JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 1, March 2007

ERROR ANALYSIS OF k-FOLD PSEUDO-HALLEY'S METHOD FINDING A SIMPLE ZERO

Young Ik Kim*

Abstract. Given a nonlinear function $f : \mathbb{R} \to \mathbb{R}$ that has a simple real zero α, a new numerical method to be called k-fold pseudo-Halley's method is proposed and it's error analysis is under investigation to confirm the convergence behavior near α. Under the assumption that f is sufficiently smooth in a small neighborhood of α, the order of convergence is found to be at least $k+3$. In addition, the corresponding asymptotic error constant is explicitly expressed in terms of k, α and f as well as the derivatives of f. A zero-finding algorithm is written and has been successfully implemented for numerous examples with Mathematica.

1. Introduction and preliminaries

The orders of convergence are known to range between 1 and 3 for many classical numerical schemes[5-9] such as Newton's method, Halley's method and the secant method as well as the bisection method. In this paper, a high-order numerical method will be developed by extending the classical method and its error analysis will be presented together with computational examples confirmed via Mathematica programming. Suppose that a given function $f : \mathbb{R} \to \mathbb{R}$ has a simple real zero α and is sufficiently smooth[1,10] in a small neighborhood of α. Let $g : \mathbb{R} \to \mathbb{R}$ be an iterative function whose fixed point is the simple zero α of f to be sought. The aim of this analysis is to find α as accurate as possible and to conduct the error analysis for a constructed iterative method

$$x_{n+1} = g(x_n),\ n = 0,\ 1,\ 2, \cdots . \tag{1.1}$$

Received December 20, 2006.
2000 Mathematics Subject Classification: Primary 65H05, 65H99.
Key words and phrases: pseudo-Halley's method, order of convergence, asymptotic error constant.
*This author was supported by the research fund of Dankook University in 2005.

In view of the detailed analysis described in [5], the *asymptotic error constant (also called the speed of convergence)* η and *order of convergence* p[2,3,11] satisfy the following relation:

$$\eta = \lim_{n\to\infty}\left|\frac{e_{n+1}}{{e_n}^p}\right| = \ |g^{(p)}(\alpha)| \ / \ p!, \tag{1.2}$$

where $e_n = x_n - \alpha$ and p is the constant such that $\frac{d^i}{dx^i}g(x)|_{x=\alpha} = g^{(i)}(\alpha) = 0$ for $0 \le i \le p-1$ and $g^{(p)}(\alpha) \neq 0$. For an arbitrarily given $x \in \mathbb{R}$, we now define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(w) = w - \frac{2f(w)\ f'(x)}{2f'(x)^2 - f(w)\ f''(x)}\ , \tag{1.3}$$

where $'$ denotes the derivative operator and $2f'(x)^2 - f(w)\ f''(x) \neq 0$. Note that F is well-defined in a sufficiently small neighborhood of α. Let $w_0(x) = F(x)$. Then for $k \in \mathbb{N}$ we recursively define a sequence of functions

$$w_k(x) = F(w_{k-1}(x)) = w_{k-1}(x) - \frac{2f(w_{k-1}(x))\ f'(x)}{2f'(x)^2 - f(w_{k-1}(x))\ f''(x)}. \tag{1.4}$$

Hence $w_k(x) = F^k(w_0) = F^{k+1}(x)$ for $k \in \mathbb{N}$, where $F^k(w_0) = F \circ F \circ \cdots \circ F(w_0)$ denotes a k-fold composite map of F evaluated at w_0. As a result of the preceding analysis, we have constructed an iterative method with $x_0 \in \mathbb{R}$

$$x_{n+1} = F^{k+1}(x_n) = g(x_n) \tag{1.5}$$

which is called *k-fold pseudo-Halley's method.* If $k = 0$, the method becomes classical *Halley's method*[2,3,11] and has the cubic convergence as shown from Halley's method and other methods of Laguerre's type[4,8]. If $k = 1$, it is simply called *pseudo-Halley's method.*

2. Error analysis

This section will extensively analyze the error of k-fold pseudo-Halley's method by careful investigation of the order of convergence and the asymptotic error constant. Since $f'(\alpha) \neq 0$ due to the simplicity of α, it can be shown from (1.4) and (1.5) that, after induction on $k \in \mathbb{N}$,

$$w_k(\alpha) = \alpha, \text{ for all } k \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

A direct computation from (1.3) finally gives

$$w_0{}'(\alpha) = \frac{d}{dx}w_0(x)\Big|_{x=\alpha} = 0, w_0{}''(\alpha) = 0, w_0{}'''(\alpha) = 6({c_2}^2 - c_3), \tag{2.2}$$

with $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$ for $j = 2$ and 3.

