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EXTREMAL LENGTH AND GEOMETRIC INEQUALITIES

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Abstract. We introduce the extremal length and examine its properties. And we consider the geometric applications of extremal length to the boundary behavior of analytic functions, conformal mappings. We derive the theorem in connection with the capacity. This theorem applies the extremal length to the analytic function defined on the domain with a number of holes. And we obtain the theorems in connection with the pure geometric problems.

1. Introduction

Using the concept of the extremal length, in [14], we established the following theorems for analytic functions.

Theorem 1.1. *Let Q be a general quadrilateral of area M. Let a be the length of the shortest arc in Q connecting one pair of opposite sides. Let b be the length of the shortest arc in Q connecting the other pair of sides. Then*

$$a \cdot b \leq M.$$

The pure geometric proof of Theorem 1.1 is difficult. But the use of extremal length makes the proof trivial.

The purpose of this paper is to apply the extremal length to the boundary behavior of analytic functions, conformal mappings and to leads a simple proof of theorem. So it shows us the usefulness of the method of extremal length.

Throughout this paper, $\mathbb{C}$ will denote the complex plane, D is a domain in $\mathbb{C}$, ∂D is a boundary of D, and $cl(D)$ is a closure of D.

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2. Extremal length and capacity

Let Γ be a family whose elements γ are locally rectifiable curves (simply, curves or arcs) in D. And let $\rho(z)$ be a non-negative Borel measurable function defined on $\mathbb{C}$. Every curve γ has a

$$L(\gamma,\rho)=\int_\gamma \rho(z)\,|dz| \tag{1}$$

which may be infinite. And D has a

$$A(D,\rho)=\iint_D \rho(z)^2\,dx\,dy\neq 0,\infty. \tag{2}$$

In order to define an invariant which depends on the whole set Γ, we introduce

$$L(\Gamma,\rho)=\inf_{\gamma\in\Gamma} L(\gamma,\rho). \tag{3}$$

Where we agree that $L(\Gamma,\rho)=\infty$ in case Γ is empty.

Definition 2.1. ([1]) To obtain a quantity that does not change when the weight function ρ is multiplied by a constant, we form the homogeneous expression $[L(\Gamma,\rho)]^2/A(D,\rho)$. The *extremal length* of Γ in D is defined as

$$\lambda(\Gamma)=\lambda_D(\Gamma)=\sup_\rho\,[L(\Gamma,\rho)]^2/A(D,\rho). \tag{4}$$

Where ρ is subject to the condition $0<A(D,\rho)<\infty$, obviously $0\le\lambda(\Gamma)\le\infty$.

We introduce the following Examples which are frequently used in our paper.

Example 2.1. Let B be a rectangle of sides a and b. Let Γ be the family of arcs in B which joins the sides of length b. Then

$$\lambda(\Gamma)=a/b.$$

Proof. For any $\rho(z)$, we have

$$\int_0^a \rho(z)\,dx\ge L(\Gamma,\rho),\qquad \iint_B \rho(z)\,dx\,dy\ge b\,L(\Gamma,\rho)$$

Then, by the Schwarz inequality,

$$\begin{aligned} b^2\,[L(\Gamma,\rho)]^2 &\le ab\iint_B \rho^2\,dx\,dy\\ &= ab\,A(B,\rho). \end{aligned}$$

This proves $\lambda(\Gamma) \le \dfrac{a}{b}$.

For $\rho = 1$, we have

$$L(\Gamma, 1) = a, \quad A(B, 1) = ab.$$

Thus $\lambda(\Gamma) \ge \dfrac{a}{b}$. □

EXAMPLE 2.2. Let Δ be the annulus $\Delta = \{z \,|\, a < |z| < b\}$. Let Γ be the family of arcs in Δ which joins the two contours. Then

$$\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{b}{a}.$$

Proof. See [14] □

EXAMPLE 2.3. ([9]) Consider the horizontal line segments $\gamma_y = \{ (x, y) | 0 \le x \le 1\}$, and let $\Gamma = \{\gamma_y \,|\, y \in E\}$ where E is a measurable set of real numbers. Then E has measure zero if and only if

$$\lambda(\Gamma) = \infty$$

Early in the development of extremal length, Ahlfors and Beurling related it to logarithmic capacity (simply, capacity). The double role of capacity as a conformal invariant and a geometric quantity permits us to gain relevant information about analytic functions. $Cap(E)$ will denote the capacity of a set E.

EXAMPLE 2.4. ([11]) For the Cantor ternary set $E(\{2/3\})$, $Cap(E) \ge 1/18$.

