JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 1, March 2007

ON SOME GENERALIZED OPERATOR EQUILIBRIUM PROBLEMS†

Won Kyu Kim*

Abstract. In this paper, we will introduce the generalized operator equlibrium problem and generalized operator quasi-equlibrium problem which generalize operator equlibrium problem due to Kazmi and Raouf into multi-valued and quasi-equlibrium problems. Using a Park's fixed point theorem, we will prove a new existence theorem on generalized operator equlibrium problem which serves as a basic existence theorem for various kinds of nonlinear problems.

1. Introduction

In a recent paper, Domokos and Kolumbán [2] gave an interesting interpretation of variational inequalities and vector variational inequalities in Banach space settings in terms of variational inequalities with operator solutions (in short, OVVI). They designed (OVVI) to provide a suitable unified approach to several kinds of variational inequality and vector variational inequality problems in Banach spaces, and successfully described these problems in a wider context of (OVVI). Inspired by their work, in recent consecutive papers [6,7], Kum and Kim developed the scheme of (OVVI) from the single-valued case into general multi-valued settings.

On the other hand, the equilibrium problem (EP) is being intensively studied, beginning with Blume and Oettli [1] where they proposed it as a generalization of optimization and variational inequality problem. It turns out that this problem includes also other problems such as the fixed point and coincidence point problem, the complementarity problem, the Nash equilibrium problem, etc. Recently, Kazmi and Raouf [4]

Received January 30, 2007.
2000 Mathematics Subject Classification: Primary 47J20, 52A07.
Key words and phrases: generalized operator equlibrium problem, natural quasi P-convex.
† This work was supported by the research grant of the Chungbuk National University in 2006.

introduced the operator equilibrium problem (OEP) which extends the notion of (OVVI) to operator equilibrium problems by using the operator solution concept due to Domokos and Kolumbán in [2]. Using the KKM lemma due to Fan [3], they first obtained two existence theorems of the solutions of (OEP) with $C(f)$-pseudo monotonicity and without $C(f)$-pseudo monotonicity. Also they presented two general version of (OEP) in general settings as main applications. However, they dealt only with the single-valued case of the bi-operator F.

In this paper, we will introduce the new definitions of generalized operator equlibrium problem (GOEP) and generalized operator quasi-equlibrium problem (GOQEP)which generalize (OEP) into multi-valued and quasi-equlibrium problems. Next, using a Park's fixed point theorem [7], we will prove a new existence theorem on generalized operator equlibrium problem (GOEP) which serves as a basic existence theorem for various kinds of nonlinear problems.

2. Preliminaries

Let E be a locally convex Hausdorff topological vector space, X a non-empty convex subset of E, F another locally convex Hausdorff topological vector space. A non-empty subset P of E is called a *convex cone* if

$$\lambda P \subseteq P, \quad \text{for all} \quad \lambda > 0, \quad \text{and} \quad P + P = P.$$

From now on, unless otherwise specified, we work under the following settings. Let $L(E, F)$ be the space of all continuous linear operators from E to F, and X' a non-empty convex subset of $L(E, F)$, and $A : X' \rightrightarrows X'$ be a multifunction. Let $C : X' \rightrightarrows F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with non-empty interior and $0 \notin C(f)$. We denote the pairing between X' and E by $\langle f, x\rangle$ for $f \in X'$ and $x \in E$. Denote $P := \bigcap_{f\in X'} C(f)$.

We first introduce the following monotone condition for multifunctions.

Definition 2.1. A multifunction $T : X' \times X' \rightrightarrows F$ is called
(1) *$C(f)$-pseudomonotone* if for any $f, g \in X'$, we have

$$T(f, g) \nsubseteq -\text{int}\, C(f) \ \text{ implies } \ T(g, f) \nsubseteq -\text{int}\, C(f); \text{ and}$$

(2) *generalized hemicontinuous* if for any $f, g \in X'$ and $\alpha \in [0, 1]$, the multifunction $\alpha \mapsto T(f + \alpha g, g)$ is upper semicontinuous at 0^+, where

$$T(f + \alpha g, g) = \{s \in F \mid s \in T(f + \alpha g, g)\}.$$

Next, we introduce the following convex condition which generalizes the Definition 1.2 in [2] to multi-valued settings:

Definition 2.2. A multifunction $T : X' \rightrightarrows F$ is called *natural quasi P-convex* if for any $f, g \in X'$ and $\lambda \in [0,1]$, we have

$$T(\lambda f + (1-\lambda)g) \subseteq co\big(T(f), T(g)\big) - P,$$

where $\mathrm{co}(A,B)$ denotes the set $\{\lambda u + (1-\lambda)v \in F \mid u \in A, v \in B, \lambda \in [0,1]\}$.

