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Estimation of the Change Point in Monitoring the Mean of Autocorrelated Processes*

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Abstract

Knowing the time of the process change could lead to quicker identification of the responsible special cause and less process down time, and it could help to reduce the probability of incorrectly identifying the special cause. In this paper, we propose the maximum likelihood estimator (MLE) for the process change point when a control chart is used in monitoring the mean of a process in which the observations can be modeled as an AR(1) process plus an additional random error. The performance of the proposed MLE is compared to the performance of the built-in estimator when they are used in EWMA charts based on the residuals. The results show that the proposed MLE provides good performance in terms of both accuracy and precision of the estimator.

Keywords: Process change point; autocorrelated process; exponentially weighted moving average chart; residual; maximum likelihood estimator.

1. Introduction

Control charts are widely used to monitor processes for the purpose of detecting the occurrence of special causes which produce changes in the process. When a control chart signals that a special cause is present, process engineers must initiate a search for and an identification of the special cause. However, the signal

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from a control chart does not provide process engineers with what caused the process to change or when the process change actually occurred. Knowing the time of the process change could lead to identify the special cause more quickly, and to take the appropriate actions immediately to improve quality. Consequently, estimating the time of the process change would be useful to process engineers.

The cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) charts provide built-in change point estimators from the behavior of the past plots on the control chart. Nishina (1992) compared the performance of built-in estimators when used in a CUSUM chart, an EWMA chart, and a moving average (MA) chart. Samuel *et al.* (1998) considered the use of the MLE for the change point in the normal process mean, and investigated its performance when used after a signal from an $\bar{X}$ chart. Pignatiello and Samuel (2001) considered the use of the MLE for the change point instead of the built-in estimator when either a CUSUM chart or an EWMA chart issued a signal. They concluded that the performance of the MLE appears to be better than the built-in estimators over the range of magnitudes of change considered. Lee and Park (2006) proposed the MLE for the process change point when a control chart with the fixed sampling rate (FSR) scheme or the variable sampling rate (VSR) scheme is used in monitoring a process to detect changes in the process mean and/or variance of a normal quality variable.

A fundamental assumption in most traditional applications of control charts is that the observations from the process are statistically independent. However, the independence assumption is often violated in some manufacturing processes, such as the chemical industry, because the dynamics of the process produce autocorrelation in the process observations. The presence of autocorrelation can have a large impact on traditional control charts developed using the independence assumption. It can result in an average false alarm rate much higher than expected or desired.

Two general approaches to dealing with autocorrelation in process monitoring have been considered. The first approach uses traditional control charts but adjusts the control limits and the techniques for estimating process parameters to account for the autocorrelation. The second approach fits a time series model to the process observations and the forecast errors or residuals from this model are used in traditional control charts. See Lu and Reynolds (1999) for more details.

In this paper, we propose the MLE for the process change point in monitoring the process mean when there is autocorrelation. We consider a first-order autoregressive process (an AR(1) process) with an additional random error as an autocorrelated process model, and consider EWMA charts based on the residuals for monitoring autocorrelated processes. The performance of the MLE is investigated by the extensive simulation, and compared to the performance of the built-in estimator.

2. The Autocorrelated Process Model

Let X_t be an observation taken from the process at time t. For the AR(1) process model with a random error, X_t can be written as

$$X_t = \mu_t + \epsilon_t \ , \ t = 1, 2, \ldots . \tag{2.1}$$

The ϵ_t's are independent normal random errors with mean 0 and variance σ_ϵ^2, and μ_t is an AR(1) process of the form

$$\mu_t = (1-\phi)\xi + \phi\mu_{t-1} + \alpha_t \ , \ t = 1, 2, \ldots , \tag{2.2}$$

where ξ is the process mean and ϕ is the AR parameter satisfying $|\phi| < 1$. The α_t's are assumed to be independent normal variables with mean 0 and variance σ_α^2 and independent of the ϵ_t's. The assumption that the starting value μ_0 follows a normal distribution with mean ξ and variance $\sigma_\mu^2 = \sigma_\alpha^2/(1-\phi^2)$ implies that the distribution of X_t is constant with mean ξ and variance $\sigma_X^2 = \sigma_\mu^2 + \sigma_\epsilon^2$ for $t = 1, 2, \ldots$. Define ψ to be the proportion of the total process variance that is due to the AR(1) process so that $\psi = \sigma_\mu^2/\sigma_X^2$. Then the proportion of the variance due to ϵ_t is $1-\psi$, and the correlation between X_t and X_{t+1} is $\phi\psi$.

