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ON THE HYERS-ULAM-RASSIAS STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION

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Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of a Cauchy-Jensen functional equation

$$2f(x+y, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w).$$

1. Introduction

In 1940, S. M. Ulam [5] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [4] gave a generalization. Recently, Găvruta [1] also obtained a further generalization of the Hyers-Ulam-Rassias theorem.

Throughout this paper, let X be normed space and Y be a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$).

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A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping[3] if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z)+f(y,z), \\ 2f(x,\frac{y+z}{2}) &= f(x,y)+f(x,z). \end{aligned} \tag{1.1}$$

It is easy to see that a mapping $f : X \times X \to Y$ is a Cauchy-Jensen mapping if and only if the mapping f satisfies the functional equation

$$2f(x+y,\frac{z+w}{2}) = f(x,z)+f(x,w)+f(y,z)+f(y,w) \tag{1.2}$$

for all $x, y, z, w \in X$. In 2006, Park and Bae [3] obtained the generalized Hyers-Ulam stability of (1) and (2). In this paper, I study the Hyers-Ulam-Rassias stability of (1) and (2).

2. Stability of (1) and (2)

For the given mapping $f : X \times X \to Y$, we define

$$\begin{aligned} Df(x,y,z,w) &:= 2f(x+y,\frac{z+w}{2})-f(x,z) \\ &\quad -f(x,w)-f(y,z)-f(y,w) \\ D_1f(x,y,z) &:= f(x+y,z)-f(x,z)-f(y,z), \\ D_2f(x,y,z) &:= 2f(x,\frac{y+z}{2})-f(x,y)-f(x,z). \end{aligned}$$

for all $x, y, z, w \in X$.

THEOREM 2.1. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_2, \delta_3, \delta_4$ be fixed positive real numbers with $p_1, p_2, p_3, q_1, q_2, q_3 < 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|D_1f(x,y,z)\| \le (\|x\|^{p_1}+\|y\|^{p_2}+\delta_1)(\|z\|^{q_1}+\delta_2), \tag{2.1}$$

$$\|D_2f(x,y,z)\| \le (\|x\|^{p_3}+\delta_3)(\|y\|^{q_2}+\|z\|^{q_3}+\delta_4) \tag{2.2}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ such that

$$\|f(x,y)-F_1(x,y)\| \le (\frac{\|x\|^{p_1}}{2-2^{p_1}}+\frac{\|x\|^{p_2}}{2-2^{p_2}}+\delta_1)(\|y\|^{q_1}+\delta_2), \tag{2.3}$$

$$\|f(x,y)-f(x,0)-F_2(x,y)\| \le (\|x\|^{p_3}+\delta_3)(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2}+\delta_4), \tag{2.4}$$

$$F_1(x,y)-F_1(x,0) = F_2(x,y) \tag{2.5}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

Proof. Letting $y = x$ and replacing z by y in (2.1),

$$\|f(2x,y) - 2f(x,y)\| \le (\|x\|^{p_1} + \|x\|^{p_2} + \delta_1)(\|y\|^{q_1} + \delta_2)$$

for all $x, y \in X$. Thus

$$\begin{aligned}\|\frac{1}{2^j} f(2^j x, y) \;-\; & \frac{1}{2^{j+1}} f(2^{j+1}x, y)\| \\ & \le \Big(\frac{2^{jp_1}}{2^{j+1}}\|x\|^{p_1} + \frac{2^{jp_2}}{2^{j+1}}\|x\|^{p_2} + \frac{\delta_1}{2^{j+1}}\Big)(\|y\|^{q_1} + \delta_2)\end{aligned}$$

for all $x, y \in X$. For given integers l, m $(0 \le l < m)$,

$$\begin{aligned}\|\frac{1}{2^l} f(2^l x, y) \;-\; & \frac{1}{2^m} f(2^m x, y)\| \\ & \le \sum_{j=l}^{m-1} \Big(\frac{2^{jp_1}}{2^{j+1}}\|x\|^{p_1} + \frac{2^{jp_2}}{2^{j+1}}\|x\|^{p_2} + \frac{\delta_1}{2^{j+1}}\Big)(\|y\|^{q_1} + \delta_2) \qquad (2.6)\end{aligned}$$

for all $x, y \in X$. By $p_1, p_2 < 1$, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.6), one can obtain the inequality (2.3). By (2.1) and (2.2),

$$\begin{aligned}\|\frac{1}{2^j} D_1 f(2^j x, 2^j y, z)\| &\le (\frac{2^{jp_1}}{2^j}\|x\|^{p_1} + \frac{2^{jp_2}}{2^j}\|y\|^{p_2} + \frac{\delta_1}{2^j})(\|z\|^{q_1} + \delta_2), \\ \|\frac{1}{2^j} D_2 f(2^j x, y, z)\| &\le (\frac{2^{jp_3}}{2^j}\|x\|^{p_3} + \frac{\delta_3}{2^j})(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4)\end{aligned}$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using $p_1, p_2, p_3 < 1$, F_1 is a Cauchy-Jensen mapping. Now, let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.3).

