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EQUIVARIANT EMBEDDING OF TWO-TORUS INTO SYMPLECTIC MANIFOLD

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Abstract. We show that there is an equivariant symplectic embedding of a two-torus with a nontrivial action into a symplectic manifold with a symplectic circle action if and only if the circle action on the manifold is non-Hamiltonian. This is a new equivalent condition for non-Hamiltonian action and gives us a new insight to solve the famous conjecture by Frankel and McDuff.

1. Introduction

As the first step in investigation of a symplectic manifold, symplectic (or algebraic) geometers embed a surface into the symplectic manifold and next observe what happens. This is a very usual process in geometry. Quantum cohomology is a typical example in this viewpoint. Here, the question follows whether this process is applied to equivariant setting. In other words, it is conceivable to try an equivariant symplectic embedding of a surface into a symplectic manifold with a group action as the first step in investigation of a symplectic group action. In fact, equivariantly embedded symplectic spheres play a very important role in dealing with a Hamiltonian action. For example, calculation of equivariant cohomology of a Hamiltonian action is reduced to count intersection numbers of equivariantly embedded symplectic spheres. In spite of this similarity, there is a difference between equivariant and inequivariant settings. In inequivariant setting, embedding of a surface with a nonzero genus has importance not less than a zero genus. But, we can not find any results on equivariant embedding of a surface with a nonzero genus. Hence, we focus on this. In the paper, we investigate

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equivariant symplectic embedding of a two-torus into a symplectic manifold with a circle action. Since a surface with a higher genus does not deliver a circle action, investigation of a torus embedding is sufficient.

In dealing with a symplectic action, there is much difference between two-sphere and two-torus. A nontrivial circle action on a two-sphere is Hamiltonian and has a fixed point. But, a nontrivial circle action on a two-torus is non-Hamiltonian and has no fixed point. Through this difference, one may suspect that an equivariant symplectic embedding of a two-torus causes a non-Hamiltonian action. More precisely, we prove the following:

THEOREM 1.1. *Let S^1 act nontrivially on a two-torus (T^2, ω_0) with a symplectic form ω_0. Let S^1 act on a compact symplectic manifold (M, ω) with a symplectic form ω. Then, there is an equivariant symplectic embedding of T^2 into M if and only if the S^1 action on M is non-Hamiltonian.*

Here, we give two remarks.

REMARK 1.2.
1. The condition of nontriviality of the action on T^2 is necessary because we can easily find fixed tori in many Hamiltonian actions.
2. The condition can be replaced by a free S^1 action on T^2 as seen in the next section.

The result gives us a new equivalent condition for non-Hamiltonian action. There are other equivalent conditions of non-Hamiltonian action by Hattori, Yoshida [5] and Ono [9] as explained in the next section. In this paper, we add one more to the list. And it seems that it is strongly related to the following famous conjecture:

CONJECTURE 1.3 (Frankel-McDuff). A compact symplectic circle action with nonempty isolated fixed points must be Hamiltonian.

For the conjecture, see [4], [7], [8]. The author hopes that the condition gives us new insight into the conjecture.

2. Proof of theorem

Let S^1 act nontrivially on a two-torus (T^2, ω_0) with a symplectic form ω_0. For the action on T^2, we admit a noneffective action. Let S^1 act also on a compact symplectic manifold (M, ω). If we give an invariant compatible almost complex structure J on (M, ω), then we have an invariant metric $\omega(\cdot, J\cdot)$ on M.

First, we prove sufficiency of the theorem.

PROPOSITION 2.1. *If the S^1 action on (M, ω) is Hamiltonian, then there is no equivariantly embedded symplectic torus with a nontrivial action in M.*

Proof. Assume that there is an equivariantly embedded symplectic torus T^2 with a nontrivial action in M. Then, we may assume that the embedded torus must have the trivial stabilizer. For if the stabilizer subgroup of T^2 is $\mathbb{Z}_k$ for some nonunit $k \in \mathbb{N}$, then we only have to consider the connected component containing T^2 of the fixed point set $M^{\mathbb{Z}_k}$ by $\mathbb{Z}_k$ in M to obtain a contradiction. Here, it is easy that the action restricted on the component is also Hamiltonian.