The AR(1) process with an additional random error in equation (2.1) and (2.2) is equivalent to an ARMA(1,1) process (Box *et al.*, 1994) that can be written as

$$(1-\phi B)X_t = (1-\phi)\xi + (1-\theta B)\gamma_t, \tag{2.3}$$

where the γ_t's are independent normal variable with mean 0 and variance σ_γ^2, θ is the moving average (MA) parameter, ϕ and ξ are the same in equation (2.2), and B is a backshift operator such that $BX_t = X_{t-1}$. Relatively simple equations are available for expressing the parameters ϕ, θ, ξ, and σ_γ^2 in the ARMA(1,1) model in terms of the parameters ϕ, ξ, σ_α^2, and σ_ϵ^2 in the AR(1) plus random error model, and vice verse (see, *e.g.*, Reynolds *et al.*, 1996; Lu and Reynolds, 1999).

The AR(1) plus random error model has been used in a number of other papers as a model for autocorrelation observations (see, *e.g.*, Padgett *et al.*, 1992;

MacGreger and Harris, 1993). In some applications it may be possible to model X_t as a simple AR(1) process, and this model can be treated as a special case of the model with $\sigma_\epsilon^2 = 0$. This model used here can be extended to the situation in which a sample of $n > 1$ observations is taken at each sample time (see, *e.g.*, Reynolds *et al.*, 1996).

3. The EWMA Chart Based on the Residuals

We consider the monitoring problem of detecting special causes that shift ξ away from the in-control value, ξ_0. It is convenient to express the shift in ξ in terms of the standardized shift $\delta = (\xi - \xi_0)/\sigma_X$. We assume that the values of ξ_0 and σ_X are known, but the value of δ is unknown. When autocorrelation is present in a process, a time series model can be fitted to process observations and the residuals from this model can be used in the control charts. If the fitted model is the same as the true process model and the parameters are estimated without error, then the residuals are independent normal random variables with mean 0 and constant variance when the process is in-control.

For the process model in equation (2.3), the residual at time t from the minimum mean square error forecast made at time $t-1$ is

$$e_t = X_t - \xi_0 - \phi(X_{t-1} - \xi_0) + \theta e_{t-1} \tag{3.1}$$

(see Box *et al.*, 1994). Suppose that there is a step change in the process mean from ξ_0 to ξ_1 between time $t = \tau$ and $t = \tau + 1$. Then the expectations of the residuals are

$$E(e_t) = \begin{cases} 0, & t = \tau, \tau - 1, \ldots, \\ c_t(\tau)\,\delta, & t = \tau + 1, \tau + 2, \ldots, \end{cases}$$

where

$$c_t(\tau) = \frac{\phi^{t-\tau-1}(\phi - \theta) - \phi + 1}{1 - \theta}\,\sigma_X \tag{3.2}$$

(see Lu and Reynolds, 1999). The residual immediately after the shift has the largest mean, and then the means of the residuals decrease to an asymptotic mean of $(1-\phi)\delta\sigma_X/(1-\theta)$. These residuals are independent and normally distributed with variance σ_γ^2.

The EWMA chart based on the residuals for detecting changes in the process mean uses the control statistic

$$Y_t = \lambda e_t + (1 - \lambda) Y_{t-1},$$

for a weight $\lambda(0 < \lambda \leq 1)$ and $Y_0 = 0$. This chart signals that the process mean has changed if $|Y_t| \geq k\sqrt{\lambda/(2-\lambda)}\,\sigma_\gamma$, where k is a constant. When $\lambda = 1$, the EWMA chart reduces to a standard Shewhart chart. Because the residuals are independent, the value of k that will give a particular in-control average run length (ARL) can be determined using methods for independent observations (see, *e.g.*, Lucas and Saccucci, 1990).

