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ON THE HYERS-ULAM-RASSIAS STABILITY OF A PEXIDERIZED MIXED TYPE QUADRATIC FUNCTIONAL EQUATION

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Abstract. We establish the Hyers-Ulam-Rassias stability of the Pexiderized mixed type quadratic equation $f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z)=0$ in the spirit of D. H. Hyers, S. M. Ulam and Th. M. Rassias.

1. Introduction

In 1940, S. M. Ulam [23] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, D. H. Hyers [5] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon$$

for all $x, y \in V$, where V and X are Banach spaces and ε is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x)-T(x)\| \le \varepsilon$$

for all $x \in V$. In 1978, Th. M. Rassias[16] gave a significant generalization of the Hyers' result. Th. M. Rassias[20] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Z. Gadja[3] following Th. M. Rassias's approach([16]) gave an affirmative solution to

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the question. Recently, P.Găvruta[4] also obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam-Rassias stability [13,15 ,17-20]. Lee and Jun [14] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y)=g(x)+h(y)$(see [7]).

Throughout this paper, let V and X be a normed space and a Banach space, respectively.

In 1983, the stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0$$

was proved by F. Skof[22] for the function $f:V\to X$. In 1984, P. W. Cholewa[1] extended V to the case of an Abelian group G in the Skof's result.

In 1992, S. Czerwik[2] gave a generalization of the Skof-Cholewa's result. Since then, the stability problem of the quadratic equation has been extensively investigated by a number of mathematicians([8,12,21]).

Jun and Lee[6, 9, 10] obtained the stability results for the Pexiderized quadratic functional equation.

$$f(x+y)+g(x-y)=2h(x)+2k(y).$$

Now we introduce the following new Pexiderized mixed type quadratic functional equation

$$\begin{aligned}&f_1(x+y+z)+f_2(x-y)+f_3(z-x)\\ &-f_4(x-y-z)-f_5(x+y)-f_6(x+z)=0.\end{aligned}\tag{1.1}$$

In this paper, we establish the Hyers-Ulam-Rassias stability for the equation (1.1).

2. Hyers-Ulam-Rassias Stability of (1.1)

We need the following lemma to prove our main results.

LEMMA 2.1. *[11] Let a be a positive real number. Let $\Phi:V\setminus\{0\}\to[0,\infty)$ be a map such that*

$$\tilde{\Phi}(x):=\sum_{l=0}^{\infty}\frac{1}{a^{l+1}}\Phi(2^l x)<\infty \text{ for all } x\in V\setminus\{0\}\quad(*)$$

or

$$\tilde{\Phi}(x):=\sum_{l=0}^{\infty}a^l\Phi(\frac{x}{2^{l+1}})<\infty \text{ for all } x\in V\setminus\{0\}.(**)$$

Suppose that the function $f : V \to X$ satisfies the inequality

$$\|f(x) - \frac{f(2x)}{a}\| \leq \frac{\Phi(x)}{a}$$

for all $x \in V \setminus \{0\}$ and $f(0) = 0$. Then there exists exactly one function $F : V \to X$ satisfying

$$\begin{aligned} &\|f(x) - F(x)\| \leq \tilde{\Phi}(x) \text{ for all } x \in V \setminus \{0\} \\ &\text{and} \quad aF(x) = F(2x) \text{ for all } x \in V. \end{aligned}$$

Moreover, the function F is given by

$$F(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{a^n} & \text{if } \Phi \text{ satisfies } (*), \\ \lim_{n\to\infty} a^n f(2^{-n}x) & \text{if } \Phi \text{ satisfies } (**) \end{cases}$$

for all $x \in V$.

We establish the stability result for the even function in the following theorem.

THEOREM 2.2. *Let $p \neq 2$ and $\varepsilon > 0$. If the even functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy*

$$\begin{aligned} (2.1) \qquad &\|f_1(x + y + z) + f_2(x - y) + f_3(x - z) - f_4(x - y - z) \\ &-f_5(x + y) - f_6(x + z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in V \setminus \{0\}$, then there exists exactly one quadratic function $Q : V \to X$ satisfying

$$(2.2) \qquad \begin{aligned} \|f_1(x) - f_1(0) - Q(x)\| &\leq M_1\varepsilon\|x\|^p, \\ \|f_4(x) - f_4(0) - Q(x)\| &\leq M_1\varepsilon\|x\|^p, \\ \|f_2(x) - f_2(0) - Q(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_3(x) - f_3(0) - Q(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_5(x) - f_5(0) - Q(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_6(x) - f_6(0) - Q(x)\| &\leq M_2\varepsilon\|x\|^p \end{aligned}$$

where $M_1 = [1 + \frac{(3^p+11)(2^p+2)}{2\cdot 2^p|2^p-4|} + \frac{7+3^p}{2\cdot 2^p} + \frac{4}{4^p}]$, $M_2 = [\frac{3^p+11}{2^p|2^p-4|} + \frac{3^p+5}{4^p}]$ for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by

$$(2.3) \qquad Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f_k(2^n x)}{4^n} & \text{if } p < 2 \\ \lim_{n\to\infty} 4^n(f_k(2^{-n}x) - f_k(0)) & \text{if } p > 2 \end{cases}$$

for all $x \in V \setminus \{0\}$ and $k = 1, 2, 3, 4, 5, 6$.