Recalling the definition of $w_k(x)$ from (1.4), we find that for $k \in \mathbb{N}$

$$\eta_k(x) = -\frac{2f(w_{k-1}(x))\ f'}{2f'^2 - f(w_{k-1}(x))\ f''}, \tag{2.3}$$

where, for notational convenience, we let

$$\eta_k(x) = w_k(x) - w_{k-1}(x),\ f' = f'(x),\ f'' = f''(x). \tag{2.4}$$

Since the current analysis suffices to investigate the convergence behavior near a sufficiently small neighborhood of α where (2.3) is well-defined, it is convenient to rewrite in the following form:

$$2f'^2 \cdot \eta_k = f(w_{k-1}(x)) \cdot \{f'' \cdot \eta_k - 2f'\}. \tag{2.5}$$

Differentiating (2.5) with respect to x and evaluating at $x = \alpha$ gives

$$\begin{aligned} 4f'f'' \cdot \eta_k + 2f'^2 \cdot \eta_k' \Big|_{x=\alpha} &= f'(w_{k-1})w_{k-1}' \cdot \{f'' \cdot \eta_k - 2f'\} \Big|_{x=\alpha} \\ &+ f(w_{k-1}) \cdot \{f''' \cdot \eta_k + f'' \cdot \eta_k' - 2f''\} \Big|_{x=\alpha}, \end{aligned} \tag{2.6}$$

from which it follows that, after simplification, $2f'(\alpha)^2 \cdot w_k'(\alpha) = 0$ leading to the relation for all $k \in \mathbb{N}$

$$w_k'(\alpha) = 0. \tag{2.7}$$

Similarly, differentiating (2.5) twice with respect to x and evaluating at $x = \alpha$ we get

$$\begin{aligned} &4(f''^2 + f'f''') \cdot \eta_k + 8f'f'' \cdot \eta_k' + 2f'^2 \cdot \eta_k'' \Big|_{x=\alpha} \\ &= (f'' \cdot \eta_k - 2f') \cdot \Big(f''(w_{k-1}) \cdot {w_{k-1}'}^2 + f'(w_{k-1}) \cdot w_{k-1}''\Big) \Big|_{x=\alpha} \\ &\quad + 2f'(w_{k-1}) \cdot w_{k-1}' \cdot \big(f''' \cdot \eta_k + f'' \cdot \eta_k' - 2f''\big) \Big|_{x=\alpha}, \end{aligned}$$

from which it follows that, after simplification, $2f'(\alpha)^2 \cdot w_k''(\alpha) = 0$ leading to the relation for all $k \in \mathbb{N}$

$$w_k''(\alpha) = 0. \tag{2.8}$$

By continuing differentiation (2.5) m times with respect to x and evaluation at $x = \alpha$, we are able to establish the following Lemma 2.1.

LEMMA 2.1. *Let* $w_k^{(m)}(\alpha) = \frac{d^m}{dx^m} w_k(x)|_{x=\alpha}$ *for any* $k, m \in \mathbb{N} \cup \{0\}$. *For the given function* f *having a simple zero* α *as stated in Section 1, we further denote* $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$ *for* $j = 2$ *and* 3. *Then the following holds.*

$$w_k^{(m)}(\alpha) = \begin{cases} \alpha, \ if \ m = 0. \\ 0, \ if \ 1 \le m \le k+2. \\ (k+3)! \ 2^k {c_2}^k ({c_2}^2 - c_3), \ if \ m = k+3. \end{cases} \tag{2.9}$$

Proof. The assertion is clear when $k = 0$ or $m = 0$ from (2.1),(2.2) and (2.7). Thus, it suffices to consider $k, \ m \in \mathbb{N}$ as follows:

$$w_k^{(m)}(\alpha) = \begin{cases} 0, \ if \ 1 \le m \le k+2. \\ (k+3)! \ 2^k {c_2}^k ({c_2}^2 - c_3), \ if \ m = k+3. \end{cases} \tag{2.10}$$

