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A Study on Box-Cox Transformed Threshold GARCH(1,1) Process*

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Abstract

In this paper, we consider a Box-Cox transformed threshold GARCH(1,1) process and find a sufficient condition under which the process is geometrically ergodic and has the β−mixing property with an exponential decay rate.

Keywords: Box-Cox transform; threshold GARCH; stationarity; geometrically ergotic; beta-mixing.

1. Introduction

Since the introduction of the seminal paper on autoregressive conditional heteroskedasticity (ARCH) process of Engle (1982) where the conditional variance is stochastic and dependent on past observation, ARCH-family process has been most adopted for modeling time varying conditional volatility. Following the natural extension of the ARMA process as a parsimonious representation of a higher order AR process, a generalized ARCH (GARCH) process is proposed by Bollerslev (1986). The classical GARCH model is formulated as a linear combination of squared observations and lagged conditional variances. However, the GARCH process fails to explain the asymmetric phenomena, since in the model, the conditional variance is a function of only the magnitudes of the lagged residuals but not their signs. A model that accounts for the asymmetric effect of the "news" is the threshold GARCH (TARCH) model of Rabemanjara and Zakoïan (1993). In the TARCH model, good news and bad news have different effects on the conditional variance. On the other hand, most of the ARCH-type models deal with the conditional variance. However, in many cases, appropriate measure of volatility

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is the standard deviation rather than the variance as noted by Barndorff-Nielsen and Shephard (2002). A class of models where the conditional standard deviation is taken as a measure of volatility is given by Ding *et al.* (1993) using the power ARCH model.

As a nonlinear and non-symmetric model, Box-Cox transformed processes are suggested and studied by many authors (see, *e.g.*, Ling and McAleer (2002), Hwang and Basawa (2004), Liu (2006) *etc.*)

We consider in this paper the Box-Cox transformed threshold GARCH(1,1) process which is defined by

$$\epsilon_t = \sqrt{h_t} e_t, \tag{1.1}$$

$$h_t^\delta = \alpha_0 + \alpha_{11}(\epsilon_{t-1}^{+2})^\delta + \alpha_{12}(\epsilon_{t-1}^{-2})^\delta + \phi h_{t-1}^\delta, \tag{1.2}$$

where $\delta > 0, \phi \geq 0, \alpha_0 > 0, \alpha_{11} \geq 0, \alpha_{12} \geq 0$ and $\{e_t\}$ is a sequence of *iid* random variables with mean zero and $E|e_t|^{2\delta} < \infty$. Here we use the notations $e_t^{+2} = (e_t^+)^2, e_t^{-2} = (e_t^-)^2$ and $e_t^+ = \max\{e_t, 0\}, e_t^- = \max\{-e_t, 0\}$.

For statistical analysis on the model, stationarity, (geometric) ergodicity, existence of moments and mixing properties are of great importance. Ling and McAleer (2002) and Hwang and Basawa (2004) find a sufficient condition for strict stationarity and moments of the process generated by (1.1) and (1.2). They show that if $\phi + \alpha_{11}E[(e_t^{+2})^\delta] + \alpha_{12}E[(e_t^{-2})^\delta] < 1$, then the process $\{h_t^\delta\}$ has the unique strictly stationary solution with finite first moment given by $h_t^{\delta *} = \alpha_0 + \alpha_0 \sum_{k=1}^\infty \Pi_{i=1}^k (\phi + \alpha_{11}(e_t^{+2})^\delta + \alpha_{12}(e_t^{-2})^\delta)$ where the infinite sum is finite almost surely. Liu (2006) and Meitz (2005) prove that the process has a unique strictly stationary ergodic solution if and only if $E[ln(\phi + \alpha_{11}(e_t^{+2})^\delta + \alpha_{12}(e_t^{-2})^\delta)] < 0$. Moments condition and tail behavior are also considered.

The goal of this paper is to find a sufficient condition for geometric ergodicity and $\beta-$mixing with exponential decay of the Box-Cox transformed process of (1.1) and (1.2).

We let $\{X_t : t \geq 0\}$ be a temporarily homogeneous Markov chain taking values in $(E, \mathcal{E})$, where E is a set and $\mathcal{E}$ is a countably generated $\sigma-$algebra of subsets of E, with transition probabilities given by $p^{(t)}(x, A) = P(X_t \in A | X_0 = x)$, $x \in E$, $A \in \mathcal{E}$. In this paper $E = R^+$ and $\mathcal{E}$ is the $\sigma-$algebra of Borel sets.

The Markov chain $\{X_t\}$ is $\phi-$irreducible if, for some $\sigma-$finite measure ϕ on $(E, \mathcal{E})$, $\sum_t p^{(t)}(x, A) > 0$ for all $x \in E$, whenever $\phi(A) > 0$.