Then we have

$$\begin{aligned}\|F_1(x,y) - F_1'(x,y)\| &\le \frac{1}{2^n}\|f(2^n x,y) - F_1(2^n x,y)\| + \frac{1}{2^n}\|f(2^n x,y) - F_1'(2^n x,y)\| \\ &\le \Big(\frac{2^{np_1}}{2^n}\frac{\|x\|^{p_1}}{2-2^{p_1}} + \frac{2^{np_2}}{2^n}\frac{\|x\|^{p_2}}{2-2^{p_2}} + \frac{\delta_1}{2^n}\Big)(\|y\|^{q_1} + \delta_2)\end{aligned}$$

for all $n \in N$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F_1(x,y) = F_1'(x,y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \to Y$ is unique.

Next, replacing y by $2y$ and z by 0 in (2.2), one can obtain

$$\begin{aligned}\|(f(x,y) - f(x,0)) &- \frac{1}{2}(f(x,2y) - f(x,0))\| \\ &\le \frac{1}{2}(\|x\|^{p_3} + \delta_3)(2^{q_2}\|y\|^{q_2} + \|0\|^{q_3} + \delta_4)\end{aligned} \tag{2.7}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (2.4), where $F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$ for all $x, y \in X$. From (2.7) and the definitions of F_1 and F_2, the equalities

$$\begin{aligned}F_1(x,y) - F_1(x,0) &= \frac{1}{2}\Big(F_1(x,2y) - F_1(x,0)\Big), \\ F_1(x,y) - F_1(x,0) &= \frac{1}{2^n}F_1(x,2^n y) - \frac{1}{2^n}F_1(x,0), \\ F_2(x,y) &= \frac{1}{2^n}F_2(x,2^n y)\end{aligned} \tag{2.8}$$

hold for all $n \in N$ and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}\|F_1(x,y) &- F_1(x,0) - F_2(x,y)\| \\ &= \|\frac{1}{2^n}F_1(x,2^n y) - \frac{1}{2^n}F_1(x,0) - \frac{1}{2^n}F_2(x,2^n y)\| \\ &= \frac{1}{2^n}\|f(x,2^n y) - F_1(x,2^n y)\| + \frac{1}{2^n}\|f(x,0) - F_1(x,0)\| \\ &+ \frac{1}{2^n}\|f(x,2^n y) - f(x,0) - F_2(x,2^n y)\| \\ &\le \Big(\frac{\|x\|^{p_1}}{2-2^{p_1}} + \frac{\|x\|^{p_2}}{2-2^{p_2}} + \delta_1\Big)(\frac{2^{nq_1}}{2^n}\|y\|^{q_1} + \frac{2\delta_2}{2^n}) \\ &+ (\|x\|^{p_3} + \delta_3)(\frac{2^{nq_2}}{2^n}\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \frac{1}{2^n}\delta_4)\end{aligned}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \to \infty$ and using $q_1, q_2 < 1$, we have (2.5). □

Theorem 2.2. *Let $p_1, p_2, p_3, q_1, q_2, q_3$ be fixed positive real numbers with $1 < p_1, p_2, p_3, q_1, q_2, q_3$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|D_1 f(x,y,z)\| \leq (\|x\|^{p_1} + \|y\|^{p_2})\|z\|^{q_1}, \tag{2.9}$$

$$\|D_2 f(x,y,z)\| \leq \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3}) \tag{2.10}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ such that

$$\|f(x,y) - F_1(x,y)\| \leq (\frac{\|x\|^{p_1}}{2^{p_1}-2} + \frac{\|x\|^{p_2}}{2^{p_2}-2})\|y\|^{q_1}, \tag{2.11}$$

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \leq \|x\|^{p_3}\frac{2^{q_2}}{2^{q_2}-2}\|y\|^{q_2}, \tag{2.12}$$

$$F_1(x,y) - f(x,0) = F_2(x,y) \tag{2.13}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) := \lim_{j\to\infty} 2^j f(\frac{x}{2^j}, y), \qquad F_2(x,y) := \lim_{j\to\infty} 2^j (f(x, \frac{y}{2^j}) - f(x,0))$$

for all $x, y \in X$.