The remaining proof will be completed by induction on $m \ge 1$. For $m = 1$ and $m = 2$, the assertion holds in view of (2.7) and (2.8). Suppose now (2.10) holds for $m \ge 1$. By differentiating $(m+1)$ times both sides of (2.5) with respect to x via Leibnitz Rule[6] and evaluating at $x = \alpha$ we obtain

$$2 \sum_{r=0}^{m+1} {}_{m+1}C_r \cdot (f'^2)^{(m+1-r)} \cdot \eta_k^{(r)}$$

$$= \sum_{r=0}^{m+1} {}_{m+1}C_r \cdot H^{(m+1-r)} \cdot \{ \left(f'' \cdot \eta_k\right)^{(r)} - 2 f^{(r+1)} \}, \tag{2.11}$$

where ${}_mC_r = \frac{m!}{(m-r)! r!}, H = f(w_{k-1}(x))$. According to the induction hypothesis, we have for $k \in \mathbb{N}$ the following relation

$$\eta_k^{(m)}(\alpha) = w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha) = \begin{cases} 0, \ if \ 1 \le m \le k+1. \\ -w_{k-1}^{(m)}(\alpha), \ if \ m = k+2. \end{cases} \tag{2.12}$$

Since $\eta_k^{(r)}(\alpha) = w_k^{(r)}(\alpha) - w_{k-1}^{(r)}(\alpha) = 0$ for $0 \le r \le m-1 \le k$, by inspection of (2.12), we find that the left side of (2.11) has possible nonvanishing terms for $r = m$ and $r = m+1$ as follows:

$$2\Big[(m+1)(f'^2)'(\alpha) \cdot \Big(w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha)\Big) + f'^2(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big)\Big]$$

$$= 2\Big[(m+1) 2 f'(\alpha) f''(\alpha) \cdot \Big(w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha)\Big) + f'^2(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big)\Big]. \tag{2.13}$$

Let us look at the factor $H^{(m+1-r)}$ in the right side of (2.11). In view of the fact that $H' = f'(w_{k-1}) \cdot w'_{k-1}$, we obtain via Leibnitz Rule:

$$H^{(m+1-r)}(x) = H'^{(m-r)}(x)$$
$$= \begin{cases} \sum_{j=0}^{m-r} {}_{m-r}C_j f'(w_{k-1})^{(m-r-j)} w_{k-1}^{(j+1)}(x), \ if \ 0 \le r \le m. \\ H(x), \ if \ r = m+1. \end{cases} \tag{2.14}$$

Since $H(\alpha) = f(w_{k-1}(\alpha)) = f(\alpha) = 0$ for $r = m+1$, it is sufficient to consider the values of r ranging $0 \le r \le m$ in (2.14). We observe that the induction hypothesis in addition to (2.10) also states

$$w_{k-1}^{(m)}(\alpha) = \begin{cases} 0, \ if \ 1 \le m \le k+1, \\ w_{k-1}^{(m)}(\alpha), \ if \ m = k+2. \end{cases} \tag{2.15}$$

Due to the fact that

$$w_{k-1}^{(j+1)}(\alpha) = \begin{cases} 0, \ if \ 1 \le j+1 \le m \le k+1, \\ w_{k-1}^{(m+1)}(\alpha), \ if \ j+1 = m+1 = k+3 \end{cases} \tag{2.16}$$

and $j+1$ ranges $1 \le j+1 \le m+1$, the summation in (2.14) has possible nonzero values for $j = m-1$ and m, i.e., for $r = 0$ and $r = 1$. Hence (2.14) yields the following relations for $x = \alpha$:

(i) when $r = 0$

$$H^{(m+1)}(\alpha) = \sum_{j=0}^{m} {}_{m}C_j f'(w_{k-1})^{(m-j)} w_{k-1}^{(j+1)}\Big|_{x=\alpha} = f'(\alpha) w_{k-1}^{(m+1)}(\alpha) \tag{2.17}$$

in view of the first relation in (2.16) with $1 \le m \le k+2$.