Proof. Let $p < 2$. Replace x by $-x$ in (2.1) to obtain

$$\begin{aligned} &\|f_1(x-y-z) + f_2(x+y) + f_3(x+z) - f_4(x+y+z) \\ &-f_5(x-y) - f_6(x-z)\| \le \varepsilon(\|-x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \tag{2.4}$$

for all $x, y, z \in V \setminus \{0\}$. It follows from (2.1) and (2.4) that

$$\begin{aligned} &\|F(x+y+z) + G(x-y) + H(x-z) - F(x-y-z) \\ &- G(x+y) - H(x+z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \tag{2.5}$$

for all $x, y, z \in V \setminus \{0\}$, where the functions $F, G, H : V \to X$ are defined by

$$\begin{aligned} F(x) &:= \tfrac{1}{2}[f_1(x) + f_4(x) - f_1(0) - f_4(0)], \\ G(x) &:= \tfrac{1}{2}[f_2(x) + f_5(x) - f_2(0) - f_5(0)], \\ H(x) &:= \tfrac{1}{2}[f_3(x) + f_6(x) - f_3(0) - f_6(0)] \end{aligned} \tag{2.6}$$

for all $x, y, z \in V$. It follows from (2.5) that

$$\begin{aligned} \| \; G(x) - \tfrac{G(2x)}{4}\| \; &\le \frac{1}{2}\| - H(x) + G(x)\| + \frac{1}{4}\| \\ &-F(\frac{3x}{2}) + F(\frac{x}{2}) + G(x) + H(x)\| \\ &+\frac{1}{4}\|F(\frac{3x}{2}) - F(\frac{x}{2}) + G(x) - G(2x) + H(x)\| \\ &\le \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|-\frac{x}{2}\|^p) \\ &+\frac{1}{4}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p) \\ &+\frac{1}{4}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{3x}{2}\|^p + \|-\frac{x}{2}\|^p) \\ &= \frac{(11+3^p)}{4 \cdot 2^p}\varepsilon\|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$. By Lemma 2.1, there exists $Q(x) : V \to X$ for all $x \in V$ satisfying

$$\|G(x) - Q(x)\| \le \frac{(11+3^p)}{2^p|4-2^p|}\varepsilon\|x\|^p \tag{2.7}$$

for all $x \in V \setminus \{0\}$ and

$$Q(x) = \lim_{n\to\infty} \frac{G(2^n x)}{4^n} \tag{2.8}$$

for all $x \in V$. By the similar method in obtaining the inequality (2.7), we get

$$\|H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{4^n}\| \leq \frac{(11+3^p)}{2^p|4-2^p|}\varepsilon\|x\|^p \tag{2.9}$$

for all $x \in V\setminus\{0\}$. Since $\|-H(2x)+G(2x)\| \leq \varepsilon(\|x\|^p+\|x\|^p+\|-x\|^p)$, we have

$$\lim_{n\to\infty} \frac{G(2^n x)}{4^n} = \lim_{n\to\infty} \frac{H(2^n x)}{4^n}$$

for all $x \in V$ and so

$$Q(x) = \lim_{n\to\infty} \frac{H(2^n x)}{4^n} 2.10 \tag{2.10}$$

for all $x \in V$. From (2.5), (2.8) and (2.9), we get

$$\begin{aligned}
\|F(x) \quad - \quad & Q(x)\| \\
\leq \ & \frac{1}{2}\|F(x)+G(x)+H(\frac{3}{2}x)-G(2x)-H(\frac{x}{2})\| \\
+ \ & \frac{1}{2}\|F(x)+G(x)+H(\frac{1}{2}x)-H(\frac{3}{2}x)\| \\
+ \ & \|-G(x)+Q(x)\| + \frac{1}{2}\|G(2x)-Q(2x)\| \\
\leq \ & \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p+\|\frac{3x}{2}\|^p+\|-x\|^p)+\frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p+\|\frac{x}{2}\|^p+\|-x\|^p) \\
+ \ & \frac{(11+3^p)}{2|4-2^p|}\varepsilon\|x\|^p + \frac{(11+3^p)}{2^p|4-2^p|}\varepsilon\|x\|^p \\
\leq \ & (\frac{(3+3^p)}{2\cdot 2^p}+1+\frac{(11+3^p)}{2|4-2^p|}+\frac{(11+3^p)}{2^p|4-2^p|})\varepsilon\|x\|^p
\end{aligned} \tag{2.11}$$

for all $x \in V\setminus\{0\}$. Replacing x by $2^n x$, dividing by 4^n in the above inequality and taking the limit in the resulted inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F(2^n x)}{4^n} = Q(x) \tag{2.12}$$

for all $x \in V$. Using (2.5), (2.8), (2.10) and (2.12), we obtain

$$\begin{aligned}
&Q(x+y+z)+Q(x-y)+Q(z-x) \\
&-Q(x-y-z)-Q(x+y)-Q(x+z)=0
\end{aligned} \tag{2.13}$$

for all $x, y, z \in V\setminus\{0\}$. Replacing x and z by $\frac{x}{2}$ in (2.13) and using the fact $Q(0)=0$, we have

$$Q(x+y)+Q(\frac{x}{2}-y))-Q(-y)-Q(\frac{x}{2}+y)-Q(x)=0 \tag{2.14}$$

for all $x, y \in V$. Replace x and z by $\frac{x}{2}$ and $\frac{-x}{2}$ in (2.13) to have

$$Q(y) + Q(\frac{x}{2} - y) + Q(x) - Q(x - y) - Q(\frac{x}{2} + y) = 0. \tag{2.15}$$

for all $x, y \in V$. Subtracting (2.14) from (2.15) and using the evenness of Q, we lead to

$$Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y) = 0,$$

for all $x, y \in V$.