$\{X_t\}$ is ergodic if there exists a probability measure π on $(E, \mathcal{E})$ such that $\lim_{t\to\infty} \|p^{(t)}(x, \cdot) - \pi(\cdot)\| = 0$ for all $x \in E$, where $\| \cdot \|$ denotes the total variation norm. If $\{X_t\}$ is ergodic and there exists a ρ, $0 < \rho < 1$ such that $\lim_{t\to\infty} \rho^{-t}\|p^{(t)}(x,\cdot) - \pi(\cdot)\| = 0$ for all $x \in E$, then $\{X_t\}$ is said to be geometrically ergodic.

If $\{X_t\}$ is a Markov process with initial distribution as its invariant measure $\pi(dx)$, then $\{X_t\}$ is stationary $\beta-$mixing with exponential decay if there exist $0 < \rho < 1$ and $c > 0$ such that $\int \|p^{(t)}(x,\cdot) - \pi(\cdot)\|\pi(dx) \le c\rho^t, \quad \forall t \in N$.

$\{X_t\}$ is called a Feller chain if for each bounded continuous function g, the function of x given by $E[g(X_t)|X_{t-1} = x]$ is also continuous.

One of the well known way to prove geometric ergodicity and mixing property is to use the Foster-Lyapounov drift condition given in the following Theorem 1.1 due to Tweedie (1983).

Theorem 1.1 *Suppose that $\{X_t\}$ is a $\phi-$irreducible aperiodic Feller chain. If there exists, for some compact set A, a nonnegative function g and $\epsilon > 0$ satisfying*

$$\int P(x,dy)g(y) \le \rho g(x) - \epsilon, \quad x \in A^c, \tag{1.3}$$

and

$$\sup_{x\in A} \int P(x,dy)g(y) < \infty, \tag{1.4}$$

then $\{X_t\}$ is geometric ergodic. If $\{X_t\}$ is initialized from an invariant initial distribution, it is strictly stationary and $\beta-$mixing with exponential decay.

The reader is referred to Meyn and Tweedie (1993) for additional definitions and properties in Markov chain context.

2. Main Results

The model (1.1)-(1.2) can be rewritten as follows:

$$\begin{aligned} \epsilon_t &= \sqrt{h_t} e_t, & (2.1)\\ h_t^\delta &= \alpha_0 + (\phi + \alpha_{11}(e_{t-1}^{+2})^\delta + \alpha_{12}(e_{t-1}^{-2})^\delta) h_{t-1}^\delta \\ &= \alpha_0 + (\phi + \eta_{t-1}) h_{t-1}^\delta, & (2.2) \end{aligned}$$

where $\eta_t = \alpha_{11}(e_t^{+2})^\delta + \alpha_{12}(e_t^{-2})^\delta$. Note that e_t is independent of $h_t^\delta, h_{t-1}^\delta, h_{t-2}^\delta, \cdots$, and $\{h_t^\delta\}$ is a Markov chain with $t-$step transition probability function $p^{(t)}(x, A) = P(h_t^\delta \in A | h_0^\delta = x)$.

Lemma 2.1 *$\{h_t^\delta\}$ is a Feller chain.*

Proof: It is straightforward to prove that, for each bounded continuous function g on R^+, $E[g(h_t^\delta)|h_{t-1}^\delta = x] = E[g(\alpha_0 + (\phi + \eta_{t-1})x)]$ is continuous in x according to the bounded convergence theorem, that is, $\{h_t^\delta\}$ is a Feller chain. □

Assumption A1. $\{e_t\}$ is a sequence of *iid* random variables with $E|e_t|^{2\delta} < \infty$. Further, its probability distribution function is absolutely continuous with respect to the Lebesgue measure and such that the probability density function $f(x)$ takes positive values almost everywhere on R^+.

Assumption A2. $\phi + E[\alpha_{11}(e_t^{+2})^\delta + \alpha_{12}(e_t^{-2})^\delta] < 1$.

Lemma 2.2 *Under the assumptions A1 and A2, $\{h_t^\delta\}$ is $\mu-$irreducible and aperiodic, where $\mu(A) = \lambda(A \cap [\alpha_0(1 + (1-r)^{-1}) + 1, \infty))$, $r = \phi + E[\eta_t]$, and λ is the Lebesgue measure on R^+.*

Proof: Note that $r = \phi + E[\eta_t] < 1$ by A2. Let A be a Borel set in R^+ with $\mu(A) > 0$ and let $a = \max\{\inf A,\ \alpha_0(1+(1-r)^{-1})+1\}$, where $\inf A = \inf\{x|x \in A\}$.