Then it is clear that the natural quasi P-convexity generalizes the ordinary convex condition by letting $C(f) := [0,\infty)$ when $E = F = X' = \mathbb{R}$. Using the induction argument, we can obtain the following condition $(*)$ which is equivalent to the natural quasi P-convexity of T as follows:

For every $n \geq 2$, whenever $g_1, \ldots, g_n \in X'$ are given and for any $\lambda_i \in [0,1], i = 1, \ldots, n,$ with $\sum_{i=1}^n \lambda_i = 1$, there exist $\mu_i \in [0,1], i = 1, \ldots, n,$ with $\sum_{i=1}^n \mu_i = 1$ such that

$$T(\sum_{i=1}^n \lambda_i\, g_i) \subseteq \mu_1 T(g_1) + \cdots + \mu_n T(g_n) - P. \qquad (*)$$

In the sequel, we shall assume that a multi-valued bi-operator $T : X' \times X' \rightrightarrows F$ such that $T(f,f) = 0$ for each $f \in X'$ is given, and $A : X' \rightrightarrows X'$ is given.

Then the *generalized operator equilibrium problem* (GOEP) is to find $f \in X'$ such that

$$T(f,g) \nsubseteq -C(f), \quad \text{for each } g \in X'; \qquad (GOEP)$$

and the *generalized operator quasi-equilibrium problem* (GOQEP) is to find $f \in X'$ such that $f \in cl\, A(f)$ and,

$$T(f,g) \nsubseteq -C(f), \quad \text{for each } g \in A(f). \qquad (GOQEP)$$

When $A(f) \equiv X'$ for each $f \in X'$, (GOQEP) reduces to (GOEP), and when $T : X' \times X' \to F$ is a single-valued mapping, (GOEP) reduces to (OEP) due to Kazmi and Raouf in [4].

In order to prove our main result in non-compact settings, we shall need the following fixed point theorem which is a particular form of Park's fixed point theorem in [8].

Lemma 2.3. *Let X be a non-empty convex subset of a Hausdorff topological vector space E, and K be a non-empty compact subset of X. Let $A : X \rightrightarrows X$ be a multifunction. Suppose that*

(1) *for each* $x \in X,\ A(x)$ *is non-empty convex;*
(2) *for each* $y \in X,\ A^{-1}(y) = \{x \in X \mid y \in A(x)\}$ *is open in* X*; and*
(3) *for each finite subset* N *of* X*, there exists a non-empty compact convex subset* L_N *of* X *containing* N *such that for each* $x \in L_N \setminus K,\ A(x) \cap L_N \neq \emptyset$.

Then A *has a fixed point* $x_0 \in X$*; that is,* $x_0 \in A(x_0)$.

3. A Solution for (GOEP) in non-compact settings

Using Lemma 2.3, we now prove a basic existence theorem of solution for (GOEP) as follows:

THEOREM 3.1. *Let* X' *be a non-empty convex subset of* $L(E,F)$ *and* K' *be a non-empty compact subset of* X'*, where* $L(E,F)$ *is endowed with the topology of compact convergence, and let* $T : X' \times X' \rightrightarrows F$ *be a non-empty compact valued multifunction such that* $T(\cdot, g)$ *is upper semicontinuous for each fixed* $g \in X'$*, and the range of* T *is contained in a compact subset of* F*. Assume that* $T(f, \cdot)$ *is natural quasi* P*-convex for each fixed* $f \in X'$*, and* $T(f,f) = \{0\}$ *for each* $f \in X'$*. Let* $W : X' \rightrightarrows F$ *be defined by* $W(f) = F \setminus -C(f)$ *such that the graph* $Gr(W)$ *of* W *is closed in* $X' \times F$*. Assume that for each finite subset* N' *of* X'*, there exists a non-empty compact convex subset* $L_{N'}$ *of* X' *containing* N' *such that for each* $f \in L_{N'} \setminus K'$*, there exists* $g \in L_{N'}$ *satisfying* $T(f,g) \subseteq -C(f)$.

