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Bayes and Empirical Bayes Estimation of the Scale Parameter of the Gamma Distribution under Balanced Loss Functions

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Abstract

The present paper investigates estimation of a scale parameter of a gamma distribution using a loss function that reflects both goodness of fit and precision of estimation. The Bayes and empirical Bayes estimators relative to balanced loss functions (BLFs) are derived and optimality of some estimators are studied.

Keywords: Admissibility; balanced loss function; Bayes and empirical Bayes estimation; minimum risk estimator.

1. Introduction

The gamma distribution is widely used and plays an important role in the reliability field and the survival analysis, therefore estimating of its parameters will be important.

Let $X_1, \ldots, X_n$ be a random sample of size n from the gamma distribution $\Gamma(\delta,\ \theta)$ with density

$$f(x|\theta) = \frac{1}{\theta^{\delta}\Gamma(\delta)} x^{\delta-1} e^{-\frac{x}{\theta}}, \quad x > 0,\ \theta > 0,\ \delta > 0, \tag{1.1}$$

where δ is known and θ is an unknown scale parameter. we consider estimation of the scale parameter of θ under the BLF, given by

$$L(\widehat{\theta}, \theta) = \frac{\omega}{n} \sum_{i=1}^{n} \left(\frac{X_i}{\delta} - \widehat{\theta} \right)^2 + (1-\omega)(\widehat{\theta} - \theta)^2, \tag{1.2}$$

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where $0 < \omega < 1$, and $\widehat{\theta}$ is an estimator of θ .

The BLF, introduced by Zellner (1994), is formulated to reflect two criteria, namely goodness of fit and precision of estimation. A goodness of fit criterion such as the sum of squared residuals in regression problems leads to an estimate which gives good fit and unbiased estimator; however, as commented by Zellner (1994), it may not be as precise as an estimator which is biased. Thus there is a need to provide a framework which combines goodness of fit and precision of estimation formally. The BLF framework meets this need. The first term on the r.h.s. of (1.2) reflects goodness of fit while the second term reflects precision of estimation. Note that if $\omega \to 0$, the loss function reduces to a squared error loss while if $\omega \to 1$, the loss function reduces to a pure goodness of fit criterion.

The BLF was first introduced in Zellner (1994) for the estimation of a scalar mean, a vector mean and a regression coefficient vector. Rodrigues and Zellner (1995) considered the estimation problem for the exponential mean time to failure under a weighted balanced loss function (WBLF). Various other authors have used this form of loss function in a number of recent studies including Chung *et al.* (1998), Dey *et al.* (1999), Sanjari Farsipour and Asgharzadeh (2003, 2004) and Gruber (2004).

Due to convexity of the loss, one needs to consider only estimators that are functions of the minimal sufficient statistics $\overline{X}$. Now, for any estimator $g(\overline{X})$ of θ ,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\delta}-g(\overline{X})\right)^2=\frac{1}{n\delta^2}\sum_{i=1}^{n}(X_i-\overline{X})^2+\left(\frac{\overline{X}}{\delta}-g(\overline{X})\right)^2 .$$

Thus, without loss of generality, one can consider the BLF as

$$L(\widehat{\theta},\theta)=\omega\left(\frac{\overline{X}}{\delta}-\widehat{\theta}\right)^2+(1-\omega)(\widehat{\theta}-\theta)^2, \tag{1.3}$$

where $\widehat{\theta}$ is a function of $\overline{X}$.

An extension of the BLF (1.3) is the WBLF, which is defined as

$$L_W(\widehat{\theta},\theta)=\omega q(\theta)\left(\frac{\overline{X}}{\delta}-\theta\right)^2+(1-\omega)q(\theta)(\widehat{\theta}-\theta)^2, \tag{1.4}$$

where $q(\theta) > 0$ is any positive function of θ which is called the weight function. It generalizes the BLF in the sense that $q(\theta) = 1$.

In this paper, the Bayes and empirical Bayes estimators of θ are considered under the BLF (1.3) and WBLF (1.4). Optimality of some estimators are studied.

2. Estimation under BLF

Let $X_1, \ldots, X_n$ be a random sample from the gamma distribution with *pdf* given by (1.1). In this section, we consider some optimal estimators of the scale parameter θ under the BLF (1.3) . The usual estimator of θ is $\overline{X}/\delta$, which is the maximum likelihood estimator (MLE) of θ.