On the other hand, it follows from (2.1) and (2.4) that

$$\begin{aligned} \|F'(x + y + z) &+ G'(x - y) + H'(x - z) + F'(x - y - z) \\ &+ G'(x + y) + H'(x + z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \tag{2.16}$$

for all $x, y, z \in V \setminus \{0\}$, where the functions $F', G', H' : V \to X$ are defined by

$$F'(x) = \frac{1}{2}[f_1(x) - f_4(x)], \quad G'(x) = \frac{1}{2}[f_2(x) - f_5(x)],$$

$$H'(x) = \frac{1}{2}[f_3(x) - f_6(x)]$$

for all $x, y, z \in V$. It follows from (2.16) that

$$\begin{aligned} \|G'(x) - G'(0)\| &\le \|F'(\frac{3x}{4}) + F'(\frac{x}{4}) + G'(\frac{x}{2}) + G'(0) + H'(\frac{x}{2}) + H'(0)\| \\ &+ \|F'(\frac{3x}{4}) + F'(\frac{x}{4}) + G'(\frac{x}{2}) + G'(x) + H'(\frac{x}{2}) + H'(0)\| \\ &\le \varepsilon(\|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{3x}{4}\|^p + \|-\frac{x}{4}\|^p) \\ &= \frac{3^p + 5}{4^p}\varepsilon\|x\|^p, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \|H'(x) - H'(0)\| &\le \|F'(\frac{3x}{4}) + F'(\frac{x}{4}) + G'(\frac{x}{2}) + G'(0) + H'(\frac{x}{2}) + H'(0)\| \\ &+ \|F'(\frac{3x}{4}) + F'(\frac{x}{4}) + G'(\frac{x}{2}) + G'(0) + H'(\frac{x}{2}) + H'(x)\| \\ &\le \varepsilon(\|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{-3x}{4}\|^p) \\ &= \frac{3^p + 5}{4^p}\varepsilon\|x\|^p, \end{aligned} \tag{2.18}$$

$$\begin{aligned}\|F'(x)-F'(0)\| &\le \|F'(0)+G'(0)+H'(\frac{3x}{4})+F'(\frac{x}{2})+G'(\frac{x}{2})+H'(\frac{x}{4})\|\\ &\quad+\|F'(x)+G'(0)+H'(\frac{x}{4})+F'(\frac{x}{2})+G'(\frac{x}{2})+H'(\frac{3x}{4})\|\\ &\le \varepsilon(\|\frac{x}{4}\|^p+\|\frac{x}{4}\|^p+\|-\frac{x}{2}\|^p+\|\frac{x}{4}\|^p+\|\frac{x}{4}\|^p+\|\frac{x}{2}\|^p)\\ &= \frac{(4+2\cdot 2^p)}{4^p}\varepsilon\|x\|^p \qquad (2.19)\end{aligned}$$

for all $x \in V \setminus \{0\}$. By the definition of F, G, H, F', G', H', we have

$$\begin{aligned}f_1(x)-f_1(0)-Q(x) &= F(x)+F'(x)-F'(0)-Q(x),\\ f_2(x)-f_2(0)-Q(x) &= G(x)+G'(x)-F'(0)-Q(x),\\ f_3(x)-f_3(0)-Q(x) &= H(x)+H'(x)-H'(0)-Q(x),\\ f_4(x)-f_4(0)-Q(x) &= F(x)-F'(x)+F'(0)-Q(x),\\ f_5(x)-f_5(0)-Q(x) &= G(x)-G'(x)+G'(0)-Q(x),\\ f_6(x)-f_6(0)-Q(x) &= H(x)-H'(x)+H'(0)-Q(x)\end{aligned}$$

hold for all $x \in V\setminus\{0\}$. Hence by using (2.7), (2.9), (2.10), (2.11), (2.17), (2.18), (2.19), the inequalities in (2.2) can be shown. The uniqueness of Q follows from Lemma 2.1.
By the similar method, we can prove the result of this theorem for the case $p > 2$.

□

THEOREM 2.3. *Let $p \neq 2$ be nonnegative real number. If the even functions $f_k : V \to X$, $k = 1, 2, \dots, 6$, satisfy (2.1) for all $x, y, z \in V$, then there exists exactly one quadratic function $Q : V \to X$ satisfying*

$$\begin{aligned}\|f_1(x)-f_1(0)-Q(x)\| &\le M_1\varepsilon\|x\|^p,\\ \|f_4(x)-f_4(0)-Q(x)\| &\le M_1\varepsilon\|x\|^p,\\ \|f_2(x)-f_2(0)-Q(x)\| &\le M_2\varepsilon\|x\|^p,\\ \|f_3(x)-f_3(0)-Q(x)\| &\le M_2\varepsilon\|x\|^p,\\ \|f_5(x)-f_5(0)-Q(x)\| &\le M_2\varepsilon\|x\|^p,\\ \|f_6(x)-f_6(0)-Q(x)\| &\le M_2\varepsilon\|x\|^p\end{aligned}$$

where $M_1 = [\frac{3}{2}+\frac{3^p+11}{2^p|2^p-4|}+\frac{2}{2^p}]$, $M_2 = (\frac{3^p+11}{2^p|2^p-4|}+\min(1, \frac{3^p+5}{4^p}))$ for all $x \in V$. Moreover, the function Q is given by (2.3) for all $x \in V$, where p is a nonnegative real number with $p \neq 2$.

