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AXES OF A MINIMAL SURFACE WITH PLANAR ENDS

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Abstract. In this article, we consider axes of a minimal surface in $\mathbf{R}^3$ of genus zero with a planar end, and then prove that two consecutive axes near the planar end must be parallel but cannot be in a same line.

1. Introduction

An immersed surface in $\mathbf{R}^3$ is said to be *minimal* if its mean curvature vanishes identically. Recall the catenoid is the unique nonplanar minimal surface of revolution, and so is really the simplest complete minimal surface after the plane. Topologically it is a sphere $\mathbf{S}^2$ minus two points, and outside of a sufficiently large compact set of $\mathbf{R}^3$ it consists of two unbounded components corresponding to the two punctures in $\mathbf{S}^2$.

In this article, we consider a minimal surface of genus zero with a planar end and try to look at that similar to the axis of the rotation of catenoid.

From the physical point of view, minimal surfaces in $\mathbf{R}^3$ are objects submitted to a balanced force system, consisting in the forces associated to non-zero one-dimensional homology classes in the surfaces. More precisely, each closed curve γ in a minimal surface M carries a force that expresses the stress produced by an unit conormal vector field ν along this curve on the whole surface. The action of ν provides a tendency of translation, or linear momentum, which we call the *flux vector*. On the other hand, another action relates a tendency of rotation around an axis, or angular momentum, which is expressed by the *torque vector* of M along γ. These objects have been deeply studied by Kusner in [3], and they and their modifications have only recently come into widespread use in the study of minimal and constant mean curvature surfaces, see

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[3] and [4]. In particular, with the simple calculation, we can compute the flux of the catenoid in the direction of the axis of rotation.

2. Flux and Torque

Let S be a compact domain of a Riemann surface, and let

$$X : S \to \mathbf{R}^3$$

be an isometic immersion. Applying Stoke's theorem we have that;

$$\int_S \Delta_S X \, dA = \int_{\partial S} \nu ds$$

where dA is the element of area on S, Δ_S is the Laplacian on S, ds is the line element on ∂S, and ν is the outward unit *conormal* which is tangent to $X(S)$ but normal to $\partial X(S)$. More precisely,

$$\nu = dX\,(\vec{n})$$

where $\vec{n}$ is the unit vector orthogonal to the unit vector $\vec{s}$ tangent to ∂S and $(\vec{n}, \vec{s})$ gives the orientation of S, see Figure 1.

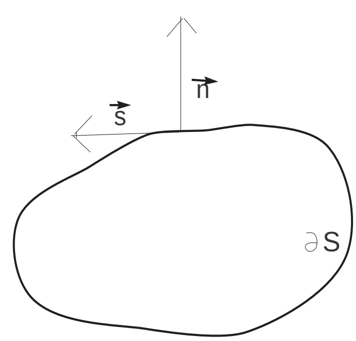

FIGURE 1.

If X is minimal and S is equipped with the metric induced by X, then

$$\Delta_S X = -H\vec{N} = (0,0,0)$$

where H is the mean curvature of S and $\vec{N}$ is the outward unit normal of $X(S)$. Thus we have;

$$\int_{\partial S} \nu ds = (0,0,0).$$

DEFINITION 2.1. Let $X : S \hookrightarrow \mathbf{R}^3$ be a minimal surface and $\gamma \subset S$ is a closed curve. Then, under the metric induced by X, we define *the flux of X along* γ as that;

$$Flux(\gamma) := \int_\gamma \nu \, ds$$

The flux is well defined on the homology class of $[\gamma]$. In fact, if $\tilde{\gamma} \in [\gamma]$ then $\gamma \cup \tilde{\gamma}$ bounds a domain Ω and we have;

$$0 = \int_\Omega \Delta_S X \, dA = \int_\gamma \nu \, ds - \int_{\tilde{\gamma}} \nu \, ds.$$

Now let $R_{\vec{u}}$ be the Killing field associated with counter-clockwise rotation about the axis $\ell_{\vec{u}}$ in the $\vec{u}$ direction. From the identity

$$(U \wedge V) \cdot W = det(U, V, W)$$

for vectors U, V, W in $\mathbf{R}^3$, we have;

$$(X \wedge \nu) \cdot \vec{u} = (\vec{u} \wedge X) \cdot \nu = R_{\vec{u}} \cdot \nu$$

where X is the position vector of a minimal surface defined on S. Because $R_{\vec{u}}$ is a Killing field, $\int_\gamma R_{\vec{u}} \cdot \nu \, ds$ is also a homology invariant and

$$\int_{\partial S} R_{\vec{u}} \cdot \nu \, ds = 0.$$

This motivates defining *the torque* of a closed curve γ on S as the vector-valued quantity.