4. Estimation of the Process Change Point

Let τ be the process change point, which is defined as the last sample from the in-control process, and $T(> \tau)$ be the time that the chart signals. Then X_1, X_2, ..., X_τ are the observations taken from the in-control process, whereas $X_{\tau+1}$, $X_{\tau+2}$, ..., X_T are from the changed process due to a step change.

Nishina (1992) proposed a built-in estimator for the process change point when an EWMA chart signals a change. Nishina's proposed change point estimator is the starting point of the rejection run (the run that eventually exceeds the upper or lower control limit). For example, following a signal that Y_T exceeds the upper control limit, the change point estimator for an increase in the process mean, would be

$$\hat{\tau}_N = \max\{t : Y_t \leq m_0\}, \tag{4.1}$$

where m_0 is the in-control value of the mean.

Samuel *et al.* (1998) proposed the MLE for the change point in an independent normal process, as

$$\hat{\tau}_S = \underset{0 \leq t < T}{\operatorname{argmax}}\{(T-t)(\bar{X}_{t+1,T} - m_0)^2\}, \tag{4.2}$$

where $\bar{X}_{t+1,T} = \sum_{i=t+1}^{T} X_i/(T-t)$ is the average of the last $T-t$ observations, and $\arg\max_{0 \leq t < T}\{B_t\}$ denotes the value of t in the range of $0 \leq t < T$ which maximizes an arbitrary function of t, B_t. Pignatiello and Samuel (2001) showed that $\hat{\tau}_S$ performs better than $\hat{\tau}_N$ over the range of magnitudes of change. In these papers, they assumed that the in-control mean m_0 and the variance are known, but the magnitude of change δ is unknown. They used the conventional charting methods, such as Shewhart, CUSUM, and EWMA, to decide that a change has occurred, and then the change point likelihood is used for the following problem of estimating τ and δ (see Hawkins *et al.*, 2003). This same modeling framework is used in this paper also.

The estimator in equation (4.2) can be used for a control chart based on the independent observations. We propose a MLE for the change point of the mean of autocorrelated processes determined by equations (2.1) and (2.2), when a control chart based on the residuals given by equation (3.1) gives a signal, as

$$\hat{\tau}_L = \underset{0 \le t < T}{\operatorname{argmax}} \left\{ \frac{\left(\sum_{i=t+1}^{T} c_i(t)\, e_i \right)^2}{\sum_{i=t+1}^{T} c_i(t)^2} \right\}, \tag{4.3}$$

where $c_i(t)$ is defined by equation (3.2). Details about the derivation of this estimator are given in the Appendix.

5. Performance of the Estimator in Monitoring the Process Mean

We compare the performance of the MLE $\hat{\tau}_L$ in equation (4.3), with the performance of the built-in estimator $\hat{\tau}_N$ in equation (4.1) when an EWMA chart based on the residuals signals a change. The Monte Carlo simulation is used in evaluating the change point estimators.

The results from the 100,000 simulation runs for various sizes of change in ξ and for $\lambda = 0.1, 0.2, 0.4, 1.0$ are given in Table 5.1 to Table 5.4. Table 5.1 is for the case of $\psi = 0.5$ and $\phi = 0.4$, Table 5.2 is for the case of $\psi = 0.5$ and $\phi = 0.8$, Table 5.3 is for the case of $\psi = 0.9$ and $\phi = 0.4$, and Table 5.4 is for the case of $\psi = 0.9$ and $\phi = 0.8$. The numerical results are for nonnegative values of ϕ because it is believed that positive autocorrelation would be much more likely in applications. The constant k for a given value of λ is set up to achieve an in-control ARL of 370.4. The process change points are generated by geometric distribution with mean $E(\tau) = 100$.