PROPOSITION 2.1. ([4]) (a) $0 \le Cap(E) < \infty$.
(b) $E_1 \subset E_2$ *implies* $Cap(E_1) \le Cap(E_2)$.
(c) *A set which is of capacity zero is of linear measure zero.*

PROPOSITION 2.2. ([9]) *Let E be a compact point set in $\{z \,|\, |z| < 1\}$, and let Γ consist of all curves which join $\{z \,|\, |z| = 1\}$ to E. Then $Cap(E) = 0$ if and only if $\lambda(\Gamma) = \infty$*

3. Some applications

For our theorem we will need the following definitions and propositions.

DEFINITION 3.1. ([7]) If every component of a set is a point, the set is called *totally disconnected*.

EXAMPLE 3.1. $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$ is a totally disconnected compact set.

PROPOSITION 3.1. ([7]) *A set E of capacity zero does not contain a continuum consisting of more than one point.*

PROPOSITION 3.2. ([7]) *Let E be a closed set of capacity zero in $\mathbb{C}$ and D a Jordan domain containing E, then $D - E$ is a domain and every point of E is a boundary point of $D - E$.*

DEFINITION 3.2. ([3]) Let Λ be a curve at $z_0 \in cl(D)$, then the *cluster set* of a function g at z_0 along Λ, denoted by $C_\Lambda(g, z_0)$, is defined to be the set of all points $\omega \in \Omega$ with the property that, for some sequence of points $\{z_n\}$ on Λ for which

$$\lim_{n\to\infty} z_n = z_0, \quad \text{we have} \quad \lim_{n\to\infty} g(z_n) = \omega.$$

Where Ω is Riemann sphere. A value ω is called a *cluster value* of g at z_0 along Λ. It follows readily that $C_\Lambda(g, z_0)$ is a nonempty closed subset of Ω.

The following theorem applies the extremal length to the analytic function defined on the domain with a number of holes. The integral of any functions on the sets of measure zero is 0. So on the set of measure zero, we can not use almost every concept or method of the analysis. But the method of extremal length can be used on the set with the zero measure (and positive capacity). So it shows us the high usefulness of the method of extremal length.

THEOREM 3.3. *Let $f(z)$ be a bounded single-valued analytic function in the complement of E, where E is a totally disconnected compact set of positive capacity in $\mathbb{C}$. Then it is not the case that for each z in E, except for those z in a set of capacity zero, there exist two curves in the complement of E at z on which $f(z)$ has the limits ω_1 and ω_2, ($\omega_1 \neq \omega_2$).*

For our proof we will need the following.

PROPOSITION 3.4. ([7]) *Let Γ be a family of curves on D and f an analytic function on D such that $f'(z) \neq 0$. Then*

$$\lambda(\Gamma) \leq \lambda[f(\Gamma)].$$

REMARK 3.1. Since the set of points z in D where $f'(z) = 0$ is countable, for our proof, we shall generalize Proposition 3.4 to the case where $f(z)$ is not constant but $f'(z) = 0$ may happen.

PROPOSITION 3.5. ([1]) (Comparison principle) *For two curve families Γ_1, Γ_2, if every $\gamma_2 \in \Gamma_2$ contains a $\gamma_1 \in \Gamma_1$, then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2).$$

Indeed, both extremal lengths can be evaluated with respect to the same D. For any ρ in D it is clear that $L(\Gamma_2, \rho) \geq L(\Gamma_1, \rho)$. These minimum lengths are compared with the same $A(D, \rho)$.

REMARK 3.2. (i) The set Γ_2 of fewer or longer curves has the larger extremal length.

(ii) There is a physical interpretation of extremal length. Think of the curve family Γ as representing a system of homogenous electric wires. Then the extremal length $\lambda(\Gamma)$ represents the resistance of Γ. So the above Proposition 3.5 reflect the fact that systems of fewer or longer wires have greater resistance (smaller conductance).

DEFINITION 3.3. ([6]) We say that the curves $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ ($n = 2, 3, \ldots$) at z_0 are *separating curves* of a function g at z_0 provided they are contained in the domain of g and no two of cluster sets $C_{\Lambda_j}(g, z_0)$ ($j = 1, 2, \ldots, n$) intersect.

LEMMA 3.6. ([6]) *Let S be a subset of $\mathbb{C}$ and g a function whose domain is S. Let E be the set of points of S^c at which there exist three separating curves of g. Then E is countable.*

LEMMA 3.7. ([6]) *It follows from Lemma 3.6 that if the domain of g is an open set G, and if E denotes the set of points of the boundary of G at which there exist three separating curves of g. Then E is countable.*