Proof. Let $0 \le p < 2$. The relations (2.4)-(2.16) holds for all $x, y, z \in V$. Hence, it follows from (2.5) and (2.7) that

$$\begin{aligned} \|F(x) - Q(x)\| &\le \|F(x) - G(x)\| + \|G(x) - Q(x)\| \\ &\le \varepsilon(\|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|0\|^p + \frac{(11+3^p)}{2^p|4-2^p|}\|x\|^p) \end{aligned} \tag{2.20}$$

for all $x \in V$. It follows from that (2.16) that

$$\begin{aligned} \|G'(x) - G'(0)\| &\le \|F'(x) + G'(x) + H'(x)\| + \|F'(x) + H'(x) + G'(0)\| \\ &\le \frac{1}{2}\varepsilon(\|x\|^p + \|0\|^p + \|0\|^p) + \frac{1}{2}\varepsilon(\|0\|^p + \|0\|^p + \|x\|^p) \\ \|H'(x) - H'(0)\| &\le \|F'(x) + G'(x) + H'(x)\| + \|F'(x) + H'(0) + G'(x)\| \\ &\le \frac{1}{2}\varepsilon(\|x\|^p + \|0\|^p + \|0\|^p) + \frac{1}{2}\varepsilon(\|0\|^p + \|x\|^p + \|0\|^p) \\ \|F'(x) - F'(0)\| &\le \|F'(x) + G'(x) + H'(x)\| + \|F'(0) + H'(x) + G'(x)\| \\ &\le \frac{1}{2}\varepsilon(\|x\|^p + \|0\|^p + \|0\|^p) + \frac{1}{2}\varepsilon(\|0\|^p + \|x\|^p + \|-x\|^p) \end{aligned}$$

for all $x \in V$. By using (2.7), (2.9), (2.20), the above inequalities and Theorem 2.2, we have the desired inequalities in this theorem.
By the similar method, we can prove the result of this theorem for the case $p > 2$.

□

We establish the stability result for the odd function in the following Theorem 2.4.

THEOREM 2.4. *Let $p \neq 1$ and $\varepsilon > 0$. If the odd functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy (2.1) for all $x, y, z \in V \setminus \{0\}$.*

Then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned} \|f_1(x) - A(x) + A_1(x) + A_2(x)\| &\le M_1\varepsilon\|x\|^p, \\ \|f_4(x) - A(x) - A_1(x) - A_2(x)\| &\le M_1\varepsilon\|x\|^p, \\ \|f_2(x) - A(x) - A_1(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_3(x) - A(x) - A_2(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_5(x) - A(x) + A_1(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_6(x) - A(x) + A_2(x)\| &\le M_2\varepsilon\|x\|^p \end{aligned} \tag{2.21}$$

where $M_1 = (\frac{2}{2^p} + \frac{4}{4^p} + \frac{2(3^p+11)+4\cdot 2^p+2\cdot 4^p}{4^p|2^p-2|})$, $M_2 = \frac{2(3^p+8)}{2^p|2^p-2|}$ *for all* $x \in V \setminus \{0\}$. *Moreover, the functions* A, A_1, A_2 *are given by*

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f_1(2^n x)+f_4(2^n x)}{2^{n+1}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-1}(f_1(\frac{x}{2^n}) + f_4(\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

$$A_1(x) = \begin{cases} \lim_{n\to\infty} \frac{f_2(2^n x)-f_5(2^n x)}{2^{n+1}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-1}(f_2(\frac{x}{2^n}) - f_5(\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

$$A_2(x) = \begin{cases} \lim_{n\to\infty} \frac{f_3(2^n x)-f_6(2^n x)}{2^{n+1}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-1}(f_3(\frac{x}{2^n}) - f_6(\frac{x}{2^n})) & \text{if } p > 1 \end{cases} \tag{2.22}$$

for all $x \in V$.

Proof. Assume that $p < 1$. Replace x by $-x$ in (2.1) to obtain

$$\begin{aligned} \|-f_1(x-y-z) &- f_2(x+y) - f_3(x+z) + f_4(x+y+z) \\ &+ f_5(x-y) + f_6(x-z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \tag{2.23}$$

for all $x, y, z \in V \setminus \{0\}$. Let the functions $F, G, H : V \to X$ be defined by

$$F(x) = \frac{1}{2}[f_1(x) + f_4(x)], \quad G(x) = \frac{1}{2}[f_2(x) + f_5(x)],$$

$$H(x) = \frac{1}{2}[f_3(x) + f_6(x)]$$

for all $x, y, z \in V$. From (2.1) and (2.23), we get

$$\begin{aligned} \|F(x+y+z) &+ G(x-y) + H(x-z) - F(x-y-z) \\ &- G(x+y) - H(x+z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \tag{2.24}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $-x$ and x in (2.24) to get

$$\|H(2x) - G(2x)\| \le \varepsilon(\|x\|^p + \|-x\|^p + \|x\|^p) \tag{2.25}$$

for all $x \in V \setminus \{0\}$. It follows from (2.24) that

$$\begin{aligned}
\|G(x) \;-\; \frac{G(2x)}{2}\| & \\
=\; & \frac{1}{2}\| - F(\frac{3x}{2}) - F(\frac{x}{2}) + G(x) + G(2x) - H(x)\| \\
+\; & \|H(x) - G(x)\| + \frac{1}{2}\|F(\frac{3x}{2}) + F(\frac{x}{2}) - G(x) - H(x)\| \\
\leq\; & \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{3x}{2}\|^p + \| - \frac{x}{2}\|^p) + \varepsilon(\|\frac{x}{2}\|^p + \| - \frac{x}{2}\|^p + \|\frac{x}{2}\|^p) \\
+\; & \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p) \\
=\; & \frac{(11 + 3^p)}{2 \cdot 2^p}\varepsilon\|x\|^p
\end{aligned}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1, there exists a function $A : V \to X$ satisfying

$$\|G(x) - A(x)\| \leq \frac{(11 + 3^p)}{2^p|2 - 2^p|}\varepsilon\|x\|^p \tag{2.26}$$