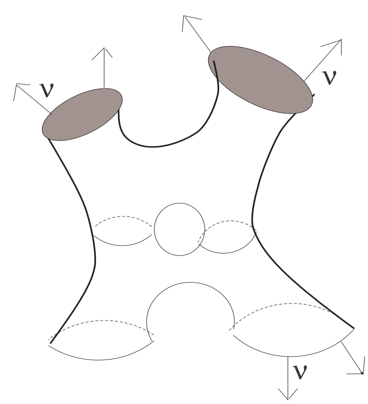

FIGURE 2.

DEFINITION 2.2. The torque of the minimal surface S along γ at $O := (0,0,0)$ is defined by;

$$Torque_O(\gamma) = \int_\gamma X \wedge \nu \, ds.$$

In general, the torque is dependent on the base point of the position vector X. If we move the base point from O to $W \in R^3$, then the position vector based on W is changed to $X - W$, and the torque is

$$\begin{aligned} Torque_W(\gamma) &= \int_\gamma (X - W) \wedge \nu \, ds \\ &= Torque_O(\gamma) - W \wedge Flux(\gamma). \end{aligned}$$

It follows that the torque of γ does not depend upon the base point of X if the flux of γ vanishes.

3. Planar ends

Let $X : S \hookrightarrow \mathbf{R}^3$ be a properly minimal immersion of finite total curvature with embedded ends. Denote by

$$X(S) = M.$$

Then it is well-known that, by Osserman [6], there exist $p_1, \cdots, p_\ell$ in the closure $\bar{S}$ of the Riemann surface S with

$$S = \bar{S} \setminus \{p_1, \cdots, p_\ell\}$$

such that the stereographic projection of the Gauss map $g : S \to \mathbf{C}$, just say *the Gauss map* of the minimal surface, extends to a holomorphic map $g : \bar{S} \to \mathbf{C}$. Let $E_1, \cdots, E_\ell$ be the ends of M corresponding to the punctures $p_1, \cdots, p_\ell$, respectively. In [8], Schoen proved that each E_i, $1 \leq i \leq \ell$, is the graph of a function u with bounded slope over the exterior of a bounded region in some horizontal plane Π_i by;

$$u(x_1, x_2) = \beta + \alpha \log r + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}) \tag{1}$$

for $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ sufficiently large, where β, α, γ_1 and γ_2 are real constants depending on E_i. If $\alpha = 0$ then the end is asymptotic to a plane, we say it a *planar end*, otherwise it is asymptotic to a catenoid.

Let E_i be a planar end, then its Gauss map g has a zero or a pole of order $k \geq 2$ at the corresponding puncture p_i, i.e., there is a coordinate $z \in \mathbf{C}$ such that;

$$g(z) = (z - p_i)^{\pm k} \tag{2}$$

in a small neighborhood of p_i. Additionally, $(E_i \cap \Pi_i) \setminus B$, B being a large ball, consists of $2k-2$ curves which are asymptotic to $2k-2$ rays on Π_i making an equal angle of $\pi/(k-1)$. In particular, if g has a zero (or a pole) of the minimum branching order 2 at the puncture, then $M \cap \Pi_i$ is an immersion of $\mathbf{R}^1$ which is asymptotically parallel to the line;

$$\gamma_1 x_1 + \gamma_2 x_2 = 0 \tag{3}$$

in the horizontal plane Π_i, see (1).

Note that we can take a representative curve of the planar end E_i. Precisely, it is the image of a boundary of a sufficiently small neighborhood of the puncture p_i in the domain, for example, $E_i \cap \partial B$.