(ii) when $r = 1$

$$H^{(m)}(\alpha) = \sum_{j=0}^{m-1} {}_{m-1}C_j f'(w_{k-1})^{(m-1-j)} w_{k-1}^{(j+1)}\Big|_{x=\alpha} = f'(\alpha) w_{k-1}^{(m)}(\alpha) \tag{2.18}$$

in view of the first relation in (2.16) with $1 \le m \le k+2$.

Substituting (2.17) and (2.18) into the right side of (2.14) we find, after the evaluation at $x = \alpha$, from (2.13) that for $1 \le m \le k+2$

$$\begin{aligned} 2\Big[(m+1)2f'(\alpha)f''(\alpha) \cdot \Big(w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha)\Big) \\ + f'^2(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big)\Big] \\ = -2f'^2(\alpha) \cdot w_{k-1}^{(m+1)}(\alpha) - 2(m+1)f'(\alpha)f''(\alpha) \cdot w_{k-1}^{(m)}(\alpha), \end{aligned} \tag{2.19}$$

using $w_k(\alpha) - w_{k-1}(\alpha) = 0 = w'_k(\alpha) - w'_{k-1}(\alpha)$ from (2.1), (2.2) and (2.7).

Upon simplification of (2.19) with $f'(\alpha) \neq 0$ we obtain for $1 \leq m \leq k+2$

$$w_k^{(m+)}(\alpha) = -(m+1) \cdot c \cdot (w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha), \tag{2.20}$$

where $c = f''(\alpha)/f'(\alpha)$. Further computation of (2.20) in view of (2.10) and (2.15) gives the following relation:

$$w_k^{(m+1)}(\alpha) = \begin{cases} 0, \ if \ 1 \leq m \leq k+1, \\ c(m+1) \cdot w_{k-1}^{(m)}(\alpha), \ if \ m = k+2. \end{cases} \tag{2.21}$$

The second relation in (2.21) yields inductively for $m = k+2$ that

$$w_k^{(m+1)}(\alpha) = w_k^{(k+3)}(\alpha) = c(k+3)\ w_{k-1}^{(k+2)}(\alpha) = c^2\ (k+3)(k+2)\ w_{k-2}^{(k)}(\alpha)$$

$$= (k+3)(k+2)(k+1)\cdots 4 \cdots c^k \cdot w_0{'''}(\alpha) = \frac{(k+3)!}{3!} c^k \cdot w_0{'''}(\alpha)$$

$$= (k+3)!\ 2^k \cdot {c_2}^k ({c_2}^2 - c_3) \tag{2.22}$$

using (2.2). Hence (2.10) also holds for $m+1$, completing the proof. □

The preceding analysis immediately leads us to the following main theorem.

THEOREM 2.2. *Let $k \in \mathbb{N} \cup \{0\}$ be given and α be a simple real zero of the smooth function f described in Section 1. Then k-fold pseudo-Halley's method defined by (1.5) is at least of order $k+3$ and its asymptotic error constant η is given by $2^k \cdot |{c_2}^k \cdot ({c_2}^2 - c3)|$, where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$ for $j = 2$ and 3.*

Proof. Let $g(x) = w_k(x) = F^{k+1}(x)$ as described in (1.3), (1.4) and (1.5). Define the iteration $x_{n+1} = g(x_n)$ with x_0 chosen in a sufficiently small compact neighborhood of α. Further we let $e_n = x_n - \alpha$ for $n \in \mathbb{N} \cup \{0\}$. Then Lemma 2.1 together with (1.2) yields the asymptotic error constant η and the order of convergence $p = k+3$ shown below

$$\eta = \lim_{n \to \infty} \left| \frac{e_{n+1}}{{e_n}^{k+3}} \right| = \frac{1}{(k+3)!}\ |g^{(k+3)}(\alpha)| = \frac{|w_k^{(k+3)}(\alpha)|}{(k+3)!} = 2^k \cdot |{c_2}^k \cdot ({c_2}^2 - c3)|,$$

which completes the proof. □

3. Numerical experiments with remarks

Based on the discussion in Sections 1 and 2, we first construct a zero-finding algorithm with the aid of symbolic and computational ability of *Mathematica*[8] as follows.