2.1. Minimum Risk Estimator

The minimum mean squared error estimator of θ can be shown as the form

$$\left(\frac{n\delta}{1+n\delta}\right)\frac{\overline{X}}{\delta}$$

which is in the class of $a(\overline{X}/\delta)$. With respect to the BLF (1.3), we propose a minimum risk estimator as $Y = a(\overline{X}/\delta)$ and find the value of a for which risk under BLF is minimum. The BLF of $a(\overline{X}/\delta)$ is

$$L\left(a\frac{\overline{X}}{\delta}, \theta\right) = \omega\left(\frac{\overline{X}}{\delta} - \frac{a\overline{X}}{\delta}\right)^2 + (1-\omega)\left(\theta - \frac{a\overline{X}}{\delta}\right)^2,$$

which has the risk function

$$R\left(a\frac{\overline{X}}{\delta}, \theta\right) = \omega E\left(\frac{\overline{X}}{\delta} - \frac{a\overline{X}}{\delta}\right)^2 + (1-\omega)E\left(\frac{a\overline{X}}{\delta} - \theta\right)^2.$$

It can be seen that

$$R\left(a\frac{\overline{X}}{\delta}, \theta\right) = (a-1)^2\theta^2 + \frac{\theta^2}{n\delta}[(a-\omega)^2 + \omega(1-\omega)].$$

The value of a for which this risk function is minimum is

$$a_0 = \frac{\omega + n\delta}{1+n\delta}.$$

Thus, the minimum risk estimator (MRE) is

$$\widehat{\theta}_{MRE} = \frac{\omega + n\delta}{1+n\delta}\frac{\overline{X}}{\delta}. \tag{2.1}$$

2.2. Bayes Estimator

For Bayesian estimation, we assume that the conjugate $IG(\alpha,\beta)$ (an inverted gamma) prior for θ with parameters α and β, and density

$$\pi(\theta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}(\frac{1}{\theta})^{\alpha+1}e^{-\beta/\theta}\qquad \theta>0, \tag{2.2}$$

where $\alpha>0$ and $\beta>0$ are known. It is easy to verify that the posterior density of θ, for a given $\mathbf{x}=(x_1,\ldots,x_n)$ is

$$\pi(\theta|\mathbf{x})=\frac{\left(\beta+\sum_{i=1}^{n}x_i\right)^{n\delta+\alpha}}{\Gamma(n\delta+\alpha)}\left(\frac{1}{\theta}\right)^{n\delta+\alpha+1}e^{\frac{-(\beta+\sum_{i=1}^{n}x_i)}{\theta}},\qquad \theta>0$$

which is the inverted gamma with parameters $n\delta+\alpha$ and $\beta+\sum_{i=1}^{n}x_i$. The posterior mean of θ can be shown that to be

$$E(\theta|\mathbf{X})=\frac{\beta+\sum_{i=1}^{n}X_i}{n\delta+\alpha-1}.$$

This posterior mean can be written as

$$E(\theta|\mathbf{X})=(1-B)\frac{\overline{X}}{\delta}+B\mu,$$

where

$$B=\frac{\alpha-1}{n\delta+\alpha-1}\qquad\text{and}\qquad \mu=\frac{\beta}{\alpha-1}.$$

Under the BLF (1.3), the Bayes estimator of θ, denoted by $\widehat{\theta}_B$, is the value $\widehat{\theta}$ which minimizes the posterior risk

$$E[L(\widehat{\theta},\theta)|X]=\omega\left(\frac{\overline{X}}{\delta}-\widehat{\theta}\right)^2+(1-\omega)E[(\theta-\widehat{\theta})^2|X].$$

Here $E(.)$ denotes posterior expectation with respect to the posterior distribution $\pi(\theta|\mathbf{x})$. Solving the equation

$$\frac{\partial E[L(\widehat{\theta},\theta)|X]}{\partial\widehat{\theta}}=0,$$

we obtain the Bayes estimator as

$$\widehat{\theta}_B=\omega\frac{\overline{X}}{\delta}+(1-\omega)E(\theta|\mathbf{X}).$$

Accordingly, the Bayes estimator $\widehat{\theta}_B$ of θ is obtained as

$$\begin{aligned}\widehat{\theta}_B &= \omega\frac{\overline{X}}{\delta} + (1-\omega)[1-B)\frac{\overline{X}}{\delta} + B\mu] \\ &= (1-u)\frac{\overline{X}}{\delta} + u\mu, \end{aligned} \tag{2.3}$$

where $u = (1-\omega)B$.

