JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 2, June 2007

CAUCHY–RASSIAS STABILITY OF DERIVATIONS ON QUASI-BANACH ALGEBRAS

Jong Su An*, Deok-Hoon Boo**, and Choonkil Park ***

Abstract. In this paper, we prove the Cauchy–Rassias stability of derivations on quasi-Banach algebras associated to the Cauchy functional equation and the Jensen functional equation. We use the Cauchy–Rassias inequality that was first introduced by Th. M. Rassias in the paper "On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300".

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphism. In the next year, Hyers [9] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th. M. Rassias [19] generalized the theorem of Hyers by considering the particular stability problem with an unbounded Cauchy difference.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Consider $f : \mathcal{X} \to \mathcal{Y}$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{X}$. Assume that there exist constants $\theta \geq 0$ and $r \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

Received May 23, 2007.
2000 Mathematics Subject Classification: Primary 39B52, 39B82, 47B47.
Key words and phrases: Cauchy functional equation, Jensen functional equation, quasi-Banach algebra, p-Banach algebra, Cauchy–Rassias stability, derivation.
The second author was supported by Korean Council for University Education, grant funded by Korean Government(MOEHRD) for 2006 Domestic Faculty Exchange.

for all $x, y \in \mathcal{X}$. Then there exists a unique $\mathbb{R}$-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in \mathcal{X}$.

The above inequality, which is known as *Cauchy–Rassias inequality*, has provided a lot of influence in the development of what is now known as *Cauchy–Rassias stability* of functional equations. More general approach was considered already by Bourgin [5] and later by Forti [7], Găvruta [8] and others. During the last decades, several problems concerning the stability of functional equations have been extensively investigated by a number of mathematicians and there are many interesting results concerning this problem (see [3, 6, 10, 11, 13, 18, 20, 21, 23]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. ([2, 22]) Let $\mathcal{X}$ be a real linear space. A *quasi-norm* is a real-valued function on $\mathcal{X}$ satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. Obviously the balls with respect to $\|\cdot\|$ define a linear topology on $\mathcal{X}$. By a *quasi-Banach space* we mean a complete quasi-normed space, i.e. a quasi-normed space in which every $\|\cdot\|$-Cauchy sequence in $\mathcal{X}$ converges (see [16]). This class includes Banach spaces and the most significant class of quasi-Banach spaces which are not Banach spaces are the L_p spaces for $0 < p < 1$ with the quasi-norm $\|\cdot\|$.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

Definition 1.2. ([1]) A quasi-normed space $(\mathcal{A}, \|\cdot\|)$ is called a *quasi-normed algebra* if $\mathcal{A}$ is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\| \cdot \|y\|$ for all $x, y \in \mathcal{A}$.

A *quasi-Banach algebra* is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a *p-norm* then the quasi-Banach algebra is called a *p-Banach algebra*.

Given a p-norm, the formula $d(x, y) := \|x-y\|^p$ gives us a translation invariant metric on $\mathcal{X}$. By the Aoki–Rolewicz theorem [22] (see also [2]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Recently, the stability of derivations on other topological structures has been recently studied by the authors; see [4, 12, 14, 15, 17].

In this paper, we prove the stability of derivations on quasi-Banach algebras associated to the Cauchy functional equation and the Jensen functional equation. Throughout the paper we assume that $\mathcal{A}$ is a p-Banach algebra with p-norm $\|\cdot\|$ and the modulus of concavity K.

2. Stability of derivations associated to the Cauchy functional equation

In this section, we establish the stability of derivations on quasi-Banach algebras associated to the Cauchy functional equation.

Theorem 2.1. *Let $r > 2$, θ be a positive real number and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{1}$$

$$\|f(xy) - f(x)y - xf(y)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{2}$$

for all $x, y \in \mathcal{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - D(x)\| \leq \frac{2\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}}\|x\|^r \tag{3}$$

for all $x \in \mathcal{A}$.

Proof. Letting $y = x$ in (1), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{4}$$

for all $x \in \mathcal{A}$. So

$$\|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\| \leq \frac{2^{j+1}\theta}{2^r}\|\frac{x}{2^j}\|^r$$

for all $x \in \mathcal{A}$ and nonnegative integers j. Since $\mathcal{A}$ is a p-Banach algebra,

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\|^p \leq \sum_{j=l}^{m-1} \|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\|^p \tag{5}$$

$$\leq \frac{2^p\theta^p}{2^{pr}} \sum_{j=l}^{m-1} \frac{2^{pj}}{2^{prj}} \|x\|^{pr} \tag{6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathcal{A}$. It follows from (5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $D : \mathcal{A} \to \mathcal{A}$ by

$$D(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in \mathcal{A}$. It follows from (1) that

$$\begin{aligned}
\|D(x+y) - D(x) - D(y)\| &= \lim_{n\to\infty} 2^n \|f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})\| \\
&\leq \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\
&= 0
\end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$D(x+y) = D(x) + D(y), \quad x, y \in \mathcal{A}$$

And letting $l = 0$ and passing the limit $m \to \infty$ in (5), we get (3).