Proof. Replacing x, y, z by $\frac{x}{2}, \frac{x}{2}, y$ in (2.9) respectively, we have

$$\|f(x,y) - 2f(\frac{x}{2}, y)\| \leq (\|\frac{x}{2}\|^{p_1} + \|\frac{x}{2}\|^{p_2})\|y\|^{q_1}$$

for all $x, y \in X$. Thus

$$\|2^j f(\frac{x}{2^j}, y) - 2^{j+1} f(\frac{x}{2^{j+1}}, y)\| \leq \big((\frac{2}{2^{p_1}})^j \|\frac{x}{2}\|^{p_1} + (\frac{2}{2^{p_2}})^j \|\frac{x}{2}\|^{p_2}\big)\|y\|^{q_1}$$

for all $x, y \in X$. For given integers l, m $(0 \leq l < m)$,

$$\begin{aligned} &\|2^l f(\frac{x}{2^l}, y) - 2^m f(\frac{x}{2^m}, y)\| \\ &\leq \sum_{j=l}^{m-1} \big((\frac{2}{2^{p_1}})^j \|\frac{x}{2}\|^{p_1} + (\frac{2}{2^{p_2}})^j \|\frac{x}{2}\|^{p_2}\big)\|y\|^{q_1} \end{aligned} \tag{2.14}$$

for all $x, y \in X$. By (2.14), the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} 2^j f(\frac{x}{2^j}, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.14), one can obtain the inequality (2.11). By (2.9) and (2.10),

$$\begin{aligned}\|2^j D_1 f(\frac{x}{2^j}, \frac{y}{2^j}, z)\| &\le \Big((\frac{2}{2^{p_1}})^j \|x\|^{p_1} + (\frac{2}{2^{p_2}})^j \|y\|^{p_2}\Big)\|z\|^{q_1},\\ \|2^j D_2 f(\frac{x}{2^j}, y, z)\| &\le (\frac{2}{2^{p_3}})^j \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3})\end{aligned}$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using $1 < p_1, p_2, p_3$, F_1 is a Cauchy-Jensen mapping. Now, let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.11). Then we have

$$\begin{aligned}\|F_1(x, y) \; - \; & F_1'(x, y)\|\\ &\le 2^n \|f(\frac{x}{2^n}, y) - F_1(\frac{x}{2^n}, y)\| + 2^n \|f(\frac{x}{2^n}, y) - F_1'(\frac{x}{2^n}, y)\|\\ &\le \Big((\frac{2}{2^{p_1}})^n \frac{2\|x\|^{p_1}}{2^{p_1} - 2} + (\frac{2}{2^{p_2}})^n \frac{2\|x\|^{p_2}}{2^{p_2} - 2}\Big)\|y\|^{q_1}\end{aligned}$$

for all $n \in N$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F_1(x, y) = F_1'(x, y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \to Y$ is unique.

Next, replacing z by 0 in (2.10), one can obtain

$$\|(f(x, y) - f(x, 0)) - 2(f(x, \frac{y}{2}) - f(x, 0))\| \le \|x\|^{p_3}\|y\|^{q_2} \tag{2.15}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (2.12), where $F_2(x, y) := \lim_{j\to\infty} 2^j \big(f(x, \frac{y}{2^j}) - f(x, 0)\big)$ for all $x, y \in X$. From (2.15) and the definitions of F_1 and F_2, the equalities

$$\begin{aligned}F_1(x, y) - F_1(x, 0) &= 2(F_1(x, \frac{y}{2}) - F_1(x, 0)),\\ F_1(x, y) - F_1(x, 0) &= 2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x, 0),\\ F_2(x, y) &= 2^n F_2(x, \frac{y}{2^n}) \qquad (2.16)\end{aligned}$$

hold for all $n \in N$ and for all $x, y \in X$. By (2.11), the equality $f(x, 0) = F_1(x, 0)$ holds for all $x \in X$. Hence, by (2.11), (2.12) and the above equalities, the inequalty