The column labeled ARL_δ denotes the out-of-control ARL for a given value of δ, the column labeled *Bias* denotes the bias, that is the average of the estimate minus the true change point, and the column labeled *S.E.* denotes the standard error of the average. The last column labeled $\Pr(|\hat{\tau} - \tau| \le \epsilon)$ denotes the proportion of the 100,000 runs where the estimated time of the change is within $\pm\epsilon$ of the actual change. This provides an indication of the precision of the two estimators. In the last three columns in Tables, the entries in the first row are for the proposed MLE, $\hat{\tau}_L$, and the entries in the second row are for the built-in estimator, $\hat{\tau}_N$.

Table 5.1: Bias, associated standard error, and precision of change point estimators when $\psi = 0.5$ and $\phi = 0.4$

δ	λ	k	ARL_δ	*Bias*	*S.E.*	$\Pr(\lvert\hat{\tau}-\tau\rvert \le \epsilon)$ $\epsilon = 0$	1	3	5
0.5	0.1	2.701	42.41	13.20	0.1031	0.04	0.11	0.21	0.29
				16.30	0.1001	0.04	0.12	0.23	0.33
	0.2	2.859	57.91	14.81	0.1063	0.04	0.11	0.21	0.28
				40.72	0.1639	0.03	0.08	0.17	0.23
	0.4	2.959	93.45	14.92	0.1124	0.04	0.11	0.21	0.29
				84.42	0.2871	0.02	0.05	0.09	0.11
	1.0	3.000	206.91	11.77	0.1112	0.05	0.12	0.23	0.31
				204.07	0.6550	0.01	0.01	0.03	0.04
1.0	0.1	2.701	13.95	−0.01	0.0528	0.14	0.31	0.51	0.64
				−2.41	0.0307	0.12	0.29	0.53	0.67
	0.2	2.859	15.46	0.82	0.0458	0.14	0.30	0.51	0.64
				0.65	0.0272	0.14	0.34	0.58	0.71
	0.4	2.959	22.78	1.55	0.0408	0.14	0.30	0.51	0.64
				10.40	0.0577	0.12	0.27	0.44	0.53
	1.0	3.000	76.64	1.73	0.0339	0.14	0.32	0.53	0.65
				72.26	0.2383	0.02	0.05	0.09	0.11
2.0	0.1	2.701	5.77	−1.17	0.0316	0.34	0.63	0.84	0.91
				−4.06	0.0278	0.22	0.46	0.67	0.75
	0.2	2.859	5.19	−0.99	0.0299	0.35	0.63	0.84	0.91
				−2.44	0.0163	0.28	0.55	0.74	0.82
	0.4	2.959	5.41	−0.67	0.0281	0.35	0.63	0.85	0.92
				−1.38	0.0098	0.35	0.62	0.82	0.90
	1.0	3.000	14.27	0.27	0.0170	0.35	0.64	0.87	0.94
				6.05	0.0371	0.25	0.43	0.59	0.67
3.0	0.1	2.701	3.80	−0.79	0.0218	0.52	0.81	0.93	0.95
				−4.44	0.0276	0.30	0.54	0.68	0.74
	0.2	2.859	3.26	−0.75	0.0243	0.53	0.81	0.93	0.96
				−2.67	0.0158	0.37	0.60	0.74	0.82
	0.4	2.959	2.97	−0.70	0.0237	0.53	0.82	0.94	0.96
				−1.68	0.0092	0.44	0.65	0.82	0.90
	1.0	3.000	4.63	−0.11	0.0160	0.54	0.84	0.96	0.98
				−0.69	0.0060	0.47	0.74	0.92	0.98

Table 5.2: Bias, associated standard error, and precision of change point estimators when $\psi = 0.5$ and $\phi = 0.8$

