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ON THE STABILITY OF AN n-DIMENSIONAL QUADRATIC EQUATION

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Abstract. Let X and Y be vector spaces. In this paper we prove that a mapping $f : X \to Y$ satisfies the following functional equation

$$\sum_{1\le k<l\le n}(f(x_k+x_l)+f(x_k-x_l))-2(n-1)\sum_{i=1}^{n}f(x_i)=0$$

if and only if the mapping f is quadratic. In addition we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation.

1. Introduction

The quadratic function $f(x) = cx^2 (x \in \mathbb{R})$, where c is a real constant, clearly satisfies the equation

$$f(x_1+x_2)+f(x_1-x_2)=2(f(x_1)+f(x_2)). \tag{1.1}$$

Hence, the equation (1.1) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation (1.1) is said to be a quadratic function. It is well known that a function $f : X \to Y$ between real vector spaces is a quadratic function (1.1) if and only if there exists a unique symmetric biadditive function $B : X^2 \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$ (see [1, 6]). The following problem of this kind had been formulated by Ulam during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940. Given a metric group $\mathrm{G}(\cdot, d)$, a number $\delta > 0$ and a mapping $h : G \to G$ which satisfies the inequality $d(h(x\cdot y), h(x)\cdot h(y)) < \delta$ for all

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$x, y \in G$, does there exist an automorphism g of G and a constant $k > 0$, depending only on G, such that $d(g(x), h(x)) \leq k\delta$ for all $x \in G$? If the answer is affirmative, we would call the equation $g(x \cdot y) = g(x) \cdot g(y)$ of automorphism stable. The Hyers-Ulam stability of the quadratic functional equation was first proved by F. Skof for a function $f : X \to Y$, where X is a normed space and Y is a Banach space [11]. P.W. Cholewa demonstrated that the theorem of Skof is also valid if X is replaced by an abelian group [2]. Later, the Hyers-Ulam-Rassias stability of the quadratic functional equation was proved by S. Czerwik [3], J.M. Rassias [8] and Th.M. Rassias ([9, 10]). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [5, 7]. In this paper, we will extend Eq.(1.1) to an n-dimensional quadratic functional equation and then investigate the generalized Hyers-Ulam-Rassias stability of the n-dimensional quadratic functional equation as follows:

$$\sum_{1 \leq k < l \leq n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^{n} f(x_i) = 0 \tag{1.2}$$

for all $x_1, \cdots, x_n \in X$, where $n \geq 2$ is an integer number.

2. Solution of the functional equation (1.2)

THEOREM 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.2) if and only if f is quadratic.*

Proof. Let f be a quadratic function. Assume the equation (1.2) is true for n by induction argument. By (1.1),

$$f(x_i + x_{n+1}) + f(x_i - x_{n+1}) - 2f(x_i) - 2f(x_{n+1}) = 0 \tag{2.1}$$

for all $i = 1, \cdots, n$. Adding up (1.2) and (2.1), we have the desired equation (1.2) for $n+1$. Conversely, let f satisfy the equation (1.2). By letting $x_i = 0$ for all $i = 1, 2, \cdots, n$, we have $f(0) = 0$. Replacing $x_i = 0$ for all $i = 3, 4, \cdots, n$, we obtain the equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2(f(x_1) + f(x_2)),$$

which implies that f is quadratic. The proof is complete. □

In Section 3, we investigate the generalized Hyers-Ulam stability of the functional equation (1.2) in the spirit of P. Găvruta [4].

3. Stability of the quadratic equation (1.2)

In this section, let X be a vector space and Y a Banach space. For the given mapping $f : X \to Y$, we define

$$Df(x_1, \cdots, x_n) := \sum_{1 \le k < l \le n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \cdots, x_n \in X$, where $n \ge 2$ is an integer number. We denote by $\varphi : X^n \to [0, \infty)$ a function such that

$$\Phi(x_1, x_2, \cdots, x_n) := \sum_{k=0}^{\infty} \left(\frac{1}{4^k}\right) \varphi(2^k x_1, \cdots, 2^k x_n) < \infty. \tag{3.1}$$

LEMMA 3.1. *Let a function $f : X \to Y$ satisfy $f(0) = 0$ and the inequality*

$$\begin{aligned} &\left\| \sum_{1 \le k < l \le n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^{n} f(x_i) \right\| \\ &\le \varphi(x_1, x_2, \cdots, x_n) \end{aligned} \tag{3.2}$$

for all $x_1, ..., x_n \in X$, where $n \ge 2$ is an integer number. Then

$$\left\| \frac{f(2^m x)}{4^m} - f(x) \right\| \le \frac{1}{2n(n-1)} \sum_{k=0}^{m-1} \left(\frac{1}{4^k}\right) \varphi \underbrace{(2^k x, \cdots, 2^k x)}_{n-times} \tag{3.3}$$

for all $m \in N$ and $x \in X$.

Proof. Now, we are going to prove our assertion by induction on $m \in N$. Put $x = x_1 = \cdots = x_n$ in (3.2). Then we obtain

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \le \frac{1}{2n(n-1)} \varphi(x, \cdots, x).$$

Thus it holds good for $m = 1$. We assume that the assertion is true for m. By replacing x by $2^m x$ in the above relation and dividing 4^m the resulting inequality, then we have

$$\left\| \frac{f(2^{m+1} x)}{4^{m+1}} - \frac{f(2^m x)}{4^m} \right\| \le \frac{1}{4^m} \frac{1}{2n(n-1)} \varphi(2^m x, \cdots, 2^m x).$$

And so

$$\begin{aligned}&\left\|\frac{f(2^{m+1}x)}{4^{m+1}}-f(x)\right\|\\&\le\left\|\frac{f(2^{m+1}x)}{4^{m+1}}-\frac{f(2^m x)}{4^m}\right\|+\left\|\frac{f(2^m x)}{4^m}-f(x)\right\|\\&\le\frac{1}{2n(n-1)}\sum_{k=0}^{m}\left(\frac{1}{4^k}\right)\varphi(2^k x,\cdots,2^k x).\end{aligned}$$

This completes the proof of the lemma. □

THEOREM 3.2. *Assume that a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality (3.2) for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x)-Q(x)\|\le\frac{1}{2n(n-1)}\Phi\underbrace{(x,\cdots,x)}_{n-times} \tag{3.4}$$

for all $x \in X$.

