JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 1, March 2007

ON STRONG C-INTEGRAL OF BANACH-VALUED FUNCTIONS

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Abstract. In this paper, we define and study the Banach-valued C-integral and strong C-integral, We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. We also study the primitive of the strong C-integral in terms of the C-variational measures.

1. Introduction

In 1996 B.Bongiorno introduced a constructive integration process of Riemann type, called C-integral, which is the smallest integral that includes the Lebesgue integral and derivatives. Some properties of the C-integral of real-valued functions were studied in [1, 2, 7].

In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval [a,b] into a Banach space X. We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. If the function $F : \mathcal{I} \to X$ is differentiable almost everywhere on $[a, b]$, then it is the indefinite strong C-integral of f if and only if the C-variational measure V_*F is absolutely continuous. Consequently, we prove that every function of bounded variation is a multiplier for the strong C-integral.

2. Definitions and basic properties

Throughout this paper, $[a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. $\mathcal{I}$ denote the family

Received November 2, 2006.
2000 Mathematics Subject Classification: Primary 28B05, 26A39.
Key words and phrases: C-integral, strong C-integral, C-variational measure.
Supported by NNSF of China (10571085) and by Science Foundation of Hohai University.

of all subintervals of $[a, b]$. $\overline{co}(Y)$ denote the closed convex hull of the set Y if $Y \subset X$.

A partition D is a finite collection of interval-point pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^{n}$, where $\{[u_i, v_i]\}_{i=1}^{n}$ are non-overlapping subintervals of $[a, b]$. $\delta(\xi)$ is a positive function on $[a, b]$, i.e. $\delta(\xi) : [a, b] \to \mathrm{R}^{+}$. We say that $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ is

(1) a partial partition of $[a, b]$ if $\bigcup_{i=1}^{n}[u_i, v_i] \subset [a, b]$,

(2) a partition of $[a, b]$ if $\bigcup_{i=1}^{n}[u_i, v_i] = [a, b]$,

(3) δ - fine *McShane partition* of $[a, b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all i=1,2,$\cdots$,n,

(4) δ - fine *C-partition* of $[a, b]$ if it is a δ - fine McShane partition of $[a, b]$ and satisfying the condition

$$\sum_{i=1}^{n} dist(\xi_i, [u_i, v_i]) < \frac{1}{\varepsilon},$$

here $dist(\xi_i, [u_i, v_i]) = inf\{|t_i - \xi_i| : t_i \in [u_i, v_i]\}$.

Given a δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i)$$

for integral sums over D, whenever $f : [a, b] \to X$.

Definition 2.1. A function $f : [a, b] \to X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^{+}$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ of $[a, b]$. A is called the *C-integral* of f on $[a, b]$, and we write $A = \int_a^b f$ or $A = (\mathrm{C}) \int_a^b f$.

The function f is C-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is C-integrable on $[a, b]$. We write$\int_E f = \int_a^b f\chi_E$.

The basic properties of the C-integral, for example, linearity and additivity with respect to intervals can be founded in [7]. We do not present them here. The reader is referred to [7] for the details.

Lemma 2.2. *(Saks-Henstock) Let $f : [a, b] \to X$ is C-integrable on $[a, b]$. Then for $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^{+}$ such that*

$$\|S(f, D) - \int_a^b f\| < \epsilon$$

for each δ *- fine C-partition* $D = \{([u, v], \xi)\}$ of $[a, b]$. Particulary, if $D^{'} = \{([u_i, v_i], \xi_i)\}_{i=1}^m$ is an arbitrary δ - fine *partial C-partition* of $[a, b]$, we have

$$\|S(f, D') - \sum_{i=1}^{m} \int_{u_i}^{v_i} f\| \leq \epsilon.$$

Proof. The proof is similar to the case for Banach-valued Henstock integrable functions and the reader is referred to [9, Lemma 3.4.1.] for details. □

THEOREM 2.3. *Let* $f : [a, b] \to X$ *is C-integrable on* $[a, b]$.

(1) for each $x^* \in X^*$*, the function* $x^* f$ *is C-integrable on* $[a, b]$ *and* $\int_a^b x^* f = x^*(\int_a^b f)$.

(2) If $T : X \to Y$ *is a continuous linear operator, then* Tf *is C-integrable on* $[a, b]$ *and* $\int_a^b Tf = T(\int_a^b f)$.

Proof. The proof is too easy and will be omitted. The reader is referred to [10, Theorem 2.9.] for details. □

DEFINITION 2.4. A function $f : [a, b] \to X$ is strongly C-integrable if there exists an additive function $F : \mathcal{I} \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\sum_{i=1}^{n} \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. We denote $F(u_i, v_i) = F(v_i) - F(u_i)$.