δ	λ	k	ARL_δ	*Bias*	*S.E.*	$\Pr(\|\hat{\tau} - \tau\| \leq \epsilon)$			
						$\epsilon = 0$	1	3	5
0.5	0.1	2.701	102.87	46.45	0.2301	0.01	0.04	0.09	0.13
				78.27	0.2990	0.02	0.04	0.08	0.12
	0.2	2.859	140.33	52.14	0.2523	0.01	0.04	0.08	0.12
				126.60	0.4308	0.01	0.03	0.06	0.08
	0.4	2.959	195.22	54.60	0.2664	0.01	0.04	0.08	0.12
				188.30	0.6101	0.01	0.02	0.03	0.05
	1.0	3.000	295.37	52.58	0.2743	0.02	0.04	0.09	0.13
				292.95	0.9301	0.00	0.01	0.02	0.02
1.0	0.1	2.701	34.06	6.92	0.0901	0.04	0.11	0.22	0.31
				9.07	0.0715	0.05	0.13	0.27	0.39
	0.2	2.859	45.08	8.65	0.0892	0.04	0.11	0.22	0.31
				27.33	0.1206	0.04	0.10	0.21	0.30
	0.4	2.959	73.09	10.26	0.0884	0.04	0.11	0.21	0.30
				63.30	0.2210	0.02	0.06	0.11	0.15
	1.0	3.000	179.10	10.45	0.0868	0.04	0.11	0.22	0.31
				176.06	0.5630	0.01	0.01	0.03	0.04
2.0	0.1	2.701	12.11	−0.83	0.0581	0.10	0.26	0.49	0.64
				−2.58	0.0296	0.10	0.27	0.54	0.69
	0.2	2.859	12.79	−0.05	0.0551	0.10	0.25	0.48	0.63
				−0.31	0.0207	0.13	0.34	0.62	0.76
	0.4	2.959	17.48	1.28	0.0470	0.10	0.25	0.48	0.63
				5.25	0.0367	0.13	0.32	0.55	0.66
	1.0	3.000	58.59	3.53	0.0335	0.09	0.24	0.46	0.61
				53.56	0.1776	0.02	0.06	0.10	0.14
3.0	0.1	2.701	7.54	−1.18	0.0458	0.16	0.40	0.69	0.83
				−3.53	0.0283	0.14	0.36	0.64	0.74
	0.2	2.859	7.14	−0.87	0.0432	0.17	0.41	0.70	0.83
				−1.89	0.0169	0.19	0.45	0.72	0.82
	0.4	2.959	8.01	−0.23	0.0411	0.16	0.40	0.70	0.84
				−0.55	0.0115	0.23	0.53	0.80	0.89
	1.0	3.000	22.48	1.93	0.0233	0.15	0.36	0.66	0.81
				14.65	0.0615	0.11	0.23	0.36	0.44

Table 5.3: Bias, associated standard error, and precision of change point estimators when $\psi = 0.9$ and $\phi = 0.4$