$$
\begin{aligned}
\|F_1(x, y) &- F_1(x, 0) - F_2(x, y)\| \\
&= \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x, 0) - 2^n F_2(x, \frac{y}{2^n})\| \\
&= 2^n \|f(x, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| + 2^n \|f(x, 0) - F_1(x, 0)\| \\
&+ 2^n \|f(x, \frac{y}{2^n}) - f(x, 0) - F_2(x, \frac{y}{2^n})\| \\
&\leq (\frac{2}{2^{q_1}})^n \Big(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2} \Big) \|y\|^{q_1} + \|x\|^{p_3} \frac{2^{q_2}}{2^{q_2} - 2} (\frac{2}{2^{q_2}})^n \|y\|^{q_2}
\end{aligned}
$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \to \infty$ and using $1 < q_1, q_2$, we have (2.13). □

Theorem 2.3. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_3$ be fixed positive real numbers with $p_1, p_2, p_3 < 1$ and $1 < q_1, q_2, q_3$. Let $f : X \times X \to Y$ be a mapping such that*

$$
\begin{aligned}
\|D_1 f(x, y, z)\| &\leq (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1) \|z\|^{q_1}, \\
\|D_2 f(x, y, z)\| &\leq (\|x\|^{p_3} + \delta_3)(\|y\|^{q_2} + \|z\|^{q_3})
\end{aligned}
$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying

$$
\|f(x, y) - F_1(x, y)\| \leq (\frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \delta_1) \|y\|^{q_1}, \tag{2.17}
$$

$$
\|f(x, y) - f(x, 0) - F_2(x, y)\| \leq (\|x\|^{p_3} + \delta_3) \frac{2^{q_2}}{2^{q_2} - 2} \|y\|^{q_2}, \tag{2.18}
$$

$$
F_1(x, y) - f(x, 0) = F_2(x, y) \tag{2.19}
$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

(2.20)

$$
F_1(x, y) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x, y) = \lim_{j \to \infty} 2^j (f(x, \frac{y}{2^j}) - f(x, 0))
$$

for all $x, y \in X$.

Proof. By the similar method in the proof of Theorem 2.1 and Theorem 2.2, one obtains that there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (2.17) and (2.18), the mappings $F_1, F_2 : X \times X \to Y$ are given by (2.20), $f(x, 0) = F_1(x, 0)$ and the equalities in (2.16) hold for all $n \in N$

and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}\|F_1(x,y) \;-\; & F_1(x,0) - F_2(x,y)\| \\ &= \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x,0) - 2^n F_2(x, \frac{y}{2^n})\| \\ &= 2^n \|f(x, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| + 2^n\|f(x,0) - F_1(x,0)\| \\ &+ 2^n\|f(x, \frac{y}{2^n}) - f(x,0) - F_2(x, \frac{y}{2^n})\| \\ &\le \Big(\frac{2}{2^{q_1}})^n (\frac{\|x\|^{p_1}}{2-2^{p_1}} + \frac{\|x\|^{p_2}}{2-2^{p_2}} + \delta_1\Big)\|y\|^{q_1} \\ &+ (\|x\|^{p_3} + \delta_3)\frac{2^{q_2}}{2^{q_2}-2}(\frac{2}{2^{q_2}})^n \|y\|^{q_2}\end{aligned}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \to \infty$ and using $1 < q_1, q_2$, we have (2.19). □

THEOREM 2.4. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_2, \delta_4$ be fixed positive real numbers with $1 < p_1, p_2, p_3$ and $q_1, q_2, q_3 < 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|D_1 f(x,y,z)\| \le (\|x\|^{p_1} + \|y\|^{p_2})(\|z\|^{q_1} + \delta_2),$$
$$\|D_2 f(x,y,z)\| \le \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4)$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ such that Let $f : X \times X \to Y$ be a mapping such that

$$\|f(x,y) - F_1(x,y)\| \le \Big(\frac{\|x\|^{p_1}}{2^{p_1}-2} + \frac{\|x\|^{p_2}}{2^{p_2}-2}\Big)(\|y\|^{q_1} + \delta_2), \tag{2.21}$$

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \|x\|^{p_3}\Big(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4\Big), \tag{2.22}$$

$$F_1(x,y) - F_1(x,0) = F_2(x,y) \tag{2.23}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) = \lim_{j\to\infty} 2^j f(\frac{x}{2^j}, y), \quad F_2(x,y) = \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y) \tag{2.24}$$

for all $x, y \in X$.