Taking $a = (1-u)$ and $b = u\mu$ in (2.3), we can obtain the admissibility result for a class of linear estimators of the form $a(\overline{X}/\delta) + b$. It is observed that $a = (1-u)$ is strictly between ω and $(n\delta-\omega)/(n\delta-1)$ and the constant $b = u\mu$ is strictly bigger than 0. Since the BLF (1.3) is strictly convex, (2.3) is the unique Bayes estimator and hence admissible. This proves that $a(\overline{X}/\delta)+b$ is admissible when $\omega < a < (n\delta-\omega)/(n\delta-1)$ and $b > 0$. According to this, the estimators

$$\widehat{\theta}_{MRE} = \frac{\omega + n\delta}{1+n\delta}\frac{\overline{X}}{\delta}$$

(MRE) and $\overline{X}/\delta$ (MLE) are admissible.

2.3. Empirical Bayes Estimator

The Bayes estimator in (2.3), *i.e.*,

$$\widehat{\theta}_B = (1-u)\frac{\overline{X}}{\delta} + u\mu = \frac{n\delta + (\alpha-1)\omega}{n\delta+\alpha-1}\frac{\overline{X}}{\delta} + \frac{(1-\omega)\beta}{n\delta+\alpha-1}$$

is seen to depend on α and β. When the prior parameters α and β are unknown, we may use the empirical Bayes approach to get their estimates by using past samples. For more details on the empirical Bayes approach, see for example Martiz and Lwin (1989) . When the current (informative) sample is observed, suppose that there are available m past similar samples $X_{j,1}, X_{j,2}, \ldots, X_{j,n}$, $j = 1, 2, \ldots, m$ with past realizations $\theta_1, \theta_2, \ldots, \theta_m$ of the random variable θ. Each sample is assumed to be a sample of size n obtained from the gamma distribution with density given by (1.1). The likelihood function of the j^{th} sample is given by

$$L(\theta_j|\underline{x}) = [\Gamma(\delta)]^{-n}\theta_j^{-n\delta}\prod_{i=1}^{n} x_{j,i}^{\delta-1} e^{-\frac{1}{\theta_j}\sum_{i=1}^{n} x_{j,i}}.$$

For a sample $j, j = 1, 2, \ldots, m$, the maximum likelihood estimate of the parameter θ_j is

$$\widehat{\theta}_j = z_j = \frac{\sum_{i=1}^{n} x_{j,i}}{n\delta}.$$

The conditional *pdf* of $Z_j = (\sum_{i=1}^{n} X_{j,i})/n\delta$, for a given θ_j is

$$f(z_j|\theta_j) = \frac{(n\delta)^{n\delta}}{\theta_j^{n\delta}\Gamma(n\delta)} z_j^{n\delta-1} e^{-\frac{n\delta z_j}{\theta_j}}, \quad z_j > 0. \tag{2.4}$$

By using (2.2) and (2.4), the marginal *pdf* of $Z_j, j = 1, 2, \ldots, m$, can be shown to be

$$\begin{aligned} f_{Z_j}(z_j) &= \int_0^\infty f(z_j|\theta_j)\pi(\theta_j)d\theta_j \\ &= \int_0^\infty \frac{(n\delta)^{n\delta}}{\theta_j^{n\delta}\Gamma(n\delta)} z_j^{n\delta-1} e^{-\frac{n\delta z_j}{\theta_j}} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\theta_j^{\alpha+1}} e^{-\frac{\beta}{\theta_j}} d\theta_j \\ &= \frac{\beta^\alpha (n\delta)^{n\delta}}{B(n\delta, \alpha)} \frac{z_j^{n\delta-1}}{(\beta + n\delta z_j)^{n\delta+\alpha}}, \quad z_j > 0, \end{aligned} \tag{2.5}$$

where $B(.,.)$ is the complete beta function. Now, since

$$E(Z_j) = \frac{\beta}{\alpha - 1} \quad \text{and} \quad E(Z_j^2) = \frac{\beta^2(n\delta + 1)}{n\delta(\alpha - 1)(\alpha - 2)},$$

the moment estimates of the parameters α and β can be shown that to be

$$\widehat{\alpha} = \frac{2n\delta S_2 - (n\delta + 1)S_1^2}{n\delta S_2 - (n\delta + 1)S_1^2}, \qquad \widehat{\beta} = \frac{n\delta S_1 S_2}{n\delta S_2 - (n\delta + 1)S_1^2}, \tag{2.6}$$

where

$$S_1 = \frac{1}{m}\sum_{j=1}^{m} z_j \quad \text{and} \quad S_2 = \frac{1}{m}\sum_{j=1}^{m} z_j^2.$$

