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GENERALIZED CUBIC MAPPINGS OF $r-$ TYPE IN SEVERAL VARIABLES

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Abstract. Let X, Y be vector spaces. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability problem for a cubic function $f : X \to Y$ satisfies

$$\begin{aligned} & r^3 f(\frac{\sum_{j=1}^{n-1} x_j + 2x_n}{r}) + r^3 f(\frac{\sum_{j=1}^{n-1} x_j - 2x_n}{r}) + 8\sum_{j=1}^{n-1} f(x_j) \\ = \quad & 2f(\sum_{j=1}^{n-1} x_j) + 4\sum_{j=1}^{n-1}\Big(f(x_j + x_n) + f(x_j - x_n)\Big) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

1. Introduction

The study of stability problems for functional equations is related to the following question originated by Ulam [13] concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy),\, h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x),\, H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D. H. Hyers [5]. Let X and Y are Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon$$

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for all $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\| f(x) - a(x) \| \leq \epsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers's theorem was generalized in various directions. In particular, Th. M. Rassias [8] and Z. Gajda [3] considered a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. They proved the following theorem by using a direct method: if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| x \|^p)$$

for some $\theta \geq 0$, $0 \leq p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\| f(x) - a(x) \| \leq \frac{2\theta}{2 - 2^p} \| x \|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear. Găvruta [4] generalized the Rassias's result above.

The quadratic function $f(x) = cx^2\,(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \,. \tag{1.1}$$

This question is called the quadratic functional equation, and every solution of the equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was first proved by Skof [12] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [2], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [9], [10], and [11].

The cubic function $f(x) = cx^3\,(c \in \mathbb{R})$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \,. \tag{1.2}$$

We promise that by a cubic function we mean every solution of the equation (1.2) is called a cubic function. The equation (1.2) was solved by Jun and Kim [6]. Also, they proved the generalized Hyers-Ulam-Rassias stability problem for the given functional equation : see [7]

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)\,. \tag{1.3}$$

Throughout this paper, we assume that $r(r^3\neq 1)$ is a real number and $n\geq 2$ is an integer number.

In this paper, for all $x_1,\cdots,x_n\in X$ the following odd functional equation $f:X\to Y$ such that

$$\begin{aligned} r^3f(\frac{\sum_{j=1}^{n-1}x_j+2x_n}{r})+r^3f(\frac{\sum_{j=1}^{n-1}x_j-2x_n}{r})+8\sum_{j=1}^{n-1}f(x_j) \\ =2f(\sum_{j=1}^{n-1}x_j)+4\sum_{j=1}^{n-1}\Big(f(x_j+x_n)+f(x_j-x_n)\Big) \end{aligned} \tag{1.4}$$

is a solution of the equation (1.3) and then investigate the generalized Hyers-Ulam-Rassias stability in Banach spaces.

2. Generalized cubic mappings of $r-$type in several variables

Lemma 2.1. *Let X and Y be real vector spaces. If an odd function $f:X\to Y$ satisfies the functional equation (1.4), then there exists functions $B:X\times X\times X\to Y$ and $A:X\to Y$ such that*

$$B(x,y,z)=\frac{1}{24}[f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)]$$

is symmetric for each fixed one variable and additive for each two variables and A is additive.

Proof. Since f is odd, we have $f(0)=0$. Now, setting $x_1=x\,,x_n=y$, and $x_j=0\,(j=2,\cdots,n-1)$, we write

$$r^3f(\frac{x+2y}{r})+r^3f(\frac{x-2y}{r})+6f(x)=4f(x+y)+4f(x-y)\,, \tag{2.1}$$

for all $x,y\in X$. By letting $y=0$, we have

$$r^3f(\frac{x}{r})=f(x)\,,$$

for all $x\in X$. Thus we may conclude that

$$\begin{aligned} 4f(x+y)+4f(x-y) &= r^3f(\frac{x+2y}{r})+r^3f(\frac{x-2y}{r})+6f(x) \\ &= f(x+2y)+f(x-2y)+6f(x)\,, \end{aligned}$$

for all $x, y \in X$, that is, the equation (1.4) satisfies the equation (1.3). The remains of proof follow from [7, Theorem 2.1]. $\square$

LEMMA 2.2. *Let X and Y be real vector spaces. If an odd function $f : X \to Y$ satisfies the functional equation (1.3) and $f(2x) = 8f(x)$, for all $x \in X$, then f is cubic.*