THEOREM 2.5. *Let* X *be a Banach space of finite dimension.* $f : [a, b] \to X$ *is C-integrable on* $[a, b]$ *if and only if* f *is strongly C-integrable on* $[a, b]$.

Proof (Sufficiency). From the definitions of the strong C-integral and C-integral, if f is strongly C-integrable on $[a, b]$, then f is C-integrable on $[a, b]$.

□

Proof (Necessity). f is C-integrable on $[a, b]$, then there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\| \sum f(\xi)(v - u) - F(u, v)\| < \epsilon$$

for each δ -fine *C-partition* $D = \{([u,v],\xi)\}$ of $[a,b]$. Let $\{e_1, e_2, \cdots, e_n\}$ be a base of X and $g_i : [a,b] \to R$ $(i = 1, 2, \cdots, n)$. By the Hahn-Banach Theorem, for each e_i there is $x_i^* \in X^*$ such that

$$x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.1}$$

for $i, j = 1, 2, \cdots, n$ and therefore $x_i^*(f) = \sum_{j=1}^{n} g_j x_i^*(e_j) = g_i$. Since $g_i : [a,b] \to R$ is C-integrable on $[a,b]$ from Theorem 2.3, for each $\varepsilon > 0$ there is a positive function $\delta_i(\xi) : [a,b] \to R^+$ such that

$$|S(g_i, D_i) - \sum \int_u^v g_i| < \epsilon$$

for each δ_i - fine *C-partition* $D_i = \{([u,v],\xi)\}$ of $[a,b]$. By an easy adaptation of Saks-Henstock Lemma we have

$$\sum |g_i(\xi)(v-u) - \int_u^v g_i| < 2\epsilon.$$

We also have

$$F(u,v) = \int_u^v f = \int_u^v \sum_{i=1}^{n} g_i e_i = \sum_{i=1}^{n} \int_u^v g_i e_i = \sum_{i=1}^{n} e_i G_i(u,v)$$

where $G_i(u,v) = \int_u^v g_i$. Let $\delta(\xi) < \delta_i(\xi)$ for $i = 1, 2, \cdots, n$ and consequently

$$\begin{aligned} & \sum \|f(\xi)(v-u) - F(u,v)\| \\ = \quad & \sum \| \sum_{i=1}^{n} g_i(\xi) e_i (v-u) - \sum_{i=1}^{n} e_i G_i(u,v)\| \\ \leq \quad & \sum_{i=1}^{n} \|e_i\| \sum |g_i(\xi)(v-u) - G_i(u,v)| \\ < \quad & \epsilon \cdot \sum_{i=1}^{n} \|e_i\| \end{aligned}$$

for each δ - fine *C-partition* $D = \{([u,v],\xi)\}$ of $[a,b]$. Hence f is strongly C-integrable on $[a,b]$.

□

3. the C-variational measure and the strong C-integral

Let $F : [a, b] \to X$, arbitrary $E \subset [a, b]$ and a positive function $\delta(\xi) : E \to R^+$, Let us set

$$V(F, \delta, E) = \sup_{D} \sum_{i} \|F(u_i, v_i)\|$$

where the supremum is take over all δ - fine *partial C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ of $[a, b]$ with $\xi_i \in E$. We put

$$V_* F(E) = \lim_{\varepsilon \to 0} \inf_{\delta} V(F, \delta, E)$$

where the infimum is take over all function $\delta(\xi) : E \to R^+$.

It is easy to know that the set function $V_* F(E)$ is a Borel metric outer measure, known as the C-variational measure generated by F.

Definition 3.1. $V_* F(E)$ is said to be absolutely continuous (AC) on a set E if for each set $N \subset E$ such that $V_* F(N) = 0$ whenever $\mu(N) = 0$.

Definition 3.2. A function $F : [a, b] \to X$ is differentiable at $\xi \in [a, b]$ if there is a $f(\xi) \in X$ such that

$$\lim_{\delta \to 0} \| \frac{F(\xi + \delta) - F(\xi)}{\delta} - f(\xi) \| = 0.$$

We denote $f(\xi) = F'(\xi)$ the derivative of F at ξ.

Theorem 3.3. *Let $F : \mathcal{I} \to X$ be differentiable almost everywhere on $[a, b]$. Then F is the indefinite strong C-integral of f if and only if the C-variational measure $V_* F$ is AC.*

Proof (Necessity). Let $E \subset [a, b]$ and $\mu(E) = 0$. Assume $E_n = \{\xi \in E : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \cdots$. Then we have $E = \bigcup E_n$ and $\mu(E_n) = 0$, so there are open sets G_n such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n}$.

By the Saks-Henstock Lemma, there exists a positive function δ_0 such that

$$\sum \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ_0 - fine *partial C-partition* $D = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$.