Proof. By the similar method in the proof of Theorem 2.1 and Theorem 2.2, one obtains that there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (2.21) and (2.22), the mappings $F_1, F_2 : X \times X \to Y$ are given by

(2.24) and the equalities in (2.8) hold for all $n \in N$ and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}
|F_1(x,y) \; - \; & F_1(x,0) - F_2(x,y)\| \\
&= \|\frac{1}{2^n}F_1(x,2^n y) - \frac{1}{2^n}F_1(x,0) - \frac{1}{2^n}F_2(x,2^n y)\| \\
&= \|\frac{1}{2^n}f(x,2^n y) - \frac{1}{2^n}F_1(x,2^n y)\| + \frac{1}{2^n}\|F_1(x,0) - f(x,0)\| \\
&+ \frac{1}{2^n}\|f(x,2^n y) - f(x,2^n 0) - F_2(x,2^n y)\| \\
&\le \Big(\frac{\|x\|^{p_1}}{2^{p_1}-2} + \frac{\|x\|^{p_2}}{2^{p_2}-2}\Big)\Big((\frac{2^{q_1}}{2})^n\|y\|^{q_1} + \frac{2}{2^n}\delta_2\Big) \\
&+ \|x\|^{p_3}\Big((\frac{2^{q_1}}{2})^n\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \frac{1}{2^n}\delta_4\Big).
\end{aligned}$$

Taking $n \to \infty$ and using $q_1, q_2 < 1$, we have (2.23). □

Corollary 2.5. *Let $0 < p_1, p_2 < 1$ and $0 < q_1, q_2 < 1$ or $1 < q_1, q_2$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Df(x,y,z,w)\| \le (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{q_1} + \|w\|^{q_2} + \delta_2) \tag{2.25}$$

for all $x, y, z, w \in X$, where $\delta_2 = 0$ for $q_1, q_2 > 1$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \Big(\frac{\|x\|^{p_1}}{2-2^{p_1}} + \frac{\|x\|^{p_2}}{2-2^{p_2}} + \delta_1\Big)\frac{1}{2}(\|y\|^{q_1} + \|y\|^{q_2} + \delta_2)$$

for all $x, y \in X$ where $\delta_2 = 0$ for $q_1, q_2 > 1$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) = \lim_{j\to\infty}\frac{1}{2^j}f(2^j x, y)$$

for all $x, y \in X$.

Proof. From (2.25), we know that

$$\begin{aligned}
D_1f(x,y,z)\| &= \|\frac{1}{2}Df(x,y,z,z)\| \\
&\le \frac{1}{2}(\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{q_1} + \|z\|^{q_2} + \delta_2), \\
\|D_2f(x,y,z)\| &= \|Df(\frac{x}{2},\frac{x}{2},y,z) - \frac{1}{2}Df(\frac{x}{2},\frac{x}{2},y,y) - \frac{1}{2}Df(\frac{x}{2},\frac{x}{2},z,z)\| \\
&\le (\|\frac{x}{2}\|^{p_1} + \|\frac{x}{2}\|^{p_2} + \delta_1)(\frac{3}{2}\|y\|^{q_1} \\
&+ \frac{3}{2}\|z\|^{q_2} + 2\delta_2 + \frac{1}{2}\|y\|^{q_2} + \frac{1}{2}\|z\|^{q_1})
\end{aligned}$$

for all $x, y \in X$. Then we can apply the similar method in the proof of Theorem 2.1 for the case $0 < q_1, q_2 < 1$ and apply the similar method in the proof of Theorem 2.3 for the case $1 < q_1, q_2$, and therefore, we get the results in this corollary. □

Corollary 2.6. *Let $1 < p_1, p_2$ and $0 < q_1, q_2 < 1$ or $1 < q_1, q_2$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Df(x, y, z, w)\| \leq (\|x\|^{p_1} + \|y\|^{p_2})(\|z\|^{q_1} + \|w\|^{q_2} + \delta_2)$$

for all $x, y, z, w \in X$, where $\delta_2 = 0$ for $q_1, q_2 > 1$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \Big(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2}\Big)\frac{1}{2}(\|y\|^{q_1} + \|y\|^{q_2} + \delta_2)$$

for all $x, y \in X$ where $\delta_2 = 0$ for $q_1, q_2 > 1$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) = \lim_{j\to\infty} 2^j f(\frac{x}{2^j}, y)$$

for all $x, y \in X$.

Proof. We can use the same method in the proof of Corollary 2.5 and we get the results in this corollary by applying Theorem 2.2 and Theorem 2.4. □

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