JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 1, March 2007

ORTHOGONAL GROUP OF CERTAIN INDEFINITE LATTICE

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Abstract. We compute the special orthogonal group of certain lattice of signature $(2,1)$.

1. Introduction

Given an even lattice M in a real quadratic space of signature $(2,n)$, Borcherds lifting [1] gives a multiplicative correspondence between vector valued modular forms F of weight $1-n/2$ with values in $\mathbb{C}[M'/M]$ (= the group ring of M'/M) and meromorphic modular forms on complex varieties $(O(2)\times O(n))\backslash O(2,n)/\mathrm{Aut}(M,F)$. Here M' denotes the dual lattice of M, $O(2,n)$ is the orthogonal group of $M\bigotimes\mathbb{R}$ and $\mathrm{Aut}(M,F)$ is the subgroup of $\mathrm{Aut}(M)$ leaving the form F stable under the natural action of $\mathrm{Aut}(M)$ on M'/M. In particular, if the signature of M is $(2,1)$, then

$$\frac{O(2,1)}{O(2)\times O(1)}\approx\mathfrak{H}.$$

and Borcherds' theory gives a lifting of vector valued modular form of weight $1/2$ to usual one variable modular form on $\mathrm{Aut}(M,F)$. In this sense in order to work out Borcherds lifting it is important to find appropriate lattice on which our wanted modular group acts. In this article we will show:

Theorem 1.1. *Let M be a 3-dimensional even lattice of all 2×2 integral symmetric matrices, that is,*

$$M=\left\{\begin{pmatrix}A & B\\ B & C\end{pmatrix}\mid A,B,C\in\mathbb{Z}\right\}$$

Received December 30, 2006.
2000 Mathematics Subject Classification: Primary 11F03, 11H56.
Key words and phrases: indefinite lattice, orthogonal group.
* This work was supported by a Bahrom research grant from Seoul Women's University (2007).

with norm $(\lambda,\lambda) = 2q(\lambda) = -2\det(\lambda)$ *for* $\lambda \in M$. *Then* $\mathrm{Aut}(M)$ *(=the special orthogonal group of* M*) is isomorphic to* $PSL_2(\mathbb{Z})$.

REMARK 1.2. Similarly, for a fixed positive integer N, if we take M to be an even lattice of all symmetric matrices $\lambda = \begin{pmatrix} A/N & B \\ B & C \end{pmatrix}$ with $A, B, C \in \mathbb{Z}$ and the norm $(\lambda,\lambda) = -2N\det(\lambda)$, then we can prove $\mathrm{Aut}(M) = \Gamma_0^*(N)$ (=the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions).

2. Proof of Theorem 1.1

For each $\gamma \in \Gamma = SL_2(\mathbb{Z})$ and $\lambda \in M$, $\gamma\lambda\gamma^t$ gives an action of Γ on M. Since $q(\gamma\lambda\gamma^t) = -\det(\gamma\lambda\gamma^t) = -\det(\lambda) = q(\lambda)$, we get a natural homomorphism f from Γ to $\mathrm{Aut}(M)$ defined by $f(\gamma)(\lambda) = \gamma\lambda\gamma^t$. Let $\gamma \in \ker(f)$. Then it holds that

$$\gamma\lambda\gamma^t = \lambda \quad \text{for all } \lambda \in M. \tag{1}$$

Applying (1) to $\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively, we obtain $\gamma = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore f induces a map from $PSL_2(\mathbb{Z})$.

Now let us show that f is surjective. To this end we first identify each element $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ of M with 3×1 integral matrix $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ and define $q\begin{pmatrix} A \\ B \\ C \end{pmatrix} = B^2 - AC$. Then for each $m \in \mathrm{Aut}(M)$ one can express

$$m = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL_3(\mathbb{R}) \quad \text{satisfying } q\left(m\begin{pmatrix} A \\ B \\ C \end{pmatrix}\right) = q\begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{2}$$

Since $m\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $m\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $m\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ have integral entries, we see that m belongs to $GL_3(\mathbb{Z})$. And it follows from (2) that $(dA + eB + fC)^2 - (aA + bB + cC)(gA + hB + iC) = B^2 - AC$, which yields $(d^2 - ag)A^2 + (e^2 - bh)B^2 + (f^2 - ci)C^2 + (2de - ah - bg)AB + (2ef - bi - ch)BC + (2df - cg - ai)AC = B^2 - AC$. Comparing the coefficients in the above equality gives rise to

$$d^2 = ag \tag{3}$$

$$f^2 = ci \tag{4}$$

$$2df - cg - ai = -1 \tag{5}$$

$$e^2 = 1 + bh \tag{6}$$

$$2de = ah + bg \tag{7}$$

$$2ef = bi + ch. \tag{8}$$

From (3), (4) and (5) we see that $\gcd(a, g) = \gcd(c, i) = 1$ and a, g, c, i have the same sign. Thus we may assume $a = p^2, g = r^2, d = pr, c = q^2, i = s^2, f = qs$ for some integers p, q, r, s with $(ps - rq)^2 = 1$. Now if we subtract $(8) \times g$ from $(7) \times i$, we obtain $(ai - cg)h = 2e(di - fg)$ and therefore $(p^2s^2 - q^2r^2)h = 2e(prs^2 - qsr^2)$, which is simplified to

$$(ps + rq)h = 2ers. \tag{9}$$

Similarly, by subtracting $(8) \times a$ from $(7) \times c$ and simplifying we get

$$(ps + rq)b = 2epq. \tag{10}$$

Multiplying sides by sides in (9) and (10) yields $(ps + rq)^2bh = 4e^2pqrs$. Now if we use (6), we come up with $e^2(ps - rq)^2 = (ps + rq)^2$. Since we know that $(ps - rq)^2 = 1$, we obtain $e^2 = (ps + rq)^2$. Hence m is of the form

$$m = \begin{pmatrix} p^2 & 2pq & q^2 \\ pr & ps + rq & qs \\ r^2 & 2rs & s^2 \end{pmatrix}.$$

On the other hand, from

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} p^2A + 2pqB + q^2C & prA + (ps + rq)B + qsC \\ prA + (ps + rq)B + qsC & r^2A + 2srB + s^2C \end{pmatrix},$$

we observe that m is induced from the action of $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, which completes the proof.

References

[1] R. Borcherds, *Automorphic forms with singularities on Grassmanians*, Invent. Math. **132** (1998), 491-562.

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