Algorithm 3.1 (Zero-Finding Algorithm)

Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct the iteration function $g = F^{k+1}$ with the given function f having a simple zero α, as stated in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and the initial value x_0. Compute $f(x_0)$ and $e_0 = |x_0 - \alpha\ |$.
Step 3. Compute $x_{n+1} = g(x_n)$ for $0 \le n \le n_{max}$ and display the computed values of $n,\ x_n, f(x_n), e_n = |x_n - \alpha|$, $|e_{n+1}/{e_n}^{k+3}|$ and η.

According to the above algorithm, we have conducted numerical experiments for a variety of test functions. The numerical results for approximated zeros of $f(x)$ are computed with the aid of Mathematica programming. The limited space allows us to illustrate only typical computational results for several test functions shown below.

(1) $f(x) = x\ \cos(\pi x) + \frac{3}{4} + \frac{1}{4}\ x^2\ e^{-(x-1)^2},\ \alpha = 1.$
(2) $f(x) = \sin^2 x - x^2 + 1,\ \alpha = 1.40449164821534\cdots 80742.$
(3) $f(x) = x^2\ \sin^2 x + e^{x^2\ \cos x\ \sin x} - 28,$
$\alpha = 4.62210416355283\cdots 82937.$

The experimental results are summarized in Tables 1-3 and apparently show a good agreement with the theory presented in this paper. The symbolic computation of $f'(x)$ and $f''(x)$in (1.4) has been easily done with the aid of Mathematica. To maintain sufficient accuracy and keep track of the asymptotic error constant requiring highly accurate arithmetic, the minimum number of precision digits was chosen as 350 by assigning *$MinPrecision=350* in Mathematica. Only the first 15 and the last 5 significant digits of the most accurate α were displayed for each test function (2) and (3) due to a limited paper space.

These two computed solutions show better results than those of Weerakoon and Fernando[9]. The reason is explained as follows: In test function (2), although 15 significant digits are listed for the solution their result is found to be accurate up to only 12 digits out of them. In test

TABLE 1. Convergence for $f(x) = x\cos(\pi x) + \frac{3}{4} + \frac{1}{4}x^2 e^{-(x-1)^2}$, $\alpha = 1$.

k	n	x_n	$f(x_n)$	$e_n = \mid x_n - \alpha \mid$	$e_{n+1}/{e_n}^{k+3}$	η
0	0	0.930000000000000	0.0575655	0.0700000		106.2786954
	1	0.992548043649202	0.00399817	0.00745196	21.72582026	
	2	0.999964985280326	0.0000175134	0.0000350147	84.61351691	
	3	0.999999999995443	2.27868×10^{-12}	4.55737×10^{-12}	106.1602832	
	4	1.000000000000000	5.02989×10^{-33}	1.00598×10^{-32}	106.2786954	
	5	1.000000000000000	5.40980×10^{-95}	1.08196×10^{-94}	106.2786954	
	6	1.000000000000000	6.73054×10^{-281}	1.34611×10^{-280}	106.2786954	
	7	1.000000000000000	8.86187×10^{-367}	8.86187×10^{-367}		
1	0	0.930000000000000	0.0575655	0.0700000		2097.85736
	1	0.996161339824199	0.00199179	0.00383866	159.8775583	
	2	0.999999626662722	1.86669×10^{-7}	3.73337×10^{-7}	1719.422827	
	3	1.000000000000000	2.03771×10^{-23}	4.07542×10^{-23}	2097.815905	
	4	1.000000000000000	2.89359×10^{-87}	5.78718×10^{-87}	2097.857360	
	5	1.000000000000000	1.17656×10^{-342}	2.35312×10^{-342}		
2	0	0.930000000000000	0.0575655	0.0700000		41410.04447
	1	0.997950127022885	0.00104563	0.00204987	1219.654297	
	2	0.999999998709524	6.45238×10^{-10}	1.29048×10^{-9}	35654.56543	
	3	1.000000000000000	7.41012×10^{-41}	1.48202×10^{-40}	41410.04052	
	4	1.000000000000000	1.48031×10^{-195}	2.96061×10^{-195}	41410.04447	
	5	1.000000000000000	$-8.86187 \times 10^{-367}$	8.86187×10^{-367}		
3	0	0.930000000000000	0.0575655	0.0700000		817401.5144
	1	0.998886423283940	0.000562902	0.00111358	9465.245910	
	2	0.999999999998597	7.01642×10^{-13}	1.40328×10^{-12}	735910.0977	
	3	1.000000000000000	3.12089×10^{-66}	6.24179×10^{-66}	817401.5143	
	4	1.000000000000000	$-8.86187 \times 10^{-367}$	8.86187×10^{-367}		
4	0	0.930000000000000	0.0575655	0.0700000		16134859.17
	1	0.999389708402777	0.000306983	0.000610292	74105.61406	
	2	1.000000000000000	2.37056×10^{-16}	4.74113×10^{-16}	15035564.45	
	3	1.000000000000000	4.34416×10^{-101}	8.68831×10^{-101}	16134859.17	
	4	1.000000000000000	0	0		