for all $x \in V \setminus \{0\}$, where

$$A(x) := \lim_{n\to\infty} \frac{G(2^n x)}{2^n}$$

for all $x \in V$. By the similar method in obtaining the inequality (2.26), we have

$$\|H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{2^n}\| \leq \frac{(11 + 3^p)}{2^p|2 - 2^p|}\varepsilon\|x\|^p \tag{2.27}$$

for all $x \in V \setminus \{0\}$. From (2.25), we get

$$A(x) = \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n} \tag{2.28}$$

for all $x \in V$. It follows from (2.24) and (2.26) that

$$\begin{aligned}\|F(x) \quad - \quad A(x)\| &= \|F(x) - H(\frac{x}{4}) + F(\frac{x}{2}) - G(\frac{x}{2}) - H(\frac{3x}{4})\| \\ &+ \|H(\frac{3x}{4}) - F(\frac{x}{2}) - G(\frac{x}{2}) + H(\frac{x}{4})\| + \|2G(\frac{x}{2}) - 2A(\frac{x}{2})\| \\ &\leq (\|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{4}\|^p + \|\frac{x}{4}\|^p \\ &+ \|-\frac{x}{2}\|^p + \frac{2(11+3^p)}{4^p|2-2^p|})\varepsilon\|x\|^p \\ &= (\frac{4+2\cdot 2^p}{4^p} + \frac{2(11+3^p)}{4^p|2-2^p|})\varepsilon\|x\|^p \qquad (2.29)\end{aligned}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^n x$, dividing by 2^n in the above inequality and taking the limit in the resulted inequality as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{F(2^n x)}{2^n} = A(x) \qquad (2.30)$$

for all $x \in V \setminus \{0\}$. It follows from (2.24), (2.28) and (2.30) that

$$\begin{aligned}A(x+y+z) \; &+ \; A(x-y) + A(x-z) - A(x-y-z) \\ &- \; A(x+y) - A(x+z) = 0 \qquad (2.31)\end{aligned}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $2y$ and x in (2.31) to obtain

$$A(2x+2y) + A(x-2y) + A(2y) - A(x+2y) - A(2x) = 0$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $-2y$ and x in (2.31) to get

$$A(2x-2y) + A(x+2y) - A(2y) - A(x-2y) - A(2x) = 0$$

for all $x, y \in V \setminus \{0\}$. Since $A(0) = 0$ and $A(2x) = 2A(x)$, using the above two equalities, we have

$$A(x-y) + A(x+y) - A(2x) = 0$$

for all $x, y \in V$. Hence, A is an additive function.

Let the functions $F', G', H' : V \to X$ be defined by

$$F'(x) = \frac{1}{2}[f_1(x) - f_4(x)], \quad G'(x) = \frac{1}{2}[f_2(x) - f_5(x)],$$

$$H'(x) = \frac{1}{2}[f_3(x) - f_6(x)]$$

for all $x, y, z \in V$. From (2.1) and (2.23), we have

$$\begin{aligned}\|F'(x+y+z) \; &+ \; G'(x-y) + H'(x-z) + F'(x-y-z) \\ &+ \; G'(x+y) + H'(x+z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \qquad (2.32)\end{aligned}$$

for all $x, y, z \in V \setminus \{0\}$. It follows from (2.32) that

$$\begin{aligned}\|G'(x) \quad &- \quad \frac{G'(2x)}{2}\| \leq \frac{1}{2}\|F'(\frac{3x}{2}) - F'(\frac{x}{2}) + G'(x) + H'(x)\| \\ &+ \quad \frac{1}{2}\| - F'(\frac{3x}{2}) + F'(\frac{x}{2}) + G'(x) - G'(2x) - H'(x)\| \\ &\leq \quad \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|\frac{x}{2}\|^p + \|\frac{3x}{2}\|^p + \| - \frac{x}{2}\|^p) \\ &= \quad \frac{(5+3^p)}{2\cdot 2^p}\varepsilon\|x\|^p\end{aligned}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1, we obtain an odd function $A_1 : V \to X$ defined by

$$A_1(x) := \lim_{n\to\infty} \frac{G'(2^n x)}{2^n}. \tag{2.33}$$

and the inequality

$$\|G'(x) - A_1(x)\| \leq \frac{(5+3^p)}{2^p|2-2^p|}\varepsilon\|x\|^p \tag{2.34}$$

holds for all $x \in V \setminus \{0\}$. Similarly, we have an odd function $A_2 : V \to X$ defined by

$$A_2(x) := \lim_{n\to\infty} \frac{H'(2^n x)}{2^n} \tag{2.35}$$

for all $x \in V$ and the inequality

$$\|H'(x) - A_2(x)\| \leq \frac{(5+3^p)}{2^p|2-2^p|}\varepsilon\|x\|^p \tag{2.36}$$

for all $x \in V \setminus \{0\}$. Replace x, y, z by x, x, $-x$ in (2.34), to get

$$\|2F'(x) + G'(2x) + H'(2x)\| \leq 3\varepsilon\|x\|^p$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^{n-1}x$ and dividing by 2^n in the above inequality, we obtain

$$\|\frac{2F'(2^{n-1}x) + G'(2^n x) + H'(2^n x)}{2^n}\| \leq 3(\frac{2^p}{2})^{n-1}\varepsilon\|x\|^p$$

for all $x \in V \setminus \{0\}$. Taking the limit in the above inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F'(2^n x)}{2^n} = -A_1(x) - A_2(x) \tag{2.37}$$

