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CHARACTERIZATIONS OF THE POWER FUNCTION DISTRIBUTION BY THE INDEPENDENCE OF RECORD VALUES

Se-Kyung Chang*

Abstract. In this paper, we present characterizations of the power function distribution by the independence of record values. We establish that $X \in POW(1,\nu)$ for $\nu > 0$, if and only if $\dfrac{X_{L(n)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$. And we prove that $X \in POW(1,\nu)$ for $\nu > 0$, if and only if $\dfrac{X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$. Also we characterize that $X \in POW(1,\nu)$ for $\nu > 0$, if and only if $\dfrac{X_{L(n)} + X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$.

1. Introduction

The record values model was introduced by Chandler([4]). Suppose that $X_1, X_2, \cdots$ is a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = \min\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is a lower record value of $\{X_n,\ n \geq 1\}$, if $Y_j < Y_{j-1}$ for $j > 1$. By definition, X_1 is a lower record value.

The indices at which the lower record values occur are given by the lower record times $L(n)$ for $n \geq 1$, where $L(n) = \min\{j | j > L(n-1),\, X_j < X_{L(n-1)},\, n \geq 2\}$ and $L(1) = 1$.

A continuous random variable X is said to have the power function distribution with two parameters $\alpha > 0$ and $\nu > 0$ if it has a cdf $F(x)$

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of the form

$$F(x) = \alpha^{-\nu} x^{\nu},\ 0 < x < \alpha,\ \alpha > 0,\ \nu > 0. \tag{1}$$

A notation that designates that X has the cdf (1) is $X \in POW(\alpha, \nu)$.

Some characterizations by the independence of record values are known. In [2] and [3], Ahsanullah studied, if $X \in PAR(\alpha, \beta)$ for $\alpha > 0$ and $\beta > 0$, then $\dfrac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$ are independent for $0 < m < n$, and if $X \in WEI(\theta, \alpha)$ for $\theta > 0$ and $\alpha > 0$, then $\dfrac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent for $0 < m < n$. The above result are not necessary and sufficient.

In this paper, we will give characterizations of necessary and sufficient for the power function distribution with the parameter $\alpha = 1$ by the independence of the lower record values.

2. Main results

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$, $F(1) = 1$ and $F(x) < 1$ for all $0 < x < 1$. Then $F(x) = x^{\nu}$ for all $0 < x < 1$ and $\nu > 0$, if and only if $\dfrac{X_{L(n)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$.*

Proof. If $F(x) = x^{\nu}$ for all $0 < x < 1$ and $\nu > 0$, then the joint pdf $f_{(n),(n+1)}(x, y)$ of $X_{L(n)}$ and $X_{L(n+1)}$ is

$$\begin{aligned} f_{(n),(n+1)}(x, y) &= \frac{(H(x))^{n-1}\ h(x)\, f(y)}{\Gamma(n)} \\ &= \frac{\nu^{n+1}\, (-ln\, x)^{n-1}\, y^{\nu-1}}{\Gamma(n)\ x} \end{aligned}$$

for all $0 < y < x < 1$, $\nu > 0$ and $n \geq 1$.

Consider the functions $U = \dfrac{X_{L(n)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$. It follows that $x_{L(n)} = w$, $x_{L(n+1)} = \dfrac{(u-1)w}{u}$ and $|J| = \dfrac{w}{u^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u,w)$ of U and W as

$$f_{(U),(W)}(u,w) = \frac{\nu^{n+1}\,(u-1)^{\nu-1}\,(-\ln w)^{n-1}\,w^{\nu-1}}{\Gamma(n)\,u^{\nu+1}} \tag{2}$$

for all $u > 1$, $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

The marginal pdf $f_{(U)}(u)$ of U is given by

$$\begin{aligned} f_{(U)}(u) &= \int_0^1 f_{(U),(W)}(u,w)\;dw \\ &= \frac{\nu\,(u-1)^{\nu-1}}{u^{\nu+1}} \end{aligned} \tag{3}$$

for all $u > 1$ and $\nu > 0$.

Also, by the above transformations, the pdf $f_{(W)}(w)$ of W is the function of $X_{L(n)}$. Thus the pdf $f_{(W)}(w)$ of W is given by

$$\begin{aligned} f_{(W)}(w) &= \frac{(H(w))^{n-1} f(w)}{\Gamma(n)} \\ &= \frac{\nu^n\,(-\ln w)^{n-1}\,w^{\nu-1}}{\Gamma(n)} \end{aligned} \tag{4}$$

for all $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_{(U)}(v) f_{(W)}(w) = f_{(U),(W)}(u,w)$.

Hence $U = \dfrac{X_{L(n)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint pdf $f_{(n),(n+1)}(x,y)$ of $X_{L(n)}$ and $X_{L(n+1)}$ is

$$f_{(n),(n+1)}(x,y) = \frac{(H(x))^{n-1}\;h(x)\,f(y)}{\Gamma(n)}$$

for all $0 < y < x < 1$, $\nu > 0$ and $n \geq 1$, where $H(x) = -\ln F(x)$ and $h(x) = -\dfrac{d}{dx}\big(H(x)\big) = \dfrac{f(x)}{F(x)}$.