Applying these estimates in (2.3), the empirical estimator of the parameter θ under the BLF (1.3) is given by

$$\widehat{\theta}_{EB} = \frac{n\delta + (\widehat{\alpha} - 1)\omega}{n\delta + \widehat{\alpha} - 1} \frac{\overline{X}}{\delta} + \frac{(1 - \omega)\widehat{\beta}}{n\delta + \widehat{\alpha} - 1} \tag{2.7}$$

3. Estimation under WBLF

In this section, we consider the Bayes estimator of θ under the WBLF (1.4). As before, it can be shown that the Bayes estimator of θ is

$$\widehat{\theta}_{WB} = \omega\widehat{\theta}_1 + (1 - \omega)\widehat{\theta}_2,$$

where $\widehat{\theta}_1 = \overline{X}/\delta$ and $\widehat{\theta}_2 = E[\theta q(\theta)|X]/E[q(\theta)|X]$.

To obtain the Bayes estimator $\widehat{\theta}_{WB}$, we now consider two examples according to noninformative prior and conjugate prior. We also consider several weight functions, $q(\theta)$ of the WBLF (1.4) for each prior.

Example 3.1 (Noninformative Prior).

The noninformative prior for θ is given by $\pi(\theta) \propto \theta^{-1}$. Thus, the posterior distribution $\pi(\theta|\mathbf{x})$ is $IG(n\delta, \sum_{i=1}^{n} x_i)$.

Case 1. $q(\theta) = \theta$:
We obtain

$$\widehat{\theta}_2 = \frac{\sum_{i=1}^{n} X_i}{n\delta - 2}, \qquad \widehat{\theta}_{WB} = \omega\frac{\overline{X}}{\delta} + (1-\omega)\frac{\sum_{i=1}^{n} X_i}{n\delta - 2}$$

Case 2. $q(\theta) = \theta^2$:

$$\widehat{\theta}_2 = \frac{\sum_{i=1}^{n} X_i}{n\delta - 3}, \qquad \widehat{\theta}_{WB} = \omega\frac{\overline{X}}{\delta} + (1-\omega)\frac{\sum_{i=1}^{n} X_i}{n\delta - 3}$$

Case 3. $q(\theta) = \theta^{-2}$:

$$\widehat{\theta}_2 = \frac{\sum_{i=1}^{n} X_i}{n\delta + 1}, \qquad \widehat{\theta}_{WB} = \omega\frac{\overline{X}}{\delta} + (1-\omega)\frac{\sum_{i=1}^{n} X_i}{n\delta + 1}$$

Example 3.2 (Conjugate Priors).

Assume that the prior of θ is $IG(\alpha, \beta)$. Then the posterior distribution is $\pi(\theta|\mathbf{x})$ is $IG(n\delta + \alpha, \beta + \sum_{i=1}^{n} x_i)$.

Case 1. $q(\theta) = \theta$:
We obtain

$$\widehat{\theta}_2 = \frac{\beta + \sum_{i=1}^{n} X_i}{n\delta + \alpha - 2}, \qquad \widehat{\theta}_{WB} = \omega\frac{\overline{X}}{\delta} + (1-\omega)\frac{\beta + \sum_{i=1}^{n} X_i}{n\delta + \alpha - 2}.$$

Case 2. $q(\theta) = \theta^2$:

$$\widehat{\theta}_2 = \frac{\beta + \sum_{i=1}^{n} X_i}{n\delta + \alpha - 3}, \qquad \widehat{\theta}_{WB} = \omega\frac{\overline{X}}{\delta} + (1-\omega)\frac{\beta + \sum_{i=1}^{n} X_i}{n\delta + \alpha - 3}$$

Case 3. $q(\theta)=\theta^{-2}$:

$$\widehat{\theta}_2=\frac{\beta+\sum_{i=1}^{n}X_i}{n\delta+\alpha+1},\qquad \widehat{\theta}_{WB}=\omega\frac{\overline{X}}{\delta}+(1-\omega)\frac{\beta+\sum_{i=1}^{n}X_i}{n\delta+\alpha+1}.$$

It should be mentioned here that when the prior parameters α and β are unknown, we may use the empirical Bayes approach to estimate them. Substitution of $\widehat{\alpha}$ and $\widehat{\beta}$, given by (2.4), in $\widehat{\theta}_{WB}$ yields the empirical Bayes estimator of θ under the WBLF (1.4).