As usual, we may assume that the symplectic form ω is integral. That is, ω is the first Chern class of a complex line bundle L. By the definition of De Rham version of equivariant cohomology, the assumption of Hamiltonian action ensures that ω is a restriction of an equivariant cohomology element [3]. Then, by [5] we may assume that L is an equivariant S^1-bundle. Since the symplectic form ω is nontrivial in De Rham cohomology, the line bundle L is inequivariantly nontrivial. Hence, the induced line bundle of L to T^2 is also inequivariantly nontrivial because T^2 is a symplectic submanifold. But, an S^1 line bundle over T^2 with the free S^1 action is always equivariantly trivial line bundle, and hence inequivariantly trivial. For there is one-to-one correspondence between S^1 complex line bundles over T^2 with the free S^1 action and inequivariant complex line bundles over circle [2], and inequivariant complex line bundles over a circle are always trivial. This gives us a contradiction and hence we obtain the proof. □

To prove necessity of the theorem, hereafter we assume that the S^1 action on M is non-Hamiltonian. By [8], there exists a generalized moment map $\mu : M \to S^1$. Here, we define two notations.

NOTATION. Let ξ_M be the fundamental vector field generated by the S^1-action.

NOTATION. Let $M_{(1)}$ be the set of points with the trivial stabilizer in M.

As noted in [1] and [8], the number of connected components of the preimage $\mu^{-1}(c)$ for any $c \in S^1$ does not change. And, it is well known that $J\xi_M$ is the gradient flow of a generalized moment map μ with respect to the metric. Then, we can easily obtain the following lemma:

LEMMA 2.2. *Assume that the S^1 action on M is non-Hamiltonian. For a regular value c in S^1, we have two points p and q in the same connected component of $M_{(1)} \cap \mu^{-1}(c)$ which are joined by an integral curve of $J\xi_M$.*

Proof. This is easily proved by the pigeonhole principle. □

It is well known that $M - M_{(1)}$ is at most codimension two. Hence, $M_{(1)}$ is path connected. In a similar way, a component of $M_{(1)} \cap \mu^{-1}(c)$ is path connected for a regular value $c \in S^1$. Here, we define a terminology.

DEFINITION 2.3. An integral curve $\gamma : [0, 1] \to M_{(1)}$ of $J\xi_M$ is *a loop up to S^1* if γ is a simple closed curve in the orbit space $M_{(1)}/S^1$.

Then, we can prove the following lemma:

LEMMA 2.4. *If an integral curve of $J\xi_M$ is a loop up to S^1, then we have an equivariant embedding of (T^2, ω_0) into (M, ω).*

Proof. Let γ be an integral curve of $J\xi_M$ which is a loop up to S^1. Then, the orbit $S^1 \cdot \gamma$ of γ is a two-torus. Since the tangent space of the two-torus is spanned by ξ_M and $J\xi_M$, it gives us an equivariant symplectic embedding of a two-torus. □

Finally, we prove necessity of the theorem.

PROPOSITION 2.5. *If the action on M is non-Hamiltonian, then there is an equivariantly embedded symplectic two-torus with a nontrivial action.*

Proof. By Lemma 2.2, we have two points p and q in a component of $M_{(1)} \cap \mu^{-1}(c)$ for a regular value c in S^1 which are joined by a integral curve of $J\xi_M$. Let us denote by C the component of $M_{(1)} \cap \mu^{-1}(c)$. Then, we consider tubular neighborhood $C \times [-\epsilon, \epsilon]$ of C in $M_{(1)}$. Here, we may regard the vector field $J\xi_M$ as $0 \times \frac{\partial}{\partial x}$. By using the arguments in [6, Lemma 3.6.], we can perturb the metric in the tubular neighborhood so that we can join the point $(p, -\epsilon)$ with $(z \cdot q, \epsilon)$ for some $z \in S^1$ through following $J\xi_M$ since C is path connected. That is, we have a loop through $J\xi_M$ up to S^1 and hence we finish the proof by Lemma 2.4. □

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