PROPOSITION 3.1 ([2]). *We define the flux and the torque associated to a planar end E with ramification order $k > 1$ in (2) as that of one representative curve for the end. If it is the graph of a function u defined in (1) over a horizontal plane, then we can compute that;*

$$\begin{aligned} Flux(E) &= (0,0,0) \\ Torque(E) &= \begin{cases} -\pi(-\gamma_2, \gamma_1, 0) & \text{if } k = 2 \\ (0,0,0) & \text{if } k > 2. \end{cases} \end{aligned} \tag{4}$$

Proof. Take a representative curve $\Gamma := E \cap C_R$ of E where

$$C_R = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 = R^2\}$$

is a right cylinder for a sufficiently large $R > 0$. Then the position vector $\Gamma(\theta)$ and the conormal vector ν of the curve are given by;

$$\begin{aligned} \Gamma(\theta) &= \left(R\cos\theta,\, R\sin\theta,\, \frac{1}{R}(\gamma_1 \cos\theta + \gamma_2 \sin\theta) + O(R^{-2}) \right) \\ \nu(\theta) &= (\cos\theta, \sin\theta, 0) + O(R^{-2}), \end{aligned}$$

where $0 \leq \theta \leq 2\pi$, respectively. First then we have;

$$Flux(E) = \int_0^{2\pi} \nu(\theta)\, R\, d\theta = (0,0,0),$$

as $R \to \infty$. It also follows that the torque of a planar end does not depend upon a base point. Also, as $R \to \infty$, we compute the torque;

$$Torque(E) = \int_0^{2\pi} \Gamma(\theta) \wedge \nu(\theta)\, R\, d\theta = \pi(-\gamma_2, \gamma_1, 0).$$

□

Note that if E has the minimum branching order, then the torque of E is the direction of $E \cap \Pi$ at infinity, see (3).

4. Main results

We denote a horizontal plane $\Pi_t := \{(x_1, x_2, x_3) |\, x_3 = t\}$ for some $t \in \mathbf{R}$. Let E be a planar end of a minimal surface M asymptotic to Π_0. Suppose that $M \cap \Pi_0$ is an immersion of $\mathbf{R}^1$ parallel to a line at infinity, i.e., E has the minimum branching order. Then there is a sufficiently small $\epsilon > 0$ such that each intermediate curve $M \cap \Pi_t$ is closed Jordan curve for all $t \in [-\epsilon, 0) \cup (0, \epsilon]$. Denote a part of M lying on the slab $S(-\epsilon, \epsilon) := \{(x_1, x_2, x_3) |\, -\epsilon < x_3 < \epsilon\}$ by

$$M_0 := M \cap S(-\epsilon, \epsilon)$$

which is a minimal annulus with a planar end. So there is $r > 1$ such that M_0 is conformally equivalent to a punctured annulus $A_r \setminus \{p\}$ where

$$A_r := \{z \in \mathbf{C} \,|\, 1/r < |z| < r\}, \quad p \in A_r.$$

Let $X : A_r \setminus \{p\} \hookrightarrow \mathbf{R}^3$ be a minimal surface with $M_0 = X(A_r \setminus \{p\})$.

Lemma 4.1. *There is a base point $Q_t \in \Pi_t$, $-\epsilon \le t \le \epsilon$ and $t \neq 0$, such that $Torque_{Q_t}(\gamma_t)$ is vertical where $\gamma_t = M \cap \Pi_t$.*

Proof. Let $X = (X^1, X^2, X^3)$ and take a point $Q = (Q^1, Q^2, Q^3)$ in $\mathbf{R}^3$, then $X^3 = Q^3$ along γ_t clearly. The torque vector of γ_t at Q is

$$\begin{aligned} Torque_Q(\gamma_t) &= \int_{\gamma_t} (X - Q) \wedge \nu \, ds \\ &= \int_{\gamma_t} \big((X^2 - Q^2)\nu_3, -(X^1 - Q^1)\nu_3.(X^1 - Q^1)\nu_2 - (X^2 - Q^2)\nu_1\big) \; ds \end{aligned}$$