function (3), however, their result appears to be very poor or exceptionally wrong since it matches our result(strongly believed to be accurate up to 335 significant digits) with only the first significant digit.

The error bound ϵ for $\mid x_n - \alpha \mid < \epsilon$ was chosen as 0.5×10^{-335} for the current experiments. As can be seen in Tables 1-3, the number of computation gets smaller due to high-order convergence as k increases. For each k, the order of convergence has been confirmed to be of at least $k+2$. The computed asymptotic error constants have shown to be in good agreement with the theoretical asymptotic error constants η up to 10 significant digits. Even though the computed root was rounded to maintain 335 significant digits, we list it only up to 15 significant digits, due to a limited space.

Although not shown here, the error analysis stated in Theorem 1 has been confirmed through many additional experiments. This new development will play a crucial role in accurate computation of zeros

TABLE 2. Convergence for $f(x) = \sin^2 x - x^2 + 1,\ \alpha = 1.40449164821534 \cdots 80742$.

k	n	x_n	$f(x_n)$	$e_n = \mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^{k+3}\rvert$	η
0	0	1.13000000000000	0.541061	0.274492		0.5262992283
	1	1.38975140172492	0.0361703	0.0147402	0.7127173662	
	2	1.40448993177358	4.26101×10^{-6}	1.71644×10^{-6}	0.5359383508	
	3	1.40449164821534	6.60702×10^{-18}	2.66147×10^{-18}	0.5263003445	
	4	1.40449164821534	2.46309×10^{-53}	9.92191×10^{-54}	0.5262992283	
	5	1.40449164821534	1.27615×10^{-159}	5.14066×10^{-160}	0.5262992283	
	6	1.40449164821534	$-8.86187 \times 10^{-367}$	1.28503×10^{-366}		
1	0	1.13000000000000	0.541061	0.274492		0.8247855728
	1	1.41342297971840	-0.0223271	0.00893133	1.573256988	
	2	1.40449165334135	-1.27252×10^{-8}	5.12601×10^{-9}	0.8055912366	
	3	1.40449164821534	-1.41366×10^{-33}	5.69455×10^{-34}	0.8247855617	
	4	1.40449164821534	$-2.15309 \times 10^{-133}$	8.67318×10^{-134}	0.8247855728	
	5	1.40449164821534	$-8.86187 \times 10^{-367}$	1.28503×10^{-366}		
2	0	1.13000000000000	0.541061	0.274492		1.29255603
	1	1.39816914793475	0.0156177	0.00632250	4.057356628	
	2	1.40449164820195	3.32555×10^{-11}	1.33961×10^{-11}	1.325971517	
	3	1.40449164821534	1.38429×10^{-54}	5.57624×10^{-55}	1.292556030	
	4	1.40449164821534	1.72999×10^{-271}	6.96880×10^{-272}	1.292556030	
	5	1.40449164821534	$-8.86187 \times 10^{-367}$	1.28503×10^{-366}		
3	0	1.13000000000000	0.541061	0.274492		2.025618713
	1	1.40854352019419	-0.0100906	0.00405187	9.472855340	
	2	1.40449164821535	-2.17682×10^{-14}	8.76876×10^{-15}	1.981545148	
	3	1.40449164821534	-2.28598×10^{-84}	9.20847×10^{-85}	2.025618713	
	4	1.40449164821534	$-8.86187 \times 10^{-367}$	1.28503×10^{-366}		
4	0	1.13000000000000	0.541061	0.274492		3.17443196
	1	1.40171240982554	0.00688436	0.00277924	23.67128614	
	2	1.40449164821534	1.02869×10^{-17}	4.14381×10^{-18}	3.235332535	
	3	1.40449164821534	1.65328×10^{-121}	6.65982×10^{-122}	3.174431960	
	4	1.40449164821534	$-8.86187 \times 10^{-367}$	1.28503×10^{-366}		

for the nonlinear algebraic equation. The future study will include the cases when zeros are multiple.