Proof. Letting $x = 2x$ in the equation (1.3), we have

$$f(2x + 2y) + f(2x - 2y) + 6f(2x) = 4f(2x + y) + 4f(2x - y)\,.$$

Since $f(2x) = 8f(x)$, for all $x \in X$, we get

$$8f(x + y) + 8f(x - y) + 48f(x) = 4f(2x + y) + 4f(2x - y)\,,$$

that is, it satisfies the equation (1.2), as desired. $\square$

3. Hyers-Ulam-Rassias stability

Throughout in this section, let X be a normed vector space with norm $||\cdot||$ and Y be a Banach space with norm $\|\cdot\|$. For the given odd mapping $f : X \to Y$, we define

$$\begin{aligned}(3.1)\quad Df(x_1, \cdots, x_n) &:= r^3 f\Big(\frac{\sum_{j=1}^{n-1} x_j + 2x_n}{r}\Big) + r^3 f\Big(\frac{\sum_{j=1}^{n-1} x_j - 2x_n}{r}\Big)\\ &+8\sum_{j=1}^{n-1} f(x_j) - 2f\Big(\sum_{j=1}^{n-1} x_j\Big) - 4\sum_{j=1}^{n-1}\Big(f(x_j + x_n) + f(x_j - x_n)\Big)\,,\end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

THEOREM 3.1. *Let $n \geq 2$ be an integer number, let $|r| < 1$, and let $f : X \to Y$ be an odd mapping for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$(3.2)\qquad \tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} r^{3j} \phi\Big(\frac{x_1}{r^j}, \cdots, \frac{x_n}{r^j}\Big) < \infty\,,$$

$$(3.3)\qquad \| Df(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n)\,,$$

and

$$||f(2x) - 8f(x)|| \leq \delta$$

for all $x_1, \cdots, x_n, x \in X$ *and for some* $\delta \geq 0$. *Then for every* $m \in \{1, 2, \cdots, n-1\}$, *there exists a generalized cubic mapping* $C : X \to Y$ *such that*

$$\| f(x) - C(x) \| \leq \tilde{\phi}(\underbrace{\frac{x}{m}, \cdots, \frac{x}{m}}_{m-terms}, 0, \cdots, 0), \tag{3.4}$$

for all $x \in X$.

Proof. Let $1 \leq m \leq n-1$ be an integer number. By setting $x_j = \frac{x}{m}$ $(j = 1, \cdots, m)$ and $x_{m+1} = \cdots = x_n = 0$, we have

$$\| f(x) - r^3 f(\frac{x}{r}) \| \leq \phi(\underbrace{\frac{x}{m}, \cdots, \frac{x}{m}}_{m}, 0, \cdots, 0), \tag{3.5}$$

for all $x \in X$. Replacing x by $\frac{x}{m}$ in the equation (3.5) and then multiplying by r^3, we have

$$\| r^3 f(\frac{x}{r}) - r^{3\cdot 2} f(\frac{x}{r^2}) \| \leq r^3 \phi(\underbrace{\frac{x}{m}, \cdots, \frac{x}{m}}_{m}, 0, \cdots, 0), \tag{3.6}$$

for all $x \in X$. Now, combining equations (3.5) and (3.6), we get

$$\begin{aligned} \| f(x) - r^{3\cdot 2} f(\frac{x}{r^2}) \| &\leq \phi(\underbrace{\frac{x}{m}, \cdots, \frac{x}{m}}_{m}, 0, \cdots, 0) \\ &+ r^3 \phi(\underbrace{\frac{x}{m \cdot r}, \cdots, \frac{x}{m \cdot r}}_{m}, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$. Continue this way, we may have

$$\| f(x) - r^{3\cdot j} f(\frac{x}{r^j}) \| \leq \sum_{k=0}^{j-1} r^{3\cdot k} \phi(\underbrace{\frac{x}{m \cdot r^k}, \cdots, \frac{x}{m \cdot r^k}}_{m}, 0, \cdots, 0), \tag{3.7}$$

for all positive integer j and all $x \in X$.

Dividing the equation (3.7) by $r^{3\cdot s}$ and then substituting x by $\frac{x}{r^s}$, we have

$$\begin{aligned} &r^{3\cdot s} \| f(\frac{x}{r^s}) - r^{3\cdot j} f(\frac{x}{r^{j+s}}) \| \\ &\leq r^{3\cdot s} \sum_{k=0}^{j-1} \cdot r^{3\cdot k} \phi(\underbrace{\frac{x}{m \cdot r^{k+s}}, \cdots, \frac{x}{m \cdot r^{k+s}}}_{m}, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$.