Then, there exists $f_0 \in K'$ *such that*

$$T(f_0, g) \nsubseteq -C(f_0), \quad \text{for all } g \in X'.$$

Proof. We first note that $L(E,F)$ equipped with the topology of compact convergence is a Hausdorff topological vector space. We now define a multifunction $A : X' \rightrightarrows X'$ by

$$A(f) := \{g \in X' \,|\, T(f,g) \subseteq -C(f)\} \quad \text{for each } f \in X'.$$

The proof is organized in the following parts.
(i) For each $f \in X'$, $A(f)$ is convex. Indeed, suppose the contrary. Then there exist g_1 and g_2 in $A(f)$, and some $t \in [0,1]$ such that $tg_1 + (1-t)g_2 \notin A(f)$. Therefore, we have

$$T(f, g_1) \subseteq -C(f),\ T(f, g_2) \subseteq -C(f), \quad \text{but} \quad T(f, tg_1 + (1-t)g_2) \nsubseteq -C(f).$$

Since $T(f,\cdot)$ is natural quasi P-convex, there exist $\mu \in [0,1]$ such that

$$\begin{aligned} T(f, tg_1 + (1-t)g_2) &\subseteq \mu T(f,g_1) + (1-\mu)T(f,g_2) - P \\ &\subseteq -\mu C(f) - (1-\mu)C(f) - P \\ &\subseteq -C(f) - C(f) - C(f) \subseteq -C(f), \end{aligned}$$

which is a contradiction so that each $A(f)$ is convex.
(ii) Clearly A has no fixed point because $T(f,f) = \{0\}$, and $0 \notin C(f)$ for all $f \in X'$.
(iii) For each $g \in X'$, $A^{-1}(g)$ is open in X'. Indeed, let $\{f_\lambda\}$ be a net in $(A^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin A(f_\lambda)$ and hence $T(f_\lambda, g) \nsubseteq -C(f_\lambda)$ so that $T(f_\lambda, g) \cap \big(F \setminus -C(f_\lambda)\big) \neq \emptyset$ for each λ. Therefore, for each λ, there exists $x_\lambda \in T(f_\lambda, g) \cap \big(F \setminus -C(f_\lambda)\big)$. Since $T(\cdot, g)$ is non-empty compact-valued and upper semicontinuous on the first variable, and the graph $W(f) = F \setminus -C(f)$ is closed, by Theorem 7.3.10 in [5], the multifunction $f \mapsto T(f,g) \cap \big(F \setminus -C(f)\big)$ is upper semicontinuous. Since $T(X' \times X')$ is compact and $\{x_\lambda\} \subset T(X' \times X')$, there exists a subnet $\{x_\mu\}$ of $\{x_\lambda\}$ converging to $\bar{x} \in T(X' \times X')$. Since the corresponding nets $\{f_\mu\} \to f$, $x_\mu \in T(f_\mu, g) \cap \big(F \setminus -C(f_\mu)\big)$ for each μ, and $\{x_\mu\} \to \bar{x}$, by virtue of Theorem 7.1.15 in [5], we have $\bar{x} \in T(f,g) \cap \big(F \setminus -C(f)\big)$ so that $T(f,g) \nsubseteq -C(f)$, i.e., $f \in (A^{-1}(g))^c$. Therefore, $(A^{-1}(g))^c$ is closed so that $A^{-1}(g)$ is open in X'.
(iv) By the given hypothesis, we know that for each finite subset N' of X', there exists a non-empty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying $g \in A(f)$, hence $L_{N'} \cap A(f) \neq \emptyset$.
(v) From (i)-(iv), we see, by Lemma 1, there must be an $f_0 \in K'$ such that $A(f_0) = \emptyset$, namely,

$$T(f_0, g) \nsubseteq -C(f_0), \quad \text{for each } g \in X'.$$

This completes the proof. □

Remark 3.2. Theorem 3.1 is very closely related with Theorem 2.1 in [4] in the following aspects:

(i) neither the C-pseudomonotonicity nor the hemicontinuity for $T(\cdot, g)$ are not needed in Theorem 2.1 in [4];

(ii) the coercivity assumption with respect to the compact convex set B is not needed in Theorem 2.1 in [4].

References

[1] E. Blume and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123-145.

[2] A. Domokos and J. Kolumbán, *Variational inequalities with operator solutions*, J. Global Optim. **23** (2002), 99-110.

[3] K. Fan, *A Generalization of Tychonoff's Fixed Point Theorem*, Math. Ann. **142** (1961), 305-310.

[4] K. R. Kazmi and A. Raouf, *A class of operator equilibrium problems*, J. Math. Anal. Appl. **308** (2005), 554-564.

[5] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley & Sons Inc., New York, 1984.

[6] S. Kum and W. K. Kim, *Generalized vector variational and quasi-variational inequalities with operator solutions*, J. Global Optim. **32** (2005), 581-595.

[7] S. Kum and W. K. Kim, *Applications of generalized variational and quasi-variational inequalities with operator solutions in a TVS*, J. Optim. Theory Appl., to appear.

[8] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Kor. Math. Soc. **31** (1994), 493-519.

*
Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: wkkim@chungbuk.ac.kr