For $\xi \in E_n$, take $\delta_n(\xi) > 0$ such that $B(\xi, \delta_n(\xi)) \subset G_n$. Let

$$\delta(\xi) = min\{\delta_0(\xi), \delta_n(\xi)\}.$$

Assume $D' = \{([u,v],\xi)\}$ is a δ - fine *partial C-partition* with $\xi \in E$. We have

$$\begin{aligned}
\sum \|F(u,v)\| &= \sum \|F(u,v) - f(\xi)(v-u) + f(\xi)(v-u)\| \\
&\leq \sum \|F(u,v) - f(\xi)(v-u)\| + \sum \|f(\xi)(v-u)\| \\
&< \epsilon + \sum_n \sum_{\xi \in E_n} \|f(\xi)(v-u)\| \\
&< \epsilon + \sum_n n \frac{\epsilon}{n \cdot 2^n} = 2\epsilon
\end{aligned}$$

This shows that $V_*F(E) < 2\epsilon$. Hence the C-variational measure V_*F is AC as desired. □

Proof (Sufficiency). There exists a set $E \subset [a,b]$ be of measure zero such that $f(\xi) \neq F'(\xi)$ or $F'(\xi)$ does not exist for $\xi \in E$. We can define a function as follows

$$f(x) = \begin{cases} F'(\xi) & \text{if } \xi \in [a,b] \backslash E, \\ \theta & \text{if } \xi \in E. \end{cases} \tag{3.1}$$

Then for $\xi \in [a,b]\backslash E$, by the definition of derivative, for each $\varepsilon > 0$, there is a positive function $\delta_1(\xi)$ such that

$$\|f(\xi)(v-u) - F(u,v)\| < \frac{\epsilon^2}{1+\epsilon(b-a)}(dist(\xi,[u,v]) + v - u)$$

for each interval $[u,v] \subset (\xi - \delta_1(\xi), \xi + \delta_1(\xi))$.

V_*F is AC, then for $\xi \in E$, there is a positive function $\delta_2(\xi)$ such that

$$\sum \|F(u,v)\| < \epsilon$$

for each δ_2 - fine *partial C-partition* $D_0 = \{([u,v],\xi)\}$ with $\xi \in E$.

Define a positive function $\delta(\xi)$ as follows

$$\delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in [a,b] \backslash E, \\ \delta_2(\xi) & \text{if } \xi \in E. \end{cases} \tag{3.2}$$

Then for each δ - fine *C-partition of* $[a, b]$, we have

$$\begin{aligned} & \sum \|f(\xi)(v-u) - F(u,v)\| \\ = & \sum_{\xi \in E} \|F(u,v) - f(\xi)(v-u)\| + \sum_{\xi \in [a,b] \setminus E} \|F(u,v) - f(\xi)(v-u)\| \\ \leq & \epsilon + \frac{\epsilon^2}{1+\epsilon(b-a)} \sum_{\xi \in [a,b] \setminus E} (dist(\xi, [u,v]) + v - u) \\ < & \epsilon + \frac{\epsilon^2}{1+\epsilon(b-a)} (\frac{1}{\epsilon} + b - a) = 2\epsilon \end{aligned}$$

Hence f is strong C-integrable on $[a, b]$ with indefinite strong C-integral F. □

DEFINITION 3.4. Let $G : [a, b] \to R$. A function $F : [a, b] \to X$ is C-Stieltjes integrable with respect to G on $[a, b]$ if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\|S(F, G, D) - A\| < \epsilon$$

for each δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$, whenever

$$S(F, G, D) = \sum_{i=1}^{n} F(\xi_i)(G(v_i) - G(u_i)).$$

A is called the *C-Stieltjes integral* of F with respect to G on $[a, b]$, and we write $A = (CS) \int_a^b F dG$.

Similar to [8,Proposition 5.], we have the following Lemma.

LEMMA 3.5. *Let $G : [a, b] \to R$ be a non decreasing function. If a function $F : [a, b] \to X$ is C-Stieltjes integrable with respect to G, then for each $[u, v] \in [a, b]$, we have*

$$(CS) \int_u^v F dG \in \overline{co}(\{G(u,v)x : x \in X \ and \ x = F(\xi) \ for \ some \ \xi \in [u, v]\})$$

THEOREM 3.6. *Let $f : [a, b] \to X$ be strongly C-integrable on $[a, b]$ and $F(x) = \int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to R$ is a function of bounded variation, then Gf is strongly C-integrable and*

$$\int_a^b Gf = G(b)F(b) - (CS) \int_a^b F dG.$$

Proof. Let $\epsilon > 0$, arbitrary $E \subset [a,b]$ with $\mu(E) = 0$. Assume $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ is an arbitrary δ - fine *partial C-partition* with $\xi_i \in E$.