for all $x \in V \setminus \{0\}$. It follows from (2.32) that

$$\begin{aligned} \|F'(x) &- \frac{F'(2x)}{2}\| \le \frac{1}{2}\|F'(x)+G'(x)-H'(\frac{x}{2})+H'(\frac{3x}{2})\| \\ &+ \frac{1}{2}\|F'(2x)-H'(\frac{x}{2})-F'(x)+G'(x)+H'(\frac{3x}{2})\| \\ &\le \frac{1}{2}\varepsilon(\|\frac{x}{2}\|^p+\|\frac{x}{2}\|^p+\|-x\|^p+\|\frac{x}{2}\|^p+\|\frac{x}{2}\|^p+\|x\|^p) \\ &= \frac{2+2^p}{2^p}\varepsilon\|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1 and (2.37), we have

$$\|F'(x)+A_1(x)+A_2(x)\| \le \frac{4+2\cdot 2^p}{2^p|2-2^p|}\varepsilon\|x\|^p \tag{2.38}$$

for all $x \in V \setminus \{0\}$. From (2.32), (2.33), (2.35) and (2.37), we have

$$\begin{aligned} &-A_1(x+y+z)-A_2(x+y+z)+A_1(x-y)+A_2(x-z) \\ &-A_1(x-y-z)-A_2(x-y-z)+A_1(x+y)+A_2(x+z)=0 \end{aligned} \tag{2.39}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $2y$ and x in (2.39), to get

$$\begin{aligned} &-A_1(2x+2y)-A_2(2x+2y)+A_1(x-2y) \\ &+A_1(2y)+A_2(2y)+A_2(2x)+A_1(x+2y)=0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$. Replace y and z by x and $2y$ in (2.39), to get

$$\begin{aligned} &-A_1(2x+2y)-A_2(2x+2y)+A_2(x-2y) \\ &+A_1(2y)+A_2(2y)+A_1(2x)+A_2(x+2y)=0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$. From the above two equalities, we get

$$(A_1-A_2)(x-2y)-(A_1-A_2)(2x)+(A_1-A_2)(x+2y)=0$$

for all $x, y \in V \setminus \{0\}$. Since $A(0)=0$, we have

$$(A_1-A_2)(x-2y)-(A_1-A_2)(2x)+(A_1-A_2)(x+2y)=0$$

for all $x, y \in V$. Hence A_1-A_2 is additive, i.e.,

$$(A_1-A_2)(x+y)=(A_1-A_2)(x)+(A_1-A_2)(y)$$

for all $x, y \in V$. Replace z by $-y$ in (2.39), to obtain

$$\begin{aligned} -A_1(x) &- A_2(x)+A_1(x-y)+A_2(x+y)-A_1(x) \\ &- A_2(x)+A_1(x+y)+A_2(x-y)=0 \end{aligned} \tag{2.40}$$

for all $x, y \in V \setminus \{0\}$. Since A_1-A_2 is additive, we have

$$A_2(2x)-A_2(x+y)-A_2(x-y)=A_1(2x)-A_1(x+y)-A_1(x-y)$$

for all $x, y \in V \setminus \{0\}$. From this and (2.40), we get

$$-A_1(4x) + 2A_1(x - y) + 2A_1(x + y) = 0$$

for all $x, y \in V \setminus \{0\}$. From this and $A_1(0) = 0$, we have

$$A_1(x + y) = A_1(x) + A_1(y)$$

for all $x, y \in V$. Since A_1 and $A_1 - A_2$ are additive, A_2 is additive.

From (2.29), (2.38) and the definition of F, F', we have

$$\begin{aligned}\|f_1(x) \;&-\; A(x) + A_1(x) + A_2(x)\| \\ &\le \|F(x) - A(x)\| + \|F'(x) + A_1(x) + A_2(x)\| \\ &\le (\frac{4 + 2 \cdot 2^p}{4^p} + \frac{2(11 + 3^p)}{4^p|2 - 2^p|} + \frac{4 + 2 \cdot 2^p}{2^p|2 - 2^p|})\varepsilon\|x\|^p\end{aligned}$$

for all $x \in V \setminus \{0\}$. The rest of inequalities in (2.21) can be shown by the similar method.

By the similar method, we can prove the result of this theorem for the case $p > 1$.

□

THEOREM 2.5. *Let $p \neq 1$ be a positive real number and $\varepsilon > 0$. If the odd functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy (2.1) for all $x, y, z \in V$.*

Then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}\|f_1(x) - A(x) + A_1(x) + A_2(x)\| &\le M_1\varepsilon\|x\|^p, \\ \|f_4(x) - A(x) - A_1(x) - A_2(x)\| &\le M_1\varepsilon\|x\|^p, \\ \|f_2(x) - A(x) - A_1(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_3(x) - A(x) - A_2(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_5(x) - A(x) + A_1(x)\| &\le M_2\varepsilon\|x\|^p, \\ \|f_6(x) - A(x) + A_2(x)\| &\le M_2\varepsilon\|x\|^p \end{aligned} \tag{2.41}$$

for all $x \in V$, where $M_1 = \min(\frac{2}{2^p} + \frac{4}{4^p} + \frac{2(3^p+11)+4\cdot 2^p+2\cdot 4^p}{4^p|2^p-2|}, \frac{10+2^p}{4|2-2^p|})$ and $M_2 = \min(\frac{2(4+2^p)}{4|2-2^p|}, \frac{1}{2} + \frac{8+2^p}{4|2-2^p|}, \frac{2(3^p+8)}{2^p|2^p-2|})$. Moreover, the functions A, A_1, A_2 are given by (2.22) for all $x \in V$, where p is a positive real number with $p \neq 1$.