Since M meets the horizontal plane Π_t transversally, we may assume that $\nu_3 > 0$ in γ_t. If $\tilde{Q}^1 < 0$ and $|\tilde{Q}^1|$ is sufficiently large, then

$$\int_{\gamma_t} (X^1 - \tilde{Q}^1)\nu_3 \, ds > 0$$

see Figure 3. On the contrary, if $\hat{Q}^1 > 0$ and $|\hat{Q}^1|$ is also large, then $\int_{\gamma_t} (X^1 - \hat{Q}^1)\nu_3 \, ds < 0$. Therefore, we have a real number Q_t^1 such that;

$$\int_{\gamma_t} (X^1 - Q_t^1)\nu_3 \, ds = 0.$$

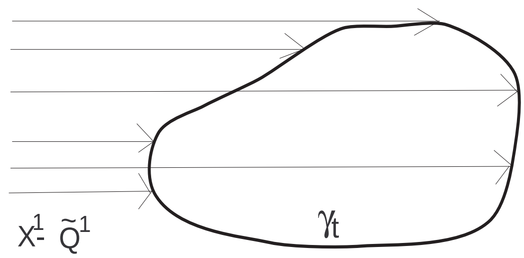

FIGURE 3.

Similarly, there is $Q_t^2 \in \mathbf{R}$ with $\int_{\gamma_t}(X^2 - Q_t^2)\nu_3\, ds = 0$. Take the point $Q_t = (Q_t^1, Q_t^2, t) \in \mathbf{R}^3$, then $Torque_{Q_t}(\gamma_t)$ must be vertical. It finishes the proof of the theorem. □

Now we will refer a flux of M_0 by; Let Γ be a representative curve of the end and let M_0^- and M_0^+ be the lower and upper parts of $M_0 \setminus \Pi_0$, respectively, defined by the followings;

$$M_0^- := M_0 \cap \{-\epsilon < x_3 < 0\}, \quad M_0^+ := M_0 \cap \{0 < x_3 < \epsilon\}.$$

Take two Jordan curves $\Gamma^- \subset M_0^-$ and $\Gamma^+ \subset M_0^+$, then $\Gamma \cup \Gamma^- \cup \Gamma^+$ bounds a compact minimal surface, precisely, it is the image of a compact subset of the domain $A_r \setminus \{p\}$, see Figure 4. So

$$Flux(\Gamma^+) - Flux(\Gamma) - Flux(\Gamma^-) = (0, 0, 0).$$

Recall the flux vector of a planar end $Flux(\Gamma)$ must be vanish, and then we have;

$$Flux(\Gamma^+) = Flux(\Gamma^-).$$

In other words, each intermediate closed Jordan curve of M_0 has the same flux. We refer it *a flux of the minimal surface* and denote by $Flux(M_0)$.

THEOREM 4.2. *Let $Q_t \in \Pi_t$, $-\epsilon \le t < 0$ (resp., $0 < t \le \epsilon$), be a base point such that $Torque_{Q_t}(\gamma_t)$, $\gamma_t = M \cap \Pi_t$, is vertical. We call it a vertical base point. Then all vertical base points of M_0^- (resp., M_0^+) are lying on a line ℓ_- (resp., ℓ_+) in the direction of $Flux(M_0)$.*

Proof. Let both $Torque_{Q_{t_1}}(\gamma_{t_1})$ and $Torque_{Q_{t_2}}(\gamma_{t_2})$ be vertical vectors where $-\epsilon \le t_1 < t_2 < 0$ or $0 < t_1 < t_2 \le \epsilon$. Since γ_{t_1} and γ_{t_2} are

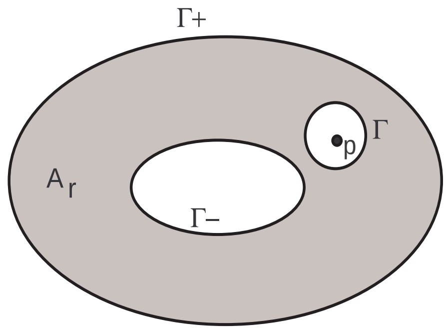

FIGURE 4.

