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EXISTENCE OF SOLUTIONS FOR GRADIENT TYPE ELLIPTIC SYSTEMS WITH LINKING METHODS

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Abstract. We study the existence of nontrivial solutions of the Gradient type Dirichlet boundary value problem for elliptic systems of the form $-\Delta U(x) = \nabla F(x, U(x)), x \in \Omega$, where $\Omega \subset R^N (N \geq 1)$ is a bounded regular domain and $U = (u, v) : \Omega \to R^2$. To study the system we use the liking theorem on product space.

1. Introduction

The elliptic system has an extensive practical background. It can be used to describe the multiplicate chemical reaction catalyzed by the catalyst grains under constant or variant temperature, and can be a simple model of tubular chemical reaction, more naturally, it can be a correspondence of the stable station of dynamical system determined by the reaction-diffusion system(see [3]).

Second order elliptic systems whose principal part is given by the differential operator $-\Delta$, where $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$, and we will discuss systems of the form

$$-\Delta u = f(x, u, v), -\Delta v = g(x, u, v) \quad in \quad \Omega.$$

We say that the system above is of Gradient type if there exists a function $F : \bar{\Omega} \times R \times R \to R$ of class C^1 such that

$$\frac{\partial F}{\partial u} = f, \frac{\partial F}{\partial v} = g.$$

The above system is said to be of Hamiltonean type if there exists a function $H : \bar{\Omega} \times R \times R \to R$ of class C^1 such that

$$\frac{\partial H}{\partial v} = f, \frac{\partial H}{\partial u} = g.$$

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In this paper we study the existence of nontrivial solutions of a Gradient type elliptic systems of the following form

$$(1.1)\qquad \begin{aligned} -\Delta U(x) &= \nabla F(x, U(x)) \quad in \quad \Omega, \\ U(x) &= 0 \quad in \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset R^n$ is a bounded regular domain, and there exists a function $f_1(x,u,v), f_2(x,u,v)$ such that

$$\nabla F(x,u,v) = \big(\frac{\partial}{\partial u}F(x,u,v), \frac{\partial}{\partial v}F(x,u,v)\big) = (f_1(x,u,v), f_2(x,u,v)).$$

without loss of generality, we set

$$F(x,u,v) = \int_{(0,0)}^{(u,v)} f_1(x,u,v)du + f_2(x,u,v)dv,$$

where , $f_1(x,u,v), f_2(x,u,v)$ are Caratheodory functions.

In this paper, we first obtain some abstract linking theorems on product space. As an application, we consider the existence of two nontrivial solutions of the elliptic systems (1.1).

2. Preliminaries

We shall work in the functional space $H \times H$ where $H := W_0^{1,2}(\Omega)$. We shall endow $H \times H$ with the Hilbert structure induced by the inner product

$$(U, V)_{H\times H} = \int_\Omega \nabla u(x)\nabla\phi(x)dx + \int_\Omega \nabla v(x)\nabla\varphi(x)dx,$$

where $U = (u,v), V = (\phi,\varphi)$. We denote the corresponding norm by $\|\cdot\|$. We define the energy functional associated to (1.1) as

$$I(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \int_\Omega F(x,u,v)dx.$$

Let $C^1(H,R)$ denote the set of all functionals which are Fréchet differentiable and whose Fréchet derivatives are continuous on H. It is easy to see that $I \in C^1(H\times H, R)$ and thus it makes sense to lock for solutions to (1.1) in the weak sense as critical points for I, i.e. $U = (u,v) \in H\times H$ such that $I'(u,v) = 0$, where

$$I'(U)V = \int_\Omega [\nabla u(x)\nabla\phi(x) + \nabla v(x)\nabla\varphi(x) - (f_1\phi(x) + f_2\varphi(x))]dx,$$

$$I'(U)V = \langle \nabla I(u,v), (\phi,\varphi)\rangle_{H\times H}, \qquad \forall V = (\phi,\varphi) \in H.$$

It is interesting that the concept of linking is of important in critical point theory.To the average person two objects are said to be linked if they cannot be pulled apart. This is basically the idea we shall use in finding critical points. Let E be a Banach space. We introduce the set Φ of mapping $\Gamma(t) \in C(E \times [0,1], E)$ with the following properties:

- for each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0,1)$
- $\Gamma(0) = I$
- for each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$ and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of E.

Definition 2.1. A subset a of E links a subset B of E if $A \cap B = \emptyset$ and for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$.

Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$. To introduce a variation of linking theorem on product space, we define the following sets. Let M be a Hilbert space and V a C^2 complete connected Finsler manifold. Suppose $M = M_1 \oplus M_2$ where M_1 are finite dimensional subspaces of M. Let $0 < \delta < R$, $e_1 \in M_1$ moreover, consider

$$Q_R = \{se_1 + u : u \in M_2, s \geq 0\|se_1 + u\| \leq R\},$$

$$S_\delta = B_\delta \cap M_1,$$

then ∂Q_R links ∂S_δ.

We recall a theorem of existence of two critical levels for a functional which is a linking theorem on product space.

Theorem 2.2. *Suppose*

$$\sup_{\partial S_\delta \times V} I < \inf_{\partial Q_R \times V} I$$

$$\inf_{Q_R \times V} I > -\infty, \qquad \sup_{S_\delta \times V} I < +\infty,$$

and that I satisfies $(PS)_c^*$ with respect to X, for every

$$c \in [\inf_{Q_R \times V} I, \sup_{S_\delta \times V} I].$$

Then I admits at least two distinct critical values c_1, c_2 such that

$$\inf_{Q_R\times V} I \le c_1 \le \sup_{\partial S_\delta\times V} I < \inf_{\partial Q_R\times V} I \le c_2 \le \sup_{S_\delta\times V} I,$$

and at least $2+2$ cuplength(V) distinct critical points.