It is easy to know that F is continuous on $[a,b]$. G is of bounded variation, then the C-Stieltjes integral $(CS)\int_a^b FdG$ exists on $[a,b]$. We can assume G is non decreasing and with upper bounded $M > 0$ on $[a,b]$, then for each i, there are $x_1^{(i)}, x_2^{(i)}, \cdots, x_{m_i}^{(i)} \in [u_i, v_i]$ and numbers $\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{m_i}^{(i)}$ with $\sum_{j=1}^{m_i} \lambda_j^{(i)} = 1$ such that

$$\|\sum_{j=1}^{m_i} \lambda_j^{(i)} G(u_i, v_i) F(x_j^{(i)}) - \int_{u_i}^{v_i} FdG\| \leq \frac{\epsilon G(u_i, v_i)}{nV(G,[a,b])}$$

where $V(G,[a,b])$ denote the variation of G over the interval $[a,b]$.

We define a function by

$$\int_a^x Gf = H(x) = G(x)F(x) - (CS)\int_a^x FdG$$

and consequently have

$$\begin{aligned}
& \|H(v_i) - H(u_i)\| \\
= \ & \|G(v_i)F(v_i) - G(u_i)F(u_i) - (CS)\int_{u_i}^{v_i} FdG\| \\
= \ & \|G(v_i)[F(v_i) - F(u_i)] + (G(v_i) - G(u_i))[F(u_i) - \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)})] + \\
& (G(v_i) - G(u_i)) \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)}) - (CS)\int_{u_i}^{v_i} FdG\| \\
\leq \ & |G(v_i)| \cdot \|F(u_i, v_i)\| + G(u_i, v_i)\|F(u_i) - \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)})\| + \\
& \|G(u_i, v_i) \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)}) - (CS)\int_{u_i}^{v_i} FdG\| \\
\leq \ & M\|F(u_i, v_i)\| + G(u_i, v_i)\|\sum_{j=1}^{m_i} \lambda_j^{(i)} [F(u_i) - F(x_j^{(i)})]\| + \frac{\epsilon G(u_i, v_i)}{nV(G,[a,b])}
\end{aligned}$$

$$
\begin{aligned}
&\leq M\|F(u_i,v_i)\| + V(G,[a,b])\sum_{j=1}^{m_i}\lambda_j^{(i)}\|F(u_i)-F(x_j^{(i)})\| + \frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])}\\
&\leq M\|F(u_i,v_i)\| + V(G,[a,b])\sum_{j=1}^{m_i}\lambda_j^{(i)}\|F(u_i)-F(x_{N^{(i)}}^{(i)})\| + \frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])}\\
&= M\|F(u_i,v_i)\| + V(G,[a,b])\|F(u_i)-F(x_{N^{(i)}}^{(i)})\| + \frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])}
\end{aligned}
$$

where

$$\|F(u_i)-F(x_{N^{(i)}}^{(i)})\| = max\{\|F(u_i)-F(x_j^{(i)})\|\}$$

for each $j \in \{1,2,\cdots,m_i^{(i)}\}$. V_*F is AC from Theorem 3.3, then there exists a positive function $\delta(\xi)$ such that

$$\sum_{i=1}^{n}\|F(u_i,v_i)\| < \frac{\epsilon}{M+V(G,[a,b])}.$$

Therefore

$$
\begin{aligned}
\sum_{i=1}^{n}\|H(v_i)-H(u_i)\| &\leq M\sum_{i=1}^{n}\|F(u_i,v_i)\| + \frac{\epsilon\sum_{i=1}^{n}G(u_i,v_i)}{nV(G,[a,b])} + \\
&\quad V(G,[a,b])\sum_{i=1}^{n}\|F(u_i)-F(x_{N^{(i)}}^{(i)})\|\\
&\leq (M+V(G,[a,b]))\frac{\epsilon}{M+V(G,[a,b])} + \epsilon = 2\epsilon
\end{aligned}
$$

and it follows that V_*H is AC. We also have that $H(x)$ is differentiable almost everywhere and $H'(x) = G(x)f(x)$ a.e. on $[a,b]$, then Gf is strongly C-integrable on $[a,b]$ from Theorem 3.3. □

Consequently, we can easily get the following theorem.

THEOREM 3.7. *Let $f : [a,b] \to X$ and $G : [a,b] \to R$. If Gf is strongly C-integrable on $[a,b]$ for every strongly C-integrable f, then G is equivalent to a function of bounded variation on $[a,b]$.*

Acknowledgement. The authors are grateful to Professor Luisa Di Piazza for her interesting papers[6,7,8].

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