Proof. Let $0 \leq p < 1$. The relations in the proof of Theorem 2.4 valid for all $x, y, z \in V$. It follows from (2.24) that

$$\|F(x) - G(x)\| \leq \frac{1}{2}\varepsilon(\|0\|^p + \|x\|^p + \|0\|^p) = \frac{1}{2}\varepsilon\|x\|^p, \tag{2.42}$$

$$\|F(x) - H(x)\| \leq \frac{1}{2}\varepsilon(\|0\|^p + \|0\|^p + \|x\|^p) = \frac{1}{2}\varepsilon\|x\|^p, \tag{2.43}$$

$$\|F(2x) - G(x) - H(x)\| \leq \frac{1}{2}\varepsilon(\|0\|^p + \|x\|^p + \|x\|^p) = \varepsilon\|x\|^p,$$

$$\|G(x) - H(x)\| \leq \frac{1}{2}\varepsilon(\|0\|^p + \|x\|^p + \|-x\|^p) = \varepsilon\|x\|^p$$

for all $x \in V$. Hence, the inequalities

$$\begin{aligned}
\|F(x) - \frac{F(2x)}{2}\| &= \frac{1}{2}\|G(x) - F(x)\| + \frac{1}{2}\|F(2x) - G(x) - H(x)\| \\
&+ \frac{1}{2}\|H(x) - F(x)\| \leq \varepsilon\|x\|^p, \\
\|G(x) - \frac{G(2x)}{2}\| &= \frac{1}{2}\|G(2x) - F(2x)\| + \frac{1}{2}\|F(2x) - G(x) - H(x)\| \\
&+ \frac{1}{2}\|H(x) - G(x)\| \leq (\frac{2^p}{4} + 1)\varepsilon\|x\|^p, \\
\|H(x) - \frac{H(2x)}{2}\| &= \frac{1}{2}\|H(2x) - F(2x)\| + \frac{1}{2}\|F(2x) - G(x) - H(x)\| \\
&+ \frac{1}{2}\|-H(x) + G(x)\| \leq (\frac{2^p}{4} + 1)\varepsilon\|x\|^p
\end{aligned}$$

hold for all $x \in V$. By the relations in the proof of Theorem 2.4, applying Lemma 2.1 to the above inequalities, we obtain the additive function $A : V \to X$ satisfying

$$\|F(x) - A(x)\| \leq \frac{1}{2 - 2^p}\varepsilon\|x\|^p, \tag{2.44}$$

$$\|G(x) - A(x)\| \leq \frac{4 + 2^p}{4 \cdot (2 - 2^p)}\varepsilon\|x\|^p, \tag{2.45}$$

$$\|H(x) - A(x)\| \leq \frac{4 + 2^p}{4 \cdot (2 - 2^p)}\varepsilon\|x\|^p \tag{2.46}$$

for all $x \in V$, where A is given by

$$A(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n}$$

for all $x \in V$.

On the other hand, it follows from (2.42),(2.43) and (2.44) that

$$\begin{aligned} \|G(x)-A(x)\| &\le \|G(x)-F(x)\|+\|F(x)-A(x)\| \\ &\le (\frac{1}{2}+\frac{1}{2-2^p})\varepsilon\|x\|^p, \quad (2.47)\\ \|H(x)-A(x)\| &\le \|H(x)-F(x)\|+\|F(x)-A(x)\| \\ &\le (\frac{1}{2}+\frac{1}{2-2^p})\varepsilon\|x\|^p \quad (2.48) \end{aligned}$$

for all $x \in V$. It follows from (2.32) that

$$\begin{aligned} &\|F'(2x)+G'(2x)+H'(2x)\| \le \frac{1}{2}\varepsilon(\|2x\|^p+\|0\|^p+\|0\|^p), \\ &\|F'(2x)+2G'(x)+H'(2x)\| \le \varepsilon(\|x\|^p+\|0\|^p+\|x\|^p), \\ &\|F'(2x)+G'(2x)+2H'(x)\| \le \varepsilon(\|x\|^p+\|x\|^p+\|0\|^p), \\ &\|2F'(x)+G'(2x)+H'(2x)\| \le \varepsilon(\|x\|^p+\|x\|^p+\|-x\|^p) \end{aligned}$$

for all $x \in V$ and hence we have

$$\begin{aligned} \|G'(x)-\frac{G'(2x)}{2}\| &\le \frac{1}{2}\|F'(2x)+G'(2x)+H'(2x)\| \\ &+ \frac{1}{2}\|F'(2x)+2G'(x)+H'(2x)\| \le \frac{4+2^p}{4}\varepsilon\|x\|^p, \\ \|H'(x)-\frac{H'(2x)}{2}\| &\le \frac{1}{2}\|F'(2x)+G'(2x)+H'(2x)\| \\ &+ \frac{1}{2}\|F'(2x)+G'(2x)+2H'(x)\|] \le \frac{4+2^p}{4}\varepsilon\|x\|^p, \\ \|F'(x)-\frac{F'(2x)}{2}\| &\le \frac{1}{2}\|F'(2x)+G'(2x)+H'(2x)\| \\ &+ \frac{1}{2}\|2F'(x)+G'(2x)+H'(2x)\| \le \frac{6+2^p}{4}\varepsilon\|x\|^p \end{aligned}$$

for all $x \in V$. Applying Lemma 2.1 and Theorem 2.4 to the above inequalities, we obtain an odd functions $A_1, A_2 : V \to X$ satisfying

$$\|G'(x)-A_1(x)\| \le \frac{4+2^p}{4(2-2^p)}\varepsilon\|x\|^p, \quad (2.49)$$

$$\|H'(x)-A_2(x)\| \le \frac{4+2^p}{4(2-2^p)}\varepsilon\|x\|^p \quad (2.50)$$

$$\|F'(x)+A_1(x)+A_2(x)\| \le \frac{6+2^p}{4(2-2^p)}\varepsilon\|x\|^p \quad (2.51)$$

for all $x \in V$. From (2.44), (2.51) and the definition of F, F', we have

$$\begin{aligned}\|f_1(x) \, - \, & A(x) + A_1(x) + A_2(x)\| \\ & \leq \|F(x) - A(x)\| + \|F'(x) + A_1(x) + A_2(x)\| \\ & \leq (\frac{1}{2-2^p} + \frac{6+2^p}{4(2-2^p)})\varepsilon\|x\|^p\end{aligned}$$

for all $x \in V$. From this and Theorem 2.4, we get the inequality (2.41). The rest of inequalities can be shown by the similar method. □

Now we establish the main theorem.