3. An application

We consider the following assumptions:

(f_1) There exist $\alpha > 2$ and $R > 0$ such that, for all $uv \neq 0$ and $\|u\| + \|v\| > R$ a.e. $x \in \Omega$,

$$f_1(x,u,v)u + f_2(x,u,v)v \ge \alpha F(x,u,v) > 0.$$

(f_2) there are $p \in (1, 2^*), q \in (2,\infty)$ such that for all (x, U),

$$F(x,U) \ge \gamma_1(|u|^p + |v|^q) - \gamma_2.$$

(f_3) When $|v| \to 0$,

$$\frac{f_2(0,v)}{v^2} \to 0.$$

A critical point of I is a solution U of (1.1) and the value of I at U is a critical value of I. Ekeland's variational principle implies the existence of a sequence (U_m) such that $I(U_n) \to c, I\prime(U_n) \to 0$. Such a sequence is called a Palais-Smale sequence at level c. The functional I satisfies the $(PS)_c$ condition if any Palais-Smale sequence at level c has a convergent subsequence. If I is bounded from below and satisfies the $(PS)_c$ condition at level c, then c is a critical value of I. Following we prove that the functional I satisfies the $(PS)_c$ condition.

LEMMA 3.1. *If (f_1), (f_2), (f_3) hold. Then functional $I(u,v)$ satisfies the $(PS)_c$ condition.*

Proof. Let $\{U_n\} = \{(u_n, v_n)\}$ be a sequence in $E = H \times H$ such that $|I(U_n)| \le C$ and $I'(U_n) \to 0$ as $n \to \infty$. First we prove that $\{U_n\} = \{(u_n, v_n)\}$ is bounded. Choose $\beta \in (\alpha^{-1}, 2^{-1})$. For large n, by condition H_1 we have

$$\begin{aligned} C_0 + o(1)\|U_n\| &\ge I(U_n) - \beta\langle\nabla I(U_n), U_n)\rangle_{H\times H} \\ &= (\frac{1}{2} - \beta)\|U_n\|^2 - \int_\Omega F(U_n)dx \\ &\quad +\beta\int_\Omega (f_1(U_n)u_n + f_2(U_n)v_n)dx \\ &\ge (\frac{1}{2} - \beta)\|U_n\|^2 + (\beta\alpha - 1)\int_\Omega F(U_n)dx. \end{aligned}$$

We obtain

$$C_0 + o(1)\|U_n\| \geq (\frac{1}{2} - \beta)\|U_n\|^2,$$

where $o(1) \to 0$ as $n \to \infty$. This implies that $\{U_n\}$ is bounded in E. Hence there exists a subsequence $\{U_{kj}\}_{j=1}^{\infty}$ and $U \in E$, with $U_{kj} \to U$ weakly in E, and $\{U_{kj}\}_{j=1}^{\infty}$ in $L^p \times L^p$ for $1 \leq p < 2^*$, by the Rellich-Kondrachov compactness theorem. We can compute that $\|U_{kj}\| \to \|U\|$, so $U_{kj} \to U$ in $H \times H$ and thus I satisfies (PS) condition. □

Let $W = H_0^1 \times \{0\} = span\{e_i^-, i \in N\}$, $Z = \{0\} \times H_0^1 = span\{e_i^+, i \in N\}$, where $e_i^{\pm}$ are the eigenfunctions associated to $\lambda_i^{\pm} = \pm\lambda_i(-\Delta)$ (where $\lambda_i(-\Delta)$ denotes the ith eigenvalue of the Laplace operator on H_0^1, with associated the eigenfunction e_i) and $e_i^+ = (0, e_i)$, $e_i^- = (e_i, 0)$. From now on $\|\cdot\|$ will denote the H_0^1 norm.

Theorem 3.2. *Suppose (f_1), (f_2), (f_3) holds. Then problem (1.1) has at least two nontrivial solutions.*

Proof. In fact, by (f_1), (f_2), for every $u \in W$

$$\begin{aligned} I(u, Re_1^+) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla Re_1^+|^2)dx - \int_\Omega F(x, u, Re_1^+)dx, \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\lambda_1 R^2 - \int_\Omega F(x, u, Re_1)dx, \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\lambda_1 R^2 - \int_\Omega [\gamma_1(|u|^p + |Re_1|^q) - \gamma_2]dx, \end{aligned}$$

since $q > 2$, by standard inequalities there exist a $R^\star > 0$ such that for $R > R^\star$ we have $\sup_{W \oplus Re_1^+} I < 0$.

Given $\epsilon > 0$, by (f_3) there exists a $\rho > 0$ such that, for every $v \in Z \ominus e_1^+$ with $\|v\| < \rho$ we have

$$\begin{aligned} I(0, v) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 - \int_\Omega F(x, 0, v)dx, \\ &\geq \frac{1}{2}\|v\|^2 - \epsilon\|v\|^2, \\ &> 0. \end{aligned}$$

Thus we have $0 < \inf_{\partial B_\rho(Z \ominus e_1^+)} I$.

We can apply Theorem 2.1 with $B_\rho \times V = B_\rho(Z \ominus e_1^+)$, $Q_R \times V = W \oplus Re_1^+$. Indeed we can find $R > 0$, $\rho > 0$ such that $R > \rho > 0$ and

$$\sup_{\partial Q_R \times V} I < 0 < \inf_{\partial B_\rho \times V} I,$$

By Theorem 2.1, $I(u,v)$ has at least two nonzero critical value c_1, c_2

$$c_1 < \sup_{\partial Q_R \times V} I < 0 < \inf_{\partial B_\rho \times V} I < c_2.$$

Therefore, (1.1) has at least two nontrivial solutions. □

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