						$\Pr(\lvert\hat{\tau}-\tau\rvert \le \epsilon)$			
δ	λ	k	ARL_δ	*Bias*	*S.E.*	$\epsilon = 0$	1	3	5
0.5	0.1	2.701	53.32	19.03	0.1254	0.04	0.09	0.17	0.24
				26.81	0.1369	0.04	0.09	0.19	0.26
	0.2	2.859	74.48	20.82	0.1317	0.04	0.09	0.17	0.24
				58.21	0.2200	0.02	0.06	0.13	0.17
	0.4	2.959	116.26	20.74	0.1399	0.04	0.09	0.18	0.24
				107.86	0.3614	0.01	0.04	0.07	0.09
	1.0	3.000	233.67	17.31	0.1411	0.04	0.10	0.19	0.26
				230.96	0.7370	0.01	0.01	0.02	0.03
1.0	0.1	2.701	16.89	0.69	0.0558	0.12	0.27	0.45	0.57
				−1.68	0.0338	0.11	0.27	0.49	0.63
	0.2	2.859	19.73	1.67	0.0511	0.12	0.26	0.45	0.57
				3.38	0.0397	0.12	0.29	0.50	0.62
	0.4	2.959	30.66	2.47	0.0473	0.12	0.26	0.44	0.57
				18.58	0.0853	0.09	0.20	0.32	0.40
	1.0	3.000	98.84	2.32	0.0428	0.13	0.28	0.46	0.59
				94.96	0.3098	0.02	0.04	0.06	0.08
2.0	0.1	2.701	6.52	−1.38	0.0348	0.34	0.58	0.79	0.87
				−4.08	0.0280	0.22	0.45	0.66	0.74
	0.2	2.859	6.03	−1.26	0.0341	0.34	0.58	0.79	0.88
				−2.41	0.0165	0.29	0.53	0.73	0.82
	0.4	2.959	6.68	−0.79	0.0312	0.35	0.59	0.80	0.89
				−1.17	0.0107	0.35	0.60	0.80	0.89
	1.0	3.000	20.38	0.31	0.0183	0.35	0.60	0.82	0.91
				12.81	0.0599	0.18	0.30	0.42	0.49
3.0	0.1	2.701	4.12	−0.96	0.0256	0.54	0.78	0.91	0.94
				−4.35	0.0272	0.31	0.54	0.68	0.74
	0.2	2.859	3.54	−0.98	0.0269	0.55	0.78	0.91	0.94
				−2.70	0.0158	0.39	0.60	0.74	0.82
	0.4	2.959	3.29	−0.92	0.0292	0.55	0.79	0.92	0.95
				−1.68	0.0092	0.45	0.65	0.82	0.90
	1.0	3.000	6.25	−0.14	0.0154	0.56	0.81	0.94	0.97
				−0.13	0.0104	0.44	0.69	0.88	0.94

Table 5.4: Bias, associated standard error, and precision of change point estimators when $\psi = 0.9$ and $\phi = 0.8$

δ	λ	k	ARL_δ	*Bias*	*S.E.*	$\Pr(\lvert\hat{\tau}-\tau\rvert \le \epsilon)$ $\epsilon = 0$	1	3	5
0.5	0.1	2.701	142.41	67.87	0.3262	0.01	0.03	0.06	0.10
				119.73	0.4302	0.01	0.03	0.06	0.09
	0.2	2.859	185.79	76.92	0.3529	0.01	0.03	0.06	0.09
				173.37	0.5768	0.01	0.02	0.04	0.06
	0.4	2.959	239.61	83.71	0.3755	0.01	0.03	0.06	0.09
				233.35	0.7516	0.01	0.01	0.03	0.04
	1.0	3.000	322.86	85.06	0.3866	0.01	0.03	0.06	0.09
				320.55	1.0167	0.00	0.01	0.02	0.02
1.0	0.1	2.701	50.67	11.46	0.1259	0.03	0.08	0.17	0.23
				24.12	0.1276	0.04	0.10	0.20	0.28
	0.2	2.859	69.90	13.80	0.1274	0.03	0.08	0.16	0.23
				53.43	0.2050	0.03	0.07	0.13	0.19
	0.4	2.959	110.19	16.52	0.1284	0.03	0.08	0.16	0.22
				101.64	0.3406	0.01	0.04	0.07	0.10
	1.0	3.000	226.05	18.38	0.1294	0.03	0.08	0.16	0.22
				223.32	0.7127	0.01	0.01	0.02	0.03
2.0	0.1	2.701	16.16	−1.36	0.0802	0.10	0.23	0.40	0.51
				−1.78	0.0326	0.11	0.27	0.50	0.64
	0.2	2.859	18.67	0.05	0.0744	0.10	0.22	0.39	0.51
				2.58	0.0361	0.12	0.30	0.52	0.65
	0.4	2.959	28.53	2.00	0.0661	0.09	0.21	0.38	0.49
				16.32	0.0780	0.09	0.21	0.35	0.43
	1.0	3.000	92.85	5.45	0.0544	0.09	0.20	0.36	0.47
				88.86	0.2916	0.02	0.04	0.07	0.09
3.0	0.1	2.701	9.10	−2.47	0.0647	0.19	0.40	0.62	0.74
				−3.60	0.0287	0.16	0.37	0.61	0.73
	0.2	2.859	9.09	−1.87	0.0624	0.19	0.39	0.61	0.74
				−1.76	0.0177	0.21	0.46	0.70	0.81
	0.4	2.959	11.55	−0.49	0.0539	0.19	0.38	0.60	0.73
				0.89	0.0207	0.24	0.48	0.70	0.81
	1.0	3.000	39.66	2.80	0.0367	0.16	0.34	0.55	0.68
				33.70	0.1217	0.07	0.13	0.20	0.24