4. Numerical Computations

In this section, we present a numerical example and a Monte Carlo simulation study to illustrate the methods of inference developed in Section 2.

4.1. Illustrative Example

The MRE, Bayes and empirical Bayes estimates of the parameter θ are compared according to the following steps:

1. For given values of prior parameters ($\alpha=2,\ \beta=1$) , we generate $\theta=1.024$ from the inverted gamma prior (2.2). The S-PLUS package is used in the generation of the inverted gamma random variates.

2. Based on the generated value $\theta=1.024$, a random sample of size $n=10$ is then generated from the gamma distribution $\Gamma(\delta=1,\ \theta=1.024)$ defined by (1.1). This sample is

   4.687 0.719 0.055 2.576 1.258 1.399 0.578 0.785 0.169
   1.429

3. Using these data, the MRE and Bayes estimates of θ are computed from (2.1) and (2.3) with $\omega=0.5$ and given, respectively by $\widehat{\theta}_{MRE}=1.303$ and $\widehat{\theta}_B=1.348$.

4. For given values of α, β and n, we generate a random sample (past data) $Z_{j,n}$, $j=1,2,\ldots,m$ of size $m=15$ from the marginal density of $Z_{j,n}$, given by (2.5), as

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0.277 0.294 0.617 0.549 1.270 0.211 0.342 0.103 1.651 5.653
0.216 0.125 0.772 0.159 0.581.
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The moment estimates $\widehat{\alpha} = 2.458$ and $\widehat{\beta} = 1.246$ are then computed by using (2.6).

5. By using the estimates of $\widehat{\alpha} = 2.458$ and $\widehat{\beta} = 1.246$ in (2.7) with $\omega = 0.5$, the empirical Bayes estimate of θ is $\widehat{\theta}_{EB} = 1.073$.

4.2. Simulation Results

The estimates obtained in Section 2 are compared based on Monte Carlo simulation as follows:

1. For given values of prior parameters α and β, we generate θ from the prior density (2.2) and then samples of different sizes n are generated from the gamma distribution $\Gamma(\delta,\ \theta)$ with $\delta = 1$.
2. The MRE estimate of θ is computed from (2.1).
3. For given value of ω, the Bayes estimate of θ is computed from (2.3).
4. The empirical Bayes estimate θ is computed from (2.7).

Table 4.1: Estimated risk (ER) of the parameter θ for different values of $n,\ m,\ \omega$ and 5,000 repetitions ($\alpha = 2,\ \beta = 1,\ \delta = 1$).

n	ω	$ER(\widehat{\theta}_{MRE})$	$ER(\widehat{\theta}_B)$	$ER(\widehat{\theta}_{EB})$	
				$m = 15$	$m = 20$
	0.4	0.1371	0.1363	0.1343	0.1315
20	0.6	0.2053	0.2052	0.1765	0.1686
	0.8	0.1639	0.1638	0.1685	0.1674
	0.4	0.0984	0.0980	0.0967	0.0961
30	0.6	0.1027	0.1024	0.1031	0.1028
	0.8	0.0740	0.0739	0.0738	0.0736
	0.4	0.0541	0.0539	0.0547	0.0543
40	0.6	0.0728	0.0727	0.0770	0.0766
	0.8	0.0577	0.0577	0.0571	0.0569

5. The square deviations $(\theta^* - \theta)^2$ are compared for different sizes n where θ^* stands an estimate (MRE, Bayes, or empirical Bayes) of the parameter θ.

6. The above steps are repeated $5,000$ times and the estimated risk (ER) is computed by averaging the the squared deviations over the $5,000$ repetitions. The computational results are displayed in Table 4.1.

It is observed from Table 4.1, the empirical Bayes estimates are almost as efficient as the Bayes estimates for all sample sizes considered. Furthermore, the estimated risks of three methods of estimation are decreasing when n and m are increasing. Different values of the prior parameters α and β rather those listed in Table 4.1 have been considered but did not change the previous conclusion.

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