By the same reasoning as in the proof of Theorem of [19], the mapping $D : \mathcal{A} \to \mathcal{A}$ is $\mathbb{R}$-linear. It follows from (2) that

$$\begin{aligned}
&\|D(xy) - D(x)y - xD(y)\| \\
&\quad = \lim_{n\to\infty} 4^n \|f(\frac{xy}{2^n \cdot 2^n}) - f(\frac{x}{2^n}) \cdot \frac{y}{2^n} - \frac{x}{2^n} \cdot f(\frac{y}{2^n})\| \\
&\quad \le \lim_{n\to\infty} \frac{4^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) = 0
\end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$D(xy) = D(x)y + xD(y) \quad x, y \in \mathcal{A}$$

Now, let $T : \mathcal{A} \to \mathcal{A}$ be another Cauchy additive mapping satisfying (3). Then we have

$$\begin{aligned}
\|D(x) - T(x)\| &= 2^n \|D(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\
&\le 2^n K(\|D(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\|) \\
&\le \frac{2^{n+2}K\theta}{(2^{pr} - 2^p)^{\frac{1}{p}} 2^{nr}} \|x\|^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $D(x) = T(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of D. □

THEOREM 2.2. *Let $r < 1$, θ be a positive real number and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying (1) and (2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique derivation $D : \mathcal{A} \to \mathcal{A}$ such that*

$$\|f(x) - D(x)\| \le \frac{2\theta}{(2^p - 2^{pr})^{\frac{1}{p}}} \|x\|^r$$

for all $x \in \mathcal{A}$.

Proof. It follows from (4) that

$$\|f(x) - \frac{1}{2}f(2x)\| \le \theta \|x\|^r$$

for all $x \in \mathcal{A}$. So

$$(7)\quad \|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\|^p \leq \sum_{j=l}^{m-1} \|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\|^p$$

$$(8)\quad \leq \theta^p \sum_{j=l}^{m-1} \frac{2^{pjr}\theta}{2^{pj}}\|x\|^{pr}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathcal{A}$. It follows from (7) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So we can define the mapping $D : \mathcal{A} \to \mathcal{A}$ by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x), \quad x \in \mathcal{A}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of derivations associated to the Jensen functional equation

We study the stability of derivations on quasi-Banach algebras associated to the Jensen functional equation.

Theorem 3.1. *Let $r < 1$, θ be a positive real number and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ satisfying*

$$(9)\quad \|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

$$(10)\quad \|f(xy) - f(x)y - xf(y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in \mathcal{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$(11)\quad \|f(x) - D(x)\| \leq \frac{K(3+3^r)\theta}{(3^p - 3^{pr})^{\frac{1}{p}}}\|x\|^r$$

for all $x \in \mathcal{A}$.

Proof. Letting $y = -x$ in (9), we get

$$\|-f(x) - f(-x)\| \leq 2\theta\|x\|^r$$

for all $x \in \mathcal{A}$. Letting $y = 3x$ and replacing x by $-x$ in (9), we get

$$\|2f(x) - f(-x) - f(3x)\| \leq (3^r + 1)\theta\|x\|^r$$

for all $x \in \mathcal{A}$. Thus

$$\|3f(x) - f(3x)\| \leq K(3^r + 3)\theta\|x\|^r \tag{12}$$

for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is a p-Banach algebra,

$$\begin{aligned} \|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}f(3^m x)\|^p &\leq \sum_{j=l}^{m-1} \|\frac{1}{3^j}f(3^j x) - \frac{1}{3^{j+1}}f(3^{j+1}x)\|^p \\ &\leq \frac{K^p(3^r+3)^p\theta^p}{3^p} \sum_{j=l}^{m-1} \frac{3^{prj}}{3^{pj}}\|x\|^{pr} \end{aligned} \tag{13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathcal{A}$. It follows from (13) that the sequence $\{\frac{1}{3^n}f(3^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{\frac{1}{3^n}f(3^n x)\}$ converges. So one can define the mapping $D : \mathcal{A} \to \mathcal{A}$ by

$$D(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in \mathcal{A}$. By (9),

$$\begin{aligned} &\|2H(\frac{x+y}{2}) - H(x) - H(y)\| \\ &\quad = \lim_{n\to\infty} \frac{1}{3^n}\|2f(3^n \cdot \frac{x+y}{2}) - f(3^n x) - f(3^n y)\| \\ &\quad \leq \lim_{n\to\infty} \frac{3^{rn}}{3^n}\theta(\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$2D(\frac{x+y}{2}) = D(x) + D(y), \quad x, y \in \mathcal{A}$$