PROPOSITION 3.8. ([4]) *Let E be a countable set of $\mathbb{C}$, then*

$$Cap(E) = 0.$$

LEMMA 3.9. ([8]) *Let E be a closed point set of capacity zero in $\mathbb{C}$, the complement E^c of E a connected set, J a Jordan curve in E^c and Γ the family of curves in E^c connecting points of J and points of E. Then*

$$\lambda(\Gamma) = \infty.$$

LEMMA 3.10. ([7]) *Let G be an open set in $\mathbb{C}$, whose boundary consists of a finite number of Jordan curves, and E a closed subset of G which is not of capacity zero. Let Γ be the family of curves in $G - E$ connecting points of ∂G and points of E. Then*

$$\lambda(\Gamma) < \infty.$$

Proof of Theorem 3.3. Since the case of constant $f(z)$ is trivial, we discuss the case of non-constant $f(z)$. Assume that the statement is not true, that is, suppose that for each z in E, there exist two curves Λ_1 and Λ_2 in E^c the complement of E at z on which $f(z)$ has the limits ω_1 and ω_2, $(\omega_1 \neq \omega_2)$.

Choose a Jordan curve J in E^c enclosing the two curves Λ_1, Λ_2 and the set E. Let ξ be a subcurve of $f(J)$ and consider the family Γ of all curves in $f(E^c)$ connecting points of ξ and points of $\partial(f(E^c)) - \{\omega_1, \omega_2\}$. Since $f(z)$ is a bounded function, by Lemma 3.10 and the comparison principle of extremal length (Proposition 3.5),

$$\lambda(\Gamma) < \infty.$$

Hence it follows at once from the Proposition 3.4 that

$$\lambda(f^{-1}(\Gamma) < \infty.$$

Let

$$E' = \{z \in E \mid \text{ a curve in } f^{-1}(\Gamma) \text{ ends at } z\}.$$

Then by the comparison principle of extremal length(Proposition 3.5) and Lemma 3.9,

$$cap(E') > 0.$$

On the other hand, each point of E' is a boundary point of E^c at which there exist three separating curves of f. Hence by Lemma 3.6 and Lemma 3.7, E' must be a countable set. Therefore by Proposition 3.8,

$$cap(E') = 0.$$

Thus we have arrived at a contradiction.

This completes the proof of the theorem. □

Using the following lemma, we have the corollary 3.12.

LEMMA 3.11. ([5]) *For any totally disconnected compact set E in $\mathbb{C}$, there exists a Jordan domain D such that the Jordan curve J bounding D passes every point of E.*

COROLLARY 3.12. *Let E be as in Theorem 3.3, and let D be a Jordan domain such that the Jordan curve bounding D passes every point of E by the Lemma 3.11. Let $N(z_0)$ denote a neighborhood of some point z_0 in E, and u a some harmonic function on D. If u is a bounded function in $N(z_0) \cap D$, then it is not the case that z_0 in E, there exist two curves in D at z_0 on which u has the limits ω_1 and ω_2, $(\omega_1 \neq \omega_2)$.*

Proof. Since u is harmonic on Jordan domain D, there exists a v the harmonic conjugate of u on D. Hence we let $f(z)$ denote a function satisfying

$$f(z) = \exp(u + iv).$$

Then $f(z)$ is a single-valued analytic function on D and $f(z)$ is bounded on $N(z_0) \cap D$. Hence applying Theorem 3.3 to $f(z)$, we obtain the above consequence for $u(x, y) = Re\, f(z)$. □

4. Geometric inequalities

The simplest examples concerns the ring domain.

THEOREM 4.1. *Let R be a ring domain in $\mathbb{C}$ and let R_0 and R_1 denote the bounded component and unbounded component of R^c the complement of R, respectively. Let ∂R_0 and ∂R_1 denote the two components of the boundary of R. Let a be the length of the shortest arc in D connecting ∂R_0 and ∂R_1. Let b be the length of the Jordan curve, ∂R_0. Then*

$$a \cdot b \leq S,$$

where S is the area of R.

The pure geometric proof of theorem 4.1 is difficult. But the use of extremal length makes the proof easy. we will need the following.

LEMMA 4.2. ([2]) *Let $R, \partial R_0$ and ∂R_1 be as in Theorem 4.1. Let Γ_R be the family of all curves in R connecting ∂R_0 and ∂R_1. Then R_0 consists of a single point if and only if*

$$\lambda(\Gamma_R) = \infty.$$

LEMMA 4.3. ([1]) *Let R, R_0, R_1 and Γ_R be as in Lemma 4.2. We say the closed curve γ in R separates R_0 and R_1 if γ has non-zero winding number about the points of R_0. Let Γ_S be the family of all closed curves in R which separates R_0 and R_1. Then*

$$\lambda(\Gamma_R) \cdot \lambda(\Gamma_S) = 1.$$

We say that $\lambda(\Gamma_S)$ is the conjugate extremal length of $\lambda(\Gamma_R)$.

