y\alpha b$, then $(x - y)\alpha b = 0$. Since C_Γ is a regular ring, $x - y = 0$, i.e., $x = y$. There exists an element x in C_Γ such that $\varphi(x) = x\alpha b$ for all $x\alpha b \in C_\Gamma \alpha b$. Thus φ is a bijection. Let us consider direct sum $A_1 = A \oplus C_\Gamma \alpha b$. Let $\{a_i\}$ be set of elements of A and $\{e_i\}$ be directed set of idempotents such that $\sup\{e_i\} = 1$ and $e_i \beta a_j = e_i \beta a_i$ for $j \geq i$, for all $\beta \in \Gamma$. In A, let us consider a submodule N by the elements $e_i\gamma a_i \oplus e_i \gamma b$. We get $N \cap A = (0)$. Indeed, if $a = \sum_i (c_i\beta e_i\gamma a_i \oplus c_i\beta e_i\gamma b) \in A$, then $\left(\sum_i c_i\beta e_i\right)\gamma b = 0$. Due to regularity of the C_Γ, $\sum_i c_i\beta e_i = 0$. Let j be the upper boundary of elements i included in the latter sum. Since $0 = \left(\sum_i c_i\beta e_i\right)\gamma a_j = \sum_i c_i\beta e_i\gamma a_j = \sum_i c_i\beta e_i\gamma a_i$, $a = 0$. Thus the natural homomorphism $\varphi : A \to A_1/N$ is an embedding. We can write the relations, $e_i\gamma\varphi(a_i) + e_i\gamma b = 0$, for all i. Indeed, $\varphi(e_i\gamma a_i) = e_i\gamma\varphi(a_i) \in A_1/N$. Therefore $e_i\gamma\varphi(a_i) = a_1 + N$, $a_1 \in A_1$. Since $e_i\gamma\varphi(a_i) = e_i\gamma a_i \oplus e_i\gamma b + N$, $e_i\gamma\varphi(a_i) + e_i\gamma b = 0$.

Let us apply the definition of injectivity: There exists a homomorphism $\Psi : A_1/N \to A$ such that $\varphi\Psi = 1$. Let $a = -\Psi(b)$. Thus

$$0 = \Psi(e_i\gamma\varphi(a_i) + e_i\gamma b) = e_i\gamma\Psi(\varphi(a_i)) + e_i\gamma\Psi(b) = e_i\gamma a_i - e_i\gamma a,$$

i.e., $e_i\gamma a_i - e_i\gamma a = 0$. Therefore $e_i\gamma a_i = e_i\gamma a$, i.e., $\lim_I a_i = a$. □

Corollary 1. *A generalized centroid of a semi-prime* Γ*-ring is a regular self-injective* Γ*-ring.*

Remark 4. Any semi-prime, self-injective, commutative Γ-ring M is same with its generalized centroid. Let Q be a Martindale Γ-ring of quotients. Let us consider Q as a right M-module. Since $M \subseteq Q$, we can write direct decomposition $Q = M \oplus A$. If $a \in A$, then, by the definition of a ring of quotients, there is an ideal $U \in F$, such that $a\gamma U \subseteq M$, $\gamma \in \Gamma$. On the other hand, $a\gamma U = (0)$. Since U is an essential, $a = 0$ and hence, $A = (0)$. Thus $M = Q$.

Theorem 11. *Let* E *be set of all idempotents in* C_Γ *and* $Q(E)$ *the quotient* Γ*-ring of* E*. Then* $Q(E) = E$*.*

Proof. Let U be essential ideal of E. Therefore $\sup U = 1$ and if $q\gamma U \subseteq E$, then $q \in Q(E)$. Let us assume that $e_u = q\gamma u$ and U be directed set of idempotents.

Since C_Γ is complete, there is limit $e = \lim_U e_u$. Since e is an idempotent and $(e-q)\gamma u = e_u - q\gamma u = 0$, $(e-q)\gamma U = (0)$ and since U is an essential ideal, $e - q = 0$. Hence, $e = q \in E$. In this case $Q(E) = E$. □

References

[1] W. E. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), no. 3, 411–422.

[2] S. Kyuno, *On prime gamma rings*, Pacific J. Math. **75** (1978), no. 1, 185–190.

[3] N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964), 81–89.

[4] M. A. Öztürk and Y. B. Jun, *On the centroid of the prime gamma rings*, Commun. Korean Math. Soc. **15** (2000), no. 3, 469–479.

[5] ———, *On the centroid of the prime gamma rings. II.*, Turkish J. Math. **25** (2001), no. 3, 367–377.

[6] ———, *Regularity of the generalized centroid of semi-prime gamma rings*, Commun. Korean Math. Soc. **19** (2004), no. 2, 233–242.

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