homologous, we have;

$$\begin{aligned} Torque_{Q_{t_2}}(\gamma_{t_2}) &= Torque_{Q_{t_2}}(\gamma_{t_1}) \\ &:= \int_{\gamma_{t_1}} (X - Q_{t_2}) \wedge \nu \, ds \\ &= \int_{\gamma_{t_1}} (X - Q_{t_1}) \wedge \nu \, ds + \int_{\gamma_{t_1}} (Q_{t_1} - Q_{t_2}) \wedge \nu \, ds \\ &= Torque_{Q_{t_1}}(\gamma_{t_1}) + (Q_{t_1} - Q_{t_2}) \wedge Flux(M_0) \end{aligned}$$

It follows that $(Q_{t_1} - Q_{t_2}) \wedge Flux(M_0)$ is also vertical vector. Observe both $Q_{t_1} - Q_{t_2}$ and $Flux(M_0)$ cannot be horizontal, so it must be;

$$(Q_{t_1} - Q_{t_2}) \wedge Flux(M_0) = (0, 0, 0).$$

Therefore, $(Q_{t_1} - Q_{t_2})$ and $Flux(M_0)$ are parallel.

We refer to these straight lines *axes* ℓ_- and ℓ_+ as the *axes* of M_0^- and M_0^+, respectively. The above theorem implies that all axes have the direction of $Flux(M_0)$. □

THEOREM 4.3. *Two consecutive axes ℓ_- and ℓ_+ near the planar end must be parallel but not be in a same line.*

Proof. The first statement is clear, so we enough to show the secondary one. Let Γ be a representative curve of the planar end E, and $\gamma_\epsilon = M \cap \Pi_\epsilon$, $\gamma_{-\epsilon} = M \cap \Pi_{-\epsilon}$. Since $\Gamma \cup \gamma_\epsilon \cup \gamma_{-\epsilon}$ bounds a compact minimal surface, we have

$$Torque_P(\gamma_\epsilon) - Torque_P(\Gamma) - Torque_P(\gamma_{-\epsilon}) = (0, 0, 0)$$

for all points $P \in \mathbf{R}^3$.

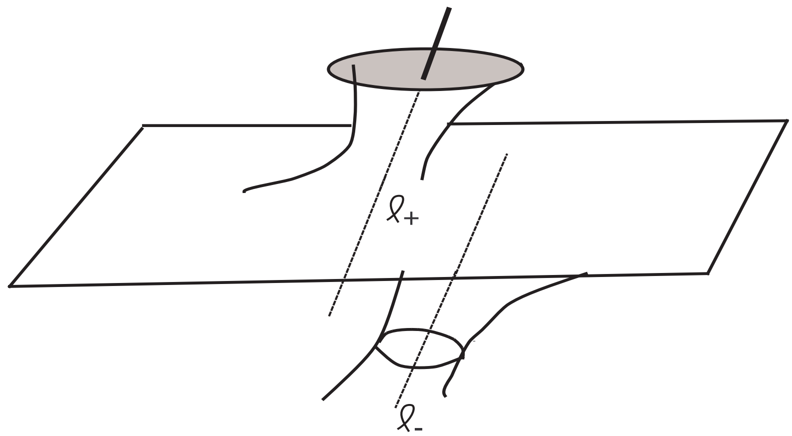

FIGURE 5.

Suppose that vertical base points $Q_\epsilon \in \Pi_\epsilon$ and $Q_{-\epsilon} \in \Pi_{-\epsilon}$ are lying on a same axis line. Then the vector $(Q_\epsilon - Q_{-\epsilon})$ must be parallel to $Flux(M_0)$, and hence;

$$\begin{aligned} Torque_P(\Gamma) &= Torque_P(\gamma_\epsilon) - Torque_P(\gamma_{-\epsilon}) \\ &= Torque_{Q_\epsilon}(\gamma_\epsilon) - Torque_{Q_{-\epsilon}}(\gamma_{-\epsilon}) + (Q_\epsilon - Q_{-\epsilon}) \wedge Flux(M_0) \\ &= Torque_{Q_\epsilon}(\gamma_\epsilon) - Torque_{Q_{-\epsilon}}(\gamma_{-\epsilon}) \end{aligned}$$

which is also vertical and independent under the base point as the torque of a planar end. However, the torque of a planar end must be horizontal, see (4). Therefore we have;

$$Torque(E) := Torque_P(\Gamma) = (0,0,0)$$

which contradicts that E has the minimum branching order, as in (4) of the proposition in Section 3, too. □

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