TABLE 3. Convergence behavior for $f(x) = x^2 \sin^2 x + e^{x^2 \cos x \sin x} - 28$, $\alpha = 4.62210416355283 \cdots 82937$.

k	n	x_n	$f(x_n)$	$e_n = \mid x_n - \alpha \mid$	$\|e_{n+1}/e_n{}^{k+3}\|$	η
0	0	4.39000000000000	316.831	0.232104		45.74465654
	1	4.51250419256673	44.1560	0.109600	8.765196287	
	2	4.59453001886434	4.58122	0.0275741	20.94453547	
	3	4.62128965684487	0.102228	0.000814507	38.84977563	
	4	4.62210413893741	3.06364×10^{-6}	2.46154×10^{-8}	45.55366887	
	5	4.62210416355284	8.49166×10^{-20}	6.82280×10^{-22}	45.74465079	
	6	4.62210416355284	1.80825×10^{-60}	1.45287×10^{-62}	45.74465654	
	7	4.62210416355284	1.74604×10^{-182}	1.40289×10^{-184}	45.74465654	
	8	4.62210416355284	1.10998×10^{-361}	6.02124×10^{-364}		
1	0	4.39000000000000	316.831	0.232104		942.1375541
	1	4.52370746660951	35.1317	0.0983967	33.90383266	
	2	4.60512100263056	2.52261	0.0169832	181.1742387	
	3	4.62204581050282	0.00726700	0.0000583531	701.4382779	
	4	4.62210416355283	1.35822×10^{-12}	1.09129×10^{-14}	941.2075845	
	5	4.62210416355284	1.66304×10^{-51}	1.33620×10^{-53}	942.1375541	
	6	4.62210416355284	3.73798×10^{-207}	3.00335×10^{-209}	942.1375541	
	7	4.62210416355284	$-2.80859 \times 10^{-362}$	6.07440×10^{-365}		
2	0	4.39000000000000	316.831	0.232104		19403.86567
	1	4.53252079854156	29.0741	0.0895834	132.9880802	
	2	4.61181230635645	1.42519	0.0102919	1783.844085	
	3	4.62210249710035	0.000207411	1.66645×10^{-6}	14431.92218	
	4	4.62210416355284	3.10358×10^{-23}	2.49363×10^{-25}	19402.93657	
	5	4.62210416355284	2.32852×10^{-117}	1.87089×10^{-119}	19403.86567	
	6	4.62210416355284	1.97664×10^{-362}	1.54680×10^{-365}		
3	0	4.39000000000000	316.831	0.232104		399633.7917
	1	4.53975981567950	24.7062	0.0823443	526.6671228	
	2	4.61603689328116	0.804029	0.00606727	19462.13920	
	3	4.62210414794708	1.94230×10^{-6}	1.56058×10^{-8}	312841.8983	
	4	4.62210416355284	7.18459×10^{-40}	5.77260×10^{-42}	399633.5389	
	5	4.62210416355284	1.84044×10^{-240}	1.47874×10^{-242}	399633.7917	
	6	4.62210416355284	$-2.70854 \times 10^{-362}$	1.69745×10^{-364}		
4	0	4.39000000000000	316.831	0.232104		8230688.162
	1	4.54587822514980	21.4005	0.0762259	2100.497952	
	2	4.61865212550726	0.445227	0.00345204	230863.9470	
	3	4.62210416351268	4.99863×10^{-9}	4.01625×10^{-11}	6875293.008	
	4	4.62210416355284	1.72667×10^{-64}	1.38732×10^{-66}	8230688.145	
	5	4.62210416355284	1.91975×10^{-362}	7.75011×10^{-365}		

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Department of Applied Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: yikbell@dreamwiz.com