Hence $D(\frac{x}{2}) = \frac{1}{2}D(x)$ for each $x \in \mathcal{A}$ and so $D(x) + D(y) = 2D(\frac{x+y}{2}) = D(x) + D(y)$ for all $x, y \in \mathcal{A}$.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (13), we get (11). It follows from (10) that

$$\begin{aligned} &\|D(xy) - D(x)y - xD(y)\| \\ &\quad = \lim_{n\to\infty} \frac{1}{9^n}\|f(9^n xy) - f(3^n x)\cdot 3^n y - 3^n x \cdot f(3^n y)\| \\ &\quad \le \lim_{n\to\infty} \frac{3^{nr}\theta}{9^n}(\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$D(xy) = D(x)y + xD(y), \quad x, y \in \mathcal{A}$$

Now, let $T : \mathcal{A} \to \mathcal{A}$ be another Jensen additive mapping satisfying (11). Then we have

$$\begin{aligned} \|D(x) - T(x)\|^p &= \frac{1}{3^{pn}}\|D(3^n x) - T(3^n x)\|^p \\ &\le \frac{1}{3^{pn}}(\|D(3^n x) - f(3^n x)\|^p + \|T(3^n x) - f(3^n x)\|^p) \\ &\le 2 \cdot \frac{3^{prn}}{3^{pn}} \cdot \frac{K^p(3+3^r)^p\theta^p}{3^p - 3^{pr}}\|x\|^{pr}, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $D(x) = T(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of D.

The rest of the proof is similar to the proof of Theorem 2.1. □

THEOREM 3.2. *Let $r > 2$, θ be a positive real number and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ satisfying (9) and (10). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique derivation $D : \mathcal{A} \to \mathcal{A}$ such that*

$$\|f(x) - D(x)\| \le \frac{K(3^r+3)\theta}{(3^{pr} - 3^p)^{\frac{1}{p}}}\|x\|^r$$

for all $x \in \mathcal{A}$.

Proof. It follows from (12) that

$$\|f(x) - 3f(\frac{x}{3})\| \le \frac{K(3^r+3)\theta}{3^r}\|x\|^r$$

for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is a p-Banach algebra,

$$\begin{aligned} \|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m})\|^p &\le \sum_{j=l}^{m-1} \|3^j f(\frac{x}{3^j}) - 3^{j+1} f(\frac{x}{3^{j+1}})\|^p \\ &\le \frac{K^p(3^r+3)^p\theta^p}{3^{pr}} \sum_{j=l}^{m-1} \frac{3^{pj}}{3^{prj}} \|x\|^{pr} \qquad (14) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathcal{A}$. It follows from (14) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $D : \mathcal{A} \to \mathcal{A}$ by

$$D(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in \mathcal{A}$.

The rest of the proof is similar to that of Theorems 2.1 and 3.1. □

References

[1] J. M. Almira and U. Luther, *Inverse closedness of approximation algebras*, J. Math. Anal. Appl. **314** (2006) 30–44.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis Vol. 1*, Colloq. Publ. **48**, Amer. Math. Soc., Providence, 2000.

[3] C. Baak, *Cauchy–Rassias stability of Cauchy–Jensen additive mappings in Banach spaces*, Acta Math. Sinica **22** (2006), 1789–1796.

[4] C. Baak and M. S. Moslehian, *On the stability of θ-derivations on JB*-triples*, Bull. Braz. Math. Soc. **38** (2007), 115–127.

[5] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[7] G. L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica **4** (1980), 23–30.

[8] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[10] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[11] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Florida, 2001.

[12] C. Park, *Lie ∗-homomorphisms between Lie C^*-algebras and Lie ∗-derivations on Lie C^*-algebras*, J. Math. Anal. Appl. **293** (2004), 419–434.

[13] C. Park, *Universal Jensen's equations in Banach modules over a C^*-algebra and its unitary group*, Acta Math. Sinica **20** (2004), 1047–1056.
[14] C. Park *Linear $*$-derivations on JB^*-algebras* Acta Math. Scientia **25** (2005) 449–454.
[15] C. Park, *Homomorphisms between Lie JC^*-algebras and Cauchy–Rassias stability of Lie JC^*-algebra derivations*, J. Lie Theory **15** (2005), 393–414.
[16] C. Park *Completion of quasi-normed algebras and quasi-normed modules*, J. Chungcheong Math. Soc. **19** (2006), 9–18.
[17] C. Park and J. Hou, *Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras*, J. Korean Math. Soc. **41** (2004), 461–477.
[18] C. Park, J. Hou and S. Oh, *Homomorphisms between JC^*-algebras and between Lie C^*-algebras*, Acta Math. Sinica **21** (2005), 1391–1398.
[19] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[20] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
[21] Th. M. Rassias (ed.), *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
[22] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Sci. Publ., Reidel and Dordrecht, 1984.
[23] P.K. Sahoo, *A generalized cubic functional equation*, Acta Math. Sinica **21** (2005), 1159–1166.
[24] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

*

Department of Mathematics Education
Pusan National University
Pusan 609–735, Republic of Korea
E-mail: jsan63@@pusan.ac.kr

**

Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: dhboo@@cnu.ac.kr

***

Department of Mathematics
Hanyang University
Seoul 133–791, Republic of Korea
E-mail: baak@@hanyang.ac.kr