Table 5.1 to Table 5.4 give the following results. As expected, the out-of-control ARL values show that small values of λ are better for detecting small shifts and large values of λ are better for detecting large shifts, and the Shewhart individuals chart (*i.e.*, $\lambda = 1$) based on the residuals is not effective for detecting small and large shifts in the process mean. Additional discussion of the EWMA chart's performance for an AR(1) process plus an additional random error can be found in Lu and Reynolds (1999).

The proposed MLE, $\hat{\tau}_L$, appears to be much less biased in estimating the process change point than $\hat{\tau}_N$, and the precision of $\hat{\tau}_L$ is better than that of $\hat{\tau}_N$ for the overall range of values of ψ, ϕ, and δ. When δ is small ($\delta \leq 1.0$), two estimators tend to overestimate the change point. As the level of autocorrelation increases, the amount of the bias increases with increases in ARL_δ for small δ. However, for these small shifts, $\hat{\tau}_L$ yields better results than $\hat{\tau}_N$. For large shift, it appears that $\hat{\tau}_L$ provides an accurate estimate of the change point. Note that for $\lambda = 1$, the values of $\hat{\tau}_N$ are meaningless because $\hat{\tau}_N$ is based on the information from the past observations in addition to the current observation, but the Shewhart control statistic uses the current observation alone. The proposed MLE can be applied when an out-of-control signal is given on any control chart based on the residuals including Shewhart, CUSUM, and EWMA charts.

6. Conclusions

This paper has considered an autocorrelated process model which is an AR(1) process plus an additional random error. We have proposed the MLE for the change point of the mean of this model when a control chart based on the residuals gives a signal. We have compared its performance with the built-in estimator in EWMA charts. The results show that the performance of the MLE appears to be better than the built-in estimator over the wide range of shifts.

Appendix : The MLE for the Change Point of the Process Mean

When there is a shift in ξ, the distribution of e_t is $N(0, \sigma_\gamma^2)$ for $t \leq \tau$ and $N(c_t(\tau)\,\delta, \sigma_\gamma^2)$ for $t \geq \tau + 1$. We assume that the values of τ and δ are unknown. Given the residuals $e_1, e_2, \ldots, e_T$, the log likelihood function can be expressed as

$$\ln L(\tau, \delta \,|\, e_1, e_2, \ldots, e_T) = A - \frac{1}{2\sigma_\gamma^2}\left[\sum_{i=1}^{\tau} e_i^2 + \sum_{i=\tau+1}^{T} (e_i - c_i(\tau)\delta)^2\right],$$

where A is a constant not depending on τ and δ.

If the change point τ is known, the MLE for δ will be

$$\hat{\delta} = \frac{\sum_{i=\tau+1}^{T} c_i(\tau)\, e_i}{\sum_{i=\tau+1}^{T} c_i(\tau)^2}.$$

Substituting this estimator into the log likelihood function, we get

$$\ln L(\tau | e_1, e_2, \ldots, e_T) = A - \frac{1}{2\sigma_\gamma^2} \sum_{i=1}^{T} e_i^2 + \frac{1}{2\sigma_\gamma^2} \frac{\left(\sum_{i=\tau+1}^{T} c_i(\tau)\, e_i \right)^2}{\sum_{i=\tau+1}^{T} c_i(\tau)^2}.$$

Thus the MLE for τ is given by

$$\hat{\tau}_L = \underset{0 \le t < T}{\operatorname{argmax}} \left\{ \frac{\left(\sum_{i=t+1}^{T} c_i(t)\, e_i \right)^2}{\sum_{i=t+1}^{T} c_i(t)^2} \right\}.$$

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