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CHARACTERIZATIONS OF THE PARETO DISTRIBUTION BY THE INDEPENDENCE OF RECORD VALUES

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Abstract. In this paper, we establish characterizations of the Pareto distribution by the independence of record values. We prove that $X \in PAR(1,\beta)$ for $\beta > 0$, if and only if $\dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $X_{U(n)}$ are independent for $n \geq 1$. And we show that $X \in PAR(1,\beta)$ for $\beta > 0$, if and only if $\dfrac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value.

The indices at which the upper record values occur are given by the record times $\{U(n),\ n \geq 1\}$, where $U(n) = \min\{j | j > U(n-1)\,, X_j > X_{U(n-1)}\,, n \geq 2\}$ with $U(1) = 1$.

A continuous random variable X is said to have the Pareto distribution with two parameters $\alpha > 0$ and $\beta > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - \left(\dfrac{x}{\alpha}\right)^{-\beta}, \ x > 1,\ \alpha > 0,\ \beta > 0 \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

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A notation that designates that X has the cdf (1) is $X \in PAR(\alpha, \beta)$.

Some characterizations by the independence of the record values are known. In [1] and [2], Ahsanullah studied, if $X \in PAR(\alpha, \beta)$ for $\alpha > 0$ and $\beta > 0$, then $\dfrac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$ are independent for $0 < m < n$. And Ahsanullah proved, if $X \in WEI(\theta, \alpha)$ for $\theta > 0$ and $\alpha > 0$, then $\dfrac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $0 < m < n$. In [4], Lee and Chang characterized that $X \in PAR(1, \beta)$ for $\beta > 0$, if and only if $\dfrac{X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $n \geq 1$ and $X \in EXP(\sigma)$ for $\sigma > 0$, if and only if $\dfrac{X_{U(n)}}{X_{U(n+1)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.

In this paper, we will give characterizations of the Pareto distribution with the parameter $\alpha = 1$ by the independence of record values.

2. Main results

THEOREM 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\beta}$ for all $x > 1$ and $\beta > 0$, if and only if $\dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.*

Proof. If $F(x) = 1 - x^{-\beta}$ for all $x > 1$ and $\beta > 0$, then the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{\beta^{n+1}\,(ln\, x)^{n-1}}{\Gamma(n)\; x\, y^{\beta+1}}$$

for all $1 < x < y$, $\beta > 0$ and $n \geq 1$.

Consider the functions $V = \dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $W = X_{U(n)}$. It follows that $x_{U(n)} = w$, $x_{U(n+1)} = \dfrac{(v-1)w}{v}$ and $|J| = \dfrac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\beta^{n+1}\, v^{\beta-1}\,(ln\, w)^{n-1}}{\Gamma(n)\,(v-1)^{\beta+1}\, w^{\beta+1}} \tag{2}$$

for all $v > 0$, $w > 1$, $\beta > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$\begin{aligned} f_V(v) &= \int_1^{\infty} f_{V,W}(v,w)\, dw \\ &= \frac{\beta^{n+1}\, v^{\beta-1}}{\Gamma(n)\,(v-1)^{\beta+1}} \int_1^{\infty} \frac{(ln\, w)^{n-1}}{w^{\beta+1}}\, dw \\ &= \frac{\beta\, v^{\beta-1}}{(v-1)^{\beta+1}} \end{aligned} \tag{3}$$

for all $v > 0$, $\beta > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$\begin{aligned} f_W(w) &= \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \\ &= \frac{\beta^n\,(ln\, w)^{n-1}}{\Gamma(n)\, w^{\beta+1}} \end{aligned} \tag{4}$$

for all $w > 1$, $\beta > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_V(v) f_W(w) = f_{V,W}(v,w)$.

Hence $V = \dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $W = X_{U(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint pdf $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{(R(x))^{n-1}\, r(x)\, f(y)}{\Gamma(n)}$$

for all $1 < x < y$, $\beta > 0$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \dfrac{d}{dx}\big(R(x)\big) = \dfrac{f(x)}{1 - F(x)}$.

Let us use the transformations $V = \dfrac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $W = X_{U(n)}$. The Jacobian of the transformations is $|J| = \dfrac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{f\left(\frac{(v-1)\,w}{v}\right)\,(R(w))^{n-1}\; r(w)\, w}{\Gamma(n)\, v^2} \tag{5}$$

for all $v > 0$, $w > 1$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1}\, f(w)}{\Gamma(n)} \tag{6}$$

for all $w > 1$ and $n \geq 1$.