Let us use the transformations $U = \dfrac{X_{L(n)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$.

The Jacobian of the transformations is $|J| = \dfrac{w}{u^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u,w)$ of U and W as

$$f_{(U),(W)}(u,w) = \frac{f\left(\frac{(u-1)\,w}{u}\right)\;(H(w))^{n-1}\;h(w)\,w}{\Gamma(n)\,u^2} \tag{5}$$

for all $u > 1$, $0 < w < 1$ and $n \geq 1$.

The pdf $f_{(W)}(w)$ of W is given by

$$f_{(W)}(w) = \frac{(H(w))^{n-1}\, f(w)}{\Gamma(n)} \tag{6}$$

for all $0 < w < 1$ and $n \geq 1$.

From (5) and (6), we obtain the pdf $f_{(U)}(u)$ of U

$$f_{(U)}(u) = \frac{f\left(\frac{(u-1)\,w}{u}\right)\, h(w)\, w}{u^2\, f(w)}$$

for all $u > 1$, $0 < w < 1$ and $n \geq 1$, where $h(x) = \dfrac{f(x)}{F(x)}$.

That is,

$$f_{(U)}(u) = \frac{\partial}{\partial u}\left(F\left(\frac{(u-1)\,w}{u}\right) / \, F(w)\right).$$

Since U and $W$are independent, we must have

$$F\left(\frac{(u-1)w}{u}\right) = F\left(\frac{u-1}{u}\right) F(w) \tag{7}$$

for all $u > 1$ and $0 < w < 1$.

Upon substituting $\dfrac{(u-1)}{u} = u_1$ in (7), then we get

$$F(u_1\, w) = F(u_1) F(w) \tag{8}$$

for all $0 < u_1 < 1$ and $0 < w < 1$.

By the functional equations (see, [1]), the only continous solution of (8) with the boundary conditions $F(0) = 0$ and $F(1) = 1$ is

$$F(x) = x^{\nu}$$

for all $0 < x < 1$ and $\nu > 0$.

This completes the proof. □

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$, $F(1) = 1$ and $F(x) < 1$ for all $0 < x < 1$. Then $F(x) = x^{\nu}$ for all $0 < x < 1$ and $\nu > 0$, if and only if $\dfrac{X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 2.1, we consider the functions $U = \dfrac{X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$. It follows that $x_{L(n)} = w$, $x_{L(n+1)} = \dfrac{uw}{(u+1)}$ and $|J| = \dfrac{w}{(u+1)^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u,w)$ of U and W as

$$f_{(U),(W)}(u,w) = \frac{\nu^{n+1}\, u^{\nu-1}\, (-ln\, w)^{n-1}\, w^{\nu-1}}{\Gamma(n)\, (u+1)^{\nu+1}} \tag{9}$$

for all $u > 0$, $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

The marginal pdf $f_{(U)}(u)$ of U is given by

$$f_{(U)}(u) = \frac{\nu u^{\nu-1}}{(u+1)^{\nu+1}} \tag{10}$$

for all $u > 0$ and $\nu > 0$.

Also, the pdf $f_{(W)}(w)$ of W is given by

$$f_{(W)}(w) = \frac{\nu^{n}\, (-ln\, w)^{n-1}\, w^{\nu-1}}{\Gamma(n)} \tag{11}$$

for all $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

From (9), (10) and (11), we obtain $f_{(U)}(u)\, f_{(W)}(w) = f_{(U),(W)}(u,w)$.

Hence $U = \dfrac{X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. By the same manner as Theorem 2.1, let us use the transformations $U = \dfrac{X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$. The Jacobian of the transformations is $|J| = \dfrac{w}{(u+1)^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u,w)$ of U and W as

$$f_{(U),(W)}(u,w) = \frac{f\left(\frac{uw}{u+1}\right)\, (H(w))^{n-1}\, h(w)\, w}{\Gamma(n)\, (u+1)^2} \tag{12}$$

for all $u > 0$, $0 < w < 1$ and $n \geq 1$.

The pdf $f_{(W)}(w)$ of W is given by

$$f_{(W)}(w) = \frac{(H(w))^{n-1} f(w)}{\Gamma(n)} \tag{13}$$

for all $0 < w < 1$ and $n \geq 1$.

From (12) and (13), we obtain the pdf $f_{(U)}(u)$ of U

$$f_{(U)}(u) = \frac{f\left(\frac{uw}{u+1}\right) h(w)\, w}{(u+1)^2 f(w)}$$

for all $u > 0$, $0 < w < 1$ and $n \geq 1$, where $h(x) = \dfrac{f(x)}{F(x)}$.