Proof. In order to prove convergence of the sequence $\left\{Q_m(x)=\frac{f(2^m x)}{4^m}\right\}$ we show that $\{Q_m(x)\}$ is a Cauchy sequence in Y. By (3.3), we have for $m_1 > m_2 > 0$,

$$\begin{aligned}&\left\|\frac{f(2^{m_1}x)}{4^{m_1}}-\frac{f(2^{m_2}x)}{4^{m_2}}\right\|\\&=\frac{1}{4^{m_2}}\left\|\frac{f(2^{m_1-m_2}\cdot 2^{m_2}x)}{4^{m_1-m_2}}-f(2^{m_2}x)\right\|\\&\le\frac{1}{2n(n-1)}\sum_{k=0}^{m_1-m_2-1}\left(\frac{1}{4^{k+m_2}}\right)\varphi(2^{k+m_2}x,\cdots,2^{k+m_2}x).\end{aligned}\tag{3.5}$$

Since the right-hand side of the inequality tends to 0 as $m_2 \to \infty$, the sequence $\{Q_m(x)\}$ is Cauchy in the Banach space Y. Therefore we may define a function $Q : X \to Y$ by

$$Q(x)=\lim_{m\to\infty}\frac{f(2^m x)}{4^m}$$

for all $x \in X$. By letting $m \to \infty$ in (3.3), we arrive at the formula (3.4).

Now we show that Q satisfies the functional equation (1.2) for all $x_1, \cdots, x_n \in X$. By definition of Q,

$$\begin{aligned}
&\Big\| \sum_{1\le k<l\le n} (Q(x_k+x_l)+Q(x_k-x_l)) - 2(n-1)\sum_{i=1}^{n} Q(x_i)\Big\| \\
&= \lim_{m\to\infty} 4^{-m}\Big\| \sum_{1\le k<l\le n} (f(2^m(x_k+x_l))+f(2^m(x_k-x_l))) \\
&\qquad\qquad -2(n-1)\sum_{i=1}^{n} f(2^m x_i)\Big\| \\
&\le \lim_{m\to\infty} 4^{-m}\varphi(2^m x_1, \cdots, 2^m x_n) = 0.
\end{aligned}$$

Thus Theorem 2.1 implies that Q is quadratic. It only remains to claim that Q is unique. Let $Q' : X \to Y$ be another quadratic function which satisfies the inequality (3.4). Since Q and Q' are quadratic function, we can easily show that

$$Q(2^m x) = 4^m Q(x) \quad \textit{and} \quad Q'(2^m x) = 4^m Q'(x) \tag{3.6}$$

for any $m \in N$. Thus, it follows from (3.4) that

$$\begin{aligned}
\|Q(x)-Q'(x)\| &\le \frac{1}{4^m}\left(\|Q(2^m x)-f(2^m x)\| + \|f(2^m x)-Q'(2^m x)\|\right) \\
&\le \frac{2}{4^m}\frac{1}{2n(n-1)}\Phi(2^m x, \cdots, 2^m x).
\end{aligned}$$

By letting $m \to \infty$, then we get that $Q(x) = Q'(x)$ for all $x \in X$, which completes the proof of the theorem. □

Corollary 3.3. *If a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\|Df(x_1, \cdots, x_n)\| \le \varepsilon(\|x_1\|^p + \cdots + \|x_n\|^p) \tag{3.7}$$

for some $0 < p < 2$ and for all $x_1, \cdots, x_n \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^{2-p}}{2(n-1)(2^{2-p}-1)}\varepsilon\|x\|^p$$

for all $x \in X$.

Corollary 3.4. *If a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\|Df(x_1, \cdots, x_n)\| \le \varepsilon$$

for all $x_1, \cdots, x_n \in X$, *then there exists a unique quadratic function* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \le \frac{2\varepsilon}{3n(n-1)}$$

for all $x \in X$.

Now, we investigate another stability question controlled by a function $\varphi : X^n \to [0, \infty)$.

THEOREM 3.5. *Assume that a function* $f : X \to Y$ *satisfies the inequality (3.2) for all* $x_1, x_2, \cdots x_n \in X$ *and* φ *satisfies the condition*

$$\hat{\Phi}(x_1, x_2, \cdots x_n) := \sum_{k=0}^{\infty} 4^k \varphi(\frac{x_1}{2^k}, \cdots, \frac{x_n}{2^k}) < \infty.$$

Then there exists a unique quadratic function $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\| \le \frac{1}{2n(n-1)} \hat{\Phi} \underbrace{(x, \cdots, x)}_{n-times}.$$

Proof. The proof is similar to that of Theorem 3.2. □

COROLLARY 3.6. *If a function* $f : X \to Y$ *satisfies the inequality (3.7) for some* $p > 2$ *and for all* $x_1, \cdots, x_n \in X$, *then there exists a unique quadratic function* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \le \frac{2^{p-2}}{2(n-1)(2^{p-2}-1)} \varepsilon \|x\|^p$$

for all $x \in X$.

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