By taking $s \to \infty$, we may conclude that $\{r^{3\cdot j} f(\frac{x}{r^j})\}$ is a Cauchy sequence in a Banach space Y. This implies that the sequence $\{r^{3\cdot j} f(\frac{x}{r^j})\}$ converges. Hence we can define a function $C : X \to Y$ by

$$C(x) = \lim_{j\to\infty} r^{3\cdot j} f(\frac{x}{r^j}),$$

for all $x \in X$. Then

$$\begin{aligned} \| DC(x_1, \cdots, x_n) \| &= \lim_{k\to\infty} r^{3\cdot k} \| Df(\frac{x_1}{r^k}, \cdots, \frac{x_n}{r^k}) \| \\ &\le \lim_{k\to\infty} r^{3\cdot k} \phi(\frac{x_1}{r^k}, \cdots, \frac{x_n}{r^k}) \\ &= 0, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. That is, $DC(x_1, \cdots, x_n) = 0$. Obviously, we have C is odd and $C(2x) = 8C(x)$, for all $x \in X$. Corollary 2.2 implies that the function $C : X \to Y$ is a cubic mapping.

It only remains to show that the function C is unique. Let $C' : X \to Y$ be another generalized cubic function satisfying the equation (3.4). Then

$$\begin{aligned} &||C(x) - C'(x)|| = r^{3k}||C((\frac{1}{r})^k x) - C'((\frac{1}{r})^k x)|| \\ \le\ & r^{3k}\Big(||C((\frac{1}{r})^k x) - f((\frac{1}{r})^k x)|| + ||C'((\frac{1}{r})^k x) - f((\frac{1}{r})^k x)||\Big) \\ \le\ & 2 \cdot r^{3k}\, \tilde{\phi}((\frac{1}{r})^k x, (\frac{1}{r})^k x, 0 \cdots, 0) \\ =\ & 2 \cdot r^{3k} \sum_{j=0}^{\infty} r^{3j} \phi((\frac{1}{r})^{j+k} x, (\frac{1}{r})^{j+k} x, 0, \cdots, 0) \\ =\ & 2 \cdot \sum_{j=k}^{\infty} r^{3j} \phi((\frac{1}{r})^j x, (\frac{1}{r})^j x, 0, \cdots, 0) \to 0, \end{aligned}$$

for all $x \in X$. As $r \to \infty$, we can conclude that $C(x) = C'(x)$, for all $x \in X$; that is, C is unique. □

Corollary 3.2. *Let $r > 1$, and let θ and $p < 3$ be positive real numbers (Or, let $r < 1$, and let θ and $p > 3$ be positive real numbers).*

Let $f : X \to Y$ be a function such that

$$||Df(x_1, \cdots, x_n)|| \le \theta \sum_{i=1}^{n} ||x_i||^p ,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized cubic mapping $C : X \to Y$ such that

$$||f(x) - C(x)|| \le \frac{m\theta}{r^{3-p} - 1} ||x||^p ,$$

for all $x \in X$, and for any $1 \le m \le n - 1$.

Proof. Define $\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p$, and apply to Theorem 3.1. □

THEOREM 3.3. *Let $n \ge 2$ be an integer number, let $|r| > 1$, and let $f : X \to Y$ be a mapping for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=1}^{\infty} r^{-3j} \phi(r^j x_1, \cdots, r^j x_n) < \infty , \tag{3.8}$$

$$\| Df(x_1, \cdots, x_n) \| \le \phi(x_1, \cdots, x_n) , \tag{3.9}$$

and

$$||f(2x) - 8f(x)|| \le \delta$$

for all $x_1, \cdots, x_n, x \in X$ and for some $\delta \ge 0$. Then for every $m \in \{1, 2, \cdots, n-1\}$, there exists a generalized cubic mapping $C : X \to Y$ such that

$$\| f(x) - C(x) \| \le \tilde{\phi}(\underbrace{\frac{x}{m}, \cdots, \frac{x}{m}}_{m-terms}, 0, \cdots, 0) , \tag{3.10}$$

for all $x \in X$.

Proof. If x is replaced by rx and dividing by r^3 in the equation (3.5) in the proof of Theorem 3.1, we have the following equation

$$\| f(x) - \frac{1}{r^3} f(rx) \| \le \frac{1}{r^3} \phi(\underbrace{\frac{r}{m}x, \cdots, \frac{r}{m}x}_{m}, 0, \cdots, 0) , \tag{3.11}$$

for all $x \in X$. The remains of the proof are similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r > 1$, and let θ and $p > 3$ be positive real numbers (Or, let $r < 1$, and let θ and $p < 3$ be positive real numbers). Let $f : X \to Y$ be a function such that*

$$||Df(x_1, \cdots, x_n)|| \leq \theta \sum_{i=1}^{n} ||x_i||^p,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized cubic mapping $C : X \to Y$ such that

$$||f(x) - C(x)|| \leq m\theta \cdot \frac{r^{p-3}}{r^{3-p} - 1} ||x||^p,$$

for all $x \in X$, and for any $1 \leq m \leq n - 1$.

Proof. Define $\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p$, and apply to Theorem 3.3. □

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