From (5) and (6), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{f\left(\frac{(v-1)\,w}{v}\right)\, r(w)\, w}{v^2\, f(w)}$$

for all $v > 0$, $w > 1$ and $n \geq 1$, where $r(x) = \dfrac{f(x)}{1-F(x)}$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(-\frac{1-F\left(\frac{(v-1)\,w}{v}\right)}{1-F(w)}\right).$$

Since V and $W$are independent, we must have

$$1-F\left(\frac{(v-1)w}{v}\right) = \left(1-F\left(\frac{v-1}{v}\right)\right)(1-F(w)) \tag{7}$$

for all $v > 0$ and $w > 1$.

Upon substituting $\dfrac{(v-1)}{v} = v_1$ in (7), then we get

$$1-F(v_1\, w) = (1-F(v_1))(1-F(w)) \tag{8}$$

for all $v_1 > 1$ and $w > 1$.

By the monotonic transformations of the exponential distribution (see, [3]), the most general solution of (8) with the boundary condition $F(1) = 0$ is

$$F(x) = 1 - x^{-\beta}$$

for all $x > 1$ and $\beta > 0$.

This completes the proof. □

THEOREM 2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(1) = 0$ and $F(x) < 1$ for all $x > 1$. Then $F(x) = 1 - x^{-\beta}$ for all $x > 1$ and $\beta > 0$, if and only if $\frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 1, we consider the functions $V = \frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $W = X_{U(n)}$. It follows that $x_{U(n)} = w$, $x_{U(n+1)} = (1-v)w$ and $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{\beta^{n+1}\,(ln\,w)^{n-1}}{\Gamma(n)\,(1-v)^{\beta+1}\,w^{\beta+1}} \tag{9}$$

for all $v < 0$, $w > 1$, $\beta > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$\begin{aligned} f_V(v) &= \int_1^\infty f_{V,W}(v,w)\; dw \\ &= \frac{\beta^{n+1}}{\Gamma(n)\,(1-v)^{\beta+1}} \int_1^\infty \frac{(ln\,w)^{n-1}}{w^{\beta+1}}\; dw \\ &= \frac{\beta}{(1-v)^{\beta+1}} \end{aligned} \tag{10}$$

for all $v < 0$, $\beta > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$\begin{aligned} f_W(w) &= \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \\ &= \frac{\beta^n\,(ln\,w)^{n-1}}{\Gamma(n)\,w^{\beta+1}} \end{aligned} \tag{11}$$

for all $w > 1$, $\beta > 0$ and $n \geq 1$.

From (9), (10) and (11), we obtain $f_V(v) f_W(w) = f_{V,W}(v,w)$.

Hence $V = \frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $W = X_{U(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. By the same manner as Theorem 1, let us use the transformations $V = \frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and

$W = X_{U(n)}$. The Jacobian of the transformations is $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{f((1-v)w)\ (R(w))^{n-1}\ r(w)\, w}{\Gamma(n)} \tag{12}$$

for all $v < 0$, $w > 1$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \tag{13}$$

for all $w > 1$ and $n \geq 1$.

From (12) and (13), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{f((1-v)w)\, r(w)\, w}{f(w)}$$

for all $v < 0$, $w > 1$ and $n \geq 1$, where $r(x) = \dfrac{f(x)}{1-F(x)}$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(\frac{1 - F((1-v)w)}{1-F(w)}\right).$$

Since V and $W$are independent, we must have

$$1 - F((1-v)w) = (1 - F(1-v))(1-F(w)) \tag{14}$$

for all $v < 0$ and $w > 1$.

Upon substituting $(1-v) = v_2$ in (14), then we get

$$1 - F(v_2\, w) = (1 - F(v_2))(1 - F(w)) \tag{15}$$

for all $v_2 > 1$ and $w > 1$.

By the monotonic transformations of the exponential distribution (see, [3]), the most general solution of (15) with the boundary condition $F(1) = 0$ is

$$F(x) = 1 - x^{-\beta}$$

for all $x > 1$ and $\beta > 0$.

This completes the proof. □

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