For any $x \in R^+$,

$$\begin{aligned} p(x, A) &= P(h_t^\delta \in A | h_{t-1}^\delta = x) \\ &= P(\alpha_0 + (\phi + \eta_{t-1})x \in A) \\ &= P(\eta_{t-1} \in \frac{1}{x}(A - \alpha_0) - \phi). \end{aligned} \tag{2.3}$$

If $0 < x \le a - \alpha_0$, then $x^{(-1)}(a - \alpha_0) - \phi > 0$ yields that $\lambda(B) > 0$ where $B = x^{(-1)}(A - \alpha_0) - \phi$. Hence we have that

$$p(x, A) = P(\eta_{t-1} \in B) = \int_B q(y)dy > 0, \tag{2.4}$$

where $q(y)$ is a probability density function of η_t.

Now let $\{h_t^\delta(x) : t \ge 0\}$ denote $\{h_t^\delta\}$ in (2.2) if $h_0^\delta = x$, $x \in R^+$. Then

$$h_t^\delta(x) = \alpha_0 + \alpha_0 \sum_{k=1}^{t-1} \Pi_{i=1}^k(\phi + \eta_{t-i}) + \Pi_{i=1}^t(\phi + \eta_{t-i})x.$$

For any $x \in R^+$, we have that

$$E[h_t^\delta(x)] = \alpha_0(1 + r + r^2 + \cdots + r^{t-1}) + r^t x$$

$$\leq \alpha_0(1-r)^{-1}+1, \tag{2.5}$$

for sufficiently large t.

From inequalities (2.5) and $\alpha_0(1-r)^{-1}+1 < a-\alpha_0$, for any fixed $x > a-\alpha_0$,

$$P(h_{t_0}^\delta(x) < a-\alpha_0) > 0, \quad \text{for some} \quad t_0 = t_0(x) \geq 1. \tag{2.6}$$

We have that

$$\begin{aligned} p^{(t_0+1)}(x,A) &= P(h_{t_0+1}^\delta(x) \in A) \\ &\geq P(h_{t_0}^\delta(x) < a-\alpha_0)P(h_{t_0+1}^\delta(x) \in A | h_{t_0}^\delta(x) < a-\alpha_0) \\ &> 0. \end{aligned} \tag{2.7}$$

Last inequality in (2.7) follows from (2.4) and (2.6). Hence if $\mu(A) > 0$, then $\sum_t p^{(t)}(x,A) > 0$ for all x, and $\{h_t^\delta\}$ is $\mu-$irreducible.

Now let $C = [a,b], a < b, a > \alpha_0(1+(1-r)^{-1})+1$. From (2.5), we may choose a positive integer t_0 such that for $t \geq t_0$, (2.5) holds for all $x \in [a,b]$, which implies that

$$P(h_t^\delta(x) \in [a,b]) > 0, \quad \text{and} \quad P(h_{t+1}^\delta(x) \in [a,b]) > 0, \tag{2.8}$$

for every $x \in [a,b]$. Aperiodicity of $\{h_t^\delta\}$ follows from, together with (2.8), the fact that every compact set is a small set, which is obtained from Feller continuity of $\{h_t^\delta\}$.

□

Theorem 2.1 *Suppose that A1 and A2 hold. Then $\{h_t^\delta\}$ is a geometrically ergodic Markov chain. Moreover, if initialized from its invariant measure, the process is strictly stationary and $\beta-$mixing with exponential decay.*

Proof: By Lemma 2.1 and 2.2, Markov process $\{h_t^\delta\}$ is a $\mu-$irreducible aperiodic Feller chain. To complete the proof, it remains to find a Lyapounov test function g such that the inequalities (1.3) and (1.4) hold. Take a test function $g : R^+ \to R^+$ by

$$g(x) = x+1, \tag{2.9}$$

then $E[g(h_t^\delta)|h_{t-1}^\delta = x] = 1+\alpha_0+(\phi+E[\eta_{t-1}])x$. For any $\epsilon > 0$, we may choose $\rho,\ 0 < \phi+E[\eta_t] < \rho < 1$ and $K,\ 0 < K < \infty$ such that

$$E[g(h_t^\delta)|h_{t-1}^\delta = x] \leq \rho g(x) - \epsilon, \quad x > K, \tag{2.10}$$

$$E[g(h_t^\delta)|h_{t-1}^\delta = x] \leq 1+\alpha_0+K < \infty, \quad x \leq K. \tag{2.11}$$

Result of the Theorem 2.1 is derived from inequalities (2.10) and (2.11) by applying Theorem 1.1. □

Remark 2.1 It is proved that if $E|e_t|^{2\delta m} < \infty$, then the necessary and sufficient condition for the existence of the $2\delta m$th moment of the solution of $\epsilon_t = \sqrt{h_t}e_t$, $h_t^\delta = \alpha_0 + \alpha_0 \sum_{k=1}^{\infty} \Pi_{j=1}^{k} (\phi + \eta_{t-i})$ is $E(\phi + \eta_t)^{2\delta m} < 1$, where m is a positive integer (see, Ling and McAleer, 2002). Moments structure for each case of $\delta = 1/2$, $\delta = 1$, or $\delta = 2$ is studied in detail by Hwang and Basawa (2004).

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