That is,

$$f_{(U)}(u) = \frac{\partial}{\partial u}\left(F\left(\frac{uw}{u+1}\right) / F(w)\right).$$

Since U and $W$are independent, we must have

$$F\left(\frac{uw}{u+1}\right) = F\left(\frac{u}{u+1}\right) F(w) \tag{14}$$

for all $u > 0$ and $0 < w < 1$.

Upon substituting $\dfrac{u}{u+1} = u_2$ in (14), then we get

$$F(u_2\, w) = F(u_2)\, F(w) \tag{15}$$

for all $0 < u_2 < 1$ and $0 < w < 1$.

By the functional equations (see, [1]), the only continous solution of (15) with the boundary conditions $F(0) = 0$ and $F(1) = 1$ is

$$F(x) = x^{\nu}$$

for all $0 < x < 1$ and $\nu > 0$.

This completes the proof. □

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$, $F(1) = 1$ and $F(x) < 1$ for all $0 < x < 1$. Then $F(x) = x^{\nu}$ for all $0 < x < 1$ and $\nu > 0$, if and only if $\dfrac{X_{L(n)} + X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $X_{L(n)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 2.1, we consider the functions $U = \dfrac{X_{L(n)} + X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$. It follows that $x_{L(n)} = w$, $x_{L(n+1)} = \dfrac{(u-1)w}{(u+1)}$ and $|J| = \dfrac{2w}{(u+1)^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u, w)$ of U and W as

$$f_{(U),(W)}(u, w) = \frac{2\nu^{n+1} (u-1)^{\nu-1} (-ln\, w)^{n-1} w^{\nu-1}}{\Gamma(n)\, (u+1)^{\nu+1}} \tag{16}$$

for all $u > 1$, $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

The marginal pdf $f_{(U)}(u)$ of U is given by

$$f_{(U)}(u) = \frac{2\nu\,(u-1)^{\nu-1}}{(u+1)^{\nu+1}} \tag{17}$$

for all $u > 1$ and $\nu > 0$.

Also, the pdf $f_{(W)}(w)$ of W is given by

$$f_{(W)}(w) = \frac{\nu^n\,(-ln\,w)^{n-1}\,w^{\nu-1}}{\Gamma(n)} \tag{18}$$

for all $0 < w < 1$, $\nu > 0$ and $n \geq 1$.

From (16), (17) and (18), we obtain $f_{(U)}(v)f_{(W)}(w) = f_{(U),(W)}(u,w)$.

Hence $U = \dfrac{X_{L(n)} + X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. By the same manner as Theorem 2.1, let us use the transformations $U = \dfrac{X_{L(n)} + X_{L(n+1)}}{X_{L(n)} - X_{L(n+1)}}$ and $W = X_{L(n)}$. The Jacobian of the transformations is $|J| = \dfrac{2w}{(u+1)^2}$. Thus we can write the joint pdf $f_{(U),(W)}(u,w)$ of U and W as

$$f_{(U),(W)}(u,w) = \frac{2\,f\left(\frac{(u-1)\,w}{u+1}\right)\,(H(w))^{n-1}\,h(w)\,w}{\Gamma(n)\,(u+1)^2} \tag{19}$$

for all $u > 1$, $0 < w < 1$ and $n \geq 1$.

The pdf $f_{(W)}(w)$ of W is given by

$$f_{(W)}(w) = \frac{(H(w))^{n-1}\,f(w)}{\Gamma(n)} \tag{20}$$

for all $0 < w < 1$ and $n \geq 1$.

From (19) and (20), we obtain the pdf $f_{(U)}(u)$ of U

$$f_{(U)}(u) = \frac{2\,f\left(\frac{(u-1)\,w}{(u+1)}\right)\,h(w)\,w}{(u+1)^2\,f(w)}$$

for all $u > 1$, $0 < w < 1$ and $n \geq 1$, where $h(x) = \dfrac{f(x)}{F(x)}$.

That is,

$$f_{(U)}(u) = \frac{\partial}{\partial u}\left(F\left(\frac{(u-1)\,w}{(u+1)}\right)/\,F(w)\right).$$

Since U and $W$are independent, we must have

$$F\left(\frac{(u-1)w}{(u+1)}\right)=F\left(\frac{u-1}{u+1}\right)F(w) \tag{21}$$

for all $u>1$ and $0<w<1$.

Upon substituting $\dfrac{u-1}{u+1}=u_3$ in (21), then we get

$$F(u_3\,w)=F(u_3)F(w) \tag{22}$$

for all $0<u_3<1$ and $0<w<1$.

By the functional equations (see, [1]), the only continous solution of (22) with the boundary conditions $F(0)=0$ and $F(1)=1$ is

$$F(x)=x^{\nu}$$

for all $0<x<1$ and $\nu>0$.

This completes the proof. □

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Department of Mathematics Education
Cheongju University
Cheongju, Chungbuk 360-764, Republic of Korea
E-mail: skchang@cju.ac.kr