Proof of Theorem 4.1. Let Γ_R and Γ_S be as in Lemmas 4.2 and 4.3 respectively. Then by Lemma 4.3,

$$\lambda(\Gamma_R) \cdot \lambda(\Gamma_S) = 1.$$

On the other hand, we choose the non-negative Borel measurable function $\rho = 1$, then $\lambda(\Gamma_R)$ and $\lambda(\Gamma_S)$ has the following lower bounds respectively. That is,

$$\begin{aligned}(a^2/S)\cdot(b^2/S) &= [\{L(\Gamma_R,1)\}^2\,/\,A(D,1)]\cdot[\{L(\Gamma_S,1)\}^2\,/\,A(D,1)]\\ &\le \lambda(\Gamma_R)\cdot\lambda(\Gamma_S)\\ &= 1.\end{aligned}$$

and the theorem follows at once. □

THEOREM 4.4. *Suppose that we have a set of n disjoint general quadrilaterals Q_k, for $k = 1, 2, \ldots, n$, that are contained in the annulus $\Delta = \{z\,|r < |z| < R\}$, $(0 < r < R,\ R \neq \infty)$ and that are bounded by Jordan curves each of which has an arc, in common with each of the circles $\{z\,|\,|z| = r\}$ and $\{z\,|\,|z| = R\}$. (The Q_k can be regarded as strips extending from the inner to the outer circle.) If these domains Q_k are mapped onto rectangles B_k with sides equal respectively to a_k and b_k in such a way that the arcs referred to are mapped into sides of lengths a_k, then*

$$\sum_{k=1}^{n} a_k/b_k \le 2\pi/\log(R/r) \tag{5}$$

with equality holding only if the Q_k are domains of the form $\{z\,|\,r < |z| < R,\ \phi_k < \arg z < \phi_{k+1}\}$ completely filling the annulus.

For our proof we will need the following.

PROPOSITION 4.5. ([11]) (Conformal invariance) *Let $f(z)$ be an 1-1 conformal mapping on D and Γ a family of curves on D, then*

$$\lambda(\Gamma) = \lambda[f(\Gamma)].$$

PROPOSITION 4.6. ([11]) *Suppose there exist disjoint open sets G_n containing the curves in Γ_n. If $\cup_n \Gamma_n \subset \Gamma$, then*

$$\sum_n 1/\lambda(\Gamma_n) \le 1/\lambda(\Gamma).$$

The method of extremal length leads to a simple proof of the inequality(5).

Proof of Theorem 4.4. We can map an arbitrary general quadrilateral conformally onto a rectangle. Let $w = f_k(z)$ be an 1-1 conformal mappings on Q_k upon B_k respectively. Let Γ be the family of arcs in Δ which join the two boundary circles, and let Γ_k be the family of arcs in Q_k which join the two sides of $Q_k \subset \partial\Delta$. Then by the conformal invariance of extremal length(Proposition 4.5) and Example 1.1,

$$\lambda(\Gamma_k) = \lambda[f_k(\Gamma_k)] = b_k/a_k. \tag{6}$$

By the hypothesis, there exist disjoint open sets $Q_k (k = 1, 2, \dots, n)$ containing Γ_k and $\cup_k \Gamma_k \subset \Gamma$. Hence by Proposition 4.6,

$$\sum_{k=1}^{n} 1/\lambda(\Gamma_k) \leq 1/\lambda(\Gamma). \tag{7}$$

Therefore by Example 1.2, (6) and (7), we obtain (5).

The proof is complete. □

5. Simple proof

We will alternatively prove the well-known result by the method of extremal length. This method shortens the length of proof significantly as we shall see by comparing the following proof with that of Theorem 14.22 in [10].

THEOREM 5.1. ([10]) *Let* $\Delta(r, R) = \{z \,|\, r < |z| < R\}$, $(0 < r < R, R \neq \infty)$. *Then* $\Delta_1(r_1, R_1)$ *and* $\Delta_2(r_2, R_2)$ *are conformally equivalent if and only if*

$$R_1/r_1 = R_2/r_2 \tag{8}$$

Proof by the Method of extremal length. Since the proof of sufficient conditions is trivial, we discuss the proof of necessary conditions. Let Γ_Δ be the family of arcs in $\Delta(r, R)$ which join the two contours. Then by Example 1.2,

$$\lambda(\Gamma_\Delta) = (1/2\pi) \log(R/r). \tag{9}$$

Suppose that $\Delta_1(r_1, R_1)$ and $\Delta_2(r_2, R_2)$ are conformally equivalent and let f be an 1-1 conformal mapping on $\Delta_1(r_1, R_1)$ upon $\Delta_2(r_2, R_2)$. Then by the conformal invariance of extremal length(Proposition 4.5),

$$\lambda(\Gamma_{\Delta_1}) = \lambda[f(\Gamma_{\Delta_1})] = \lambda(\Gamma_{\Delta_2}). \tag{10}$$

Hence by (9), (10), we obtain (8).

The proof is now complete. □

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