THEOREM 2.6. *Let $p \neq 1, 2$ and $\varepsilon > 0$. Suppose that the functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy (2.1) for all $x, y, z \in V \setminus \{0\}$.*

Then there exist exactly one quadratic function $Q : V \to X$ and three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}\|f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x)\| &\leq M_1\varepsilon\|x\|^p, \\ \|f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x)\| &\leq M_1\varepsilon\|x\|^p, \\ \|f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x)\| &\leq M_2\varepsilon\|x\|^p, \\ \|f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x)\| &\leq M_2\varepsilon\|x\|^p\end{aligned}$$

where $M_1 = [\frac{(3^p+11)(2^p+2)}{2\cdot 2^p|2^p-4|} + \frac{11+3^p}{2\cdot 2^p} + 1 + \frac{8}{4^p} + \frac{2(3^p+11)+4\cdot 2^p+2\cdot 4^p}{4^p|2^p-2|}]$, $M_2 = [\frac{(3^p+11)}{2^p|2^p-4|} + \frac{3^p+5}{4^p} + \frac{2(3^p+8)}{2^p|2^p-2|}]$ for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f_k(2^n x) + f_k(-2^n x)}{2\cdot 4^n} & \text{if } p < 2 \\ \lim_{n\to\infty} 4^n(\frac{f_k(2^{-n}x) + f_k(-2^{-n}x)}{2} - f_k(0)) & \text{if } p > 2 \end{cases}$$

for $k = 1, 2, 3, 4, 5, 6$ and the functions A, A_1, A_2 $(k = 1, 2, 3)$ are given by

$$A(x) = \begin{cases} \lim_{n\to\infty} \frac{f_1(2^n x)+f_4(2^n x)-f_1(-2^n x)-f_4(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_1(\frac{x}{2^n}) + f_4(\frac{x}{2^n}) - f_1(-\frac{x}{2^n}) & \\ \qquad -f_4(-\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

$$A_1(x) = \begin{cases} \lim_{n\to\infty} \frac{f_2(2^n x)-f_5(2^n x)-f_2(-2^n x)+f_5(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_2(\frac{x}{2^n}) - f_5(\frac{x}{2^n}) - f_2(-\frac{x}{2^n}) & \\ \qquad +f_5(-\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

$$A_2(x) = \begin{cases} \lim_{n\to\infty} \frac{f_3(2^n x)-f_6(2^n x)-f_3(-2^n x)+f_6(-2^n x)}{2^{n+2}} & \text{if } p < 1, \\ \lim_{n\to\infty} 2^{n-2}(f_3(\frac{x}{2^n}) - f_6(\frac{x}{2^n}) - f_3(-\frac{x}{2^n}) & \\ \qquad +f_6(-\frac{x}{2^n})) & \text{if } p > 1 \end{cases}$$

for all $x \in V$.

Proof. From (2.1), we obtain

$$\begin{aligned} \|f_1(-x-y-z) \,&+\, f_2(-x+y) + f_3(-x+z) - f_4(-x+y+z) \\ &-\, f_5(-x-y) - f_6(-x-z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.1) and this inequality, one gets

$$\begin{aligned} \|f_{1e}(x+y+z) \,&+\, f_{2e}(x-y) + f_{3e}(x-z) - f_{4e}(x-y-z) \\ &-\, f_{5e}(x+y) - f_{6e}(x+z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p), \\ \|f_{1o}(x+y+z) \,&+\, f_{2o}(x-y) + f_{3o}(x-z) - f_{4o}(x-y-z) \\ &-\, f_{5o}(x+y) - f_{6o}(x+z)\| \le \varepsilon\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in V \setminus \{0\}$, where $f_{ke}(x) = \frac{f_k(x)+f_k(-x)}{2}$, $f_{ko}(x) = \frac{f_k(x)-f_k(-x)}{2}$ for all $x \in V \setminus \{0\}$, $k = 1, 2, 3, 4, 5, 6$. Since f_{ke} is an even function, f_{ko} is an odd function and $f_k = f_{ke} + f_{ko}$, we can apply Theorem 2.2 and Theorem 2.4. □

THEOREM 2.7. *Let $p \neq 1, 2$ be a positive number and $\varepsilon > 0$. Suppose that the functions $f_k : V \to X$, $k = 1, 2, \ldots, 6$, satisfy (2.1) for all $x, y, z \in V$.*

Then there exist exactly one quadratic function $Q : V \to X$ and three *additive functions* $A, A_1, A_2 : V \to X$ *satisfying*

$$\|f_1(x) - Q(x) - f_1(0) - A(x) + A_1(x) + A_2(x)\| \le M_1'\varepsilon\|x\|^p,$$
$$\|f_4(x) - Q(x) - f_4(0) - A(x) - A_1(x) - A_2(x)\| \le M_1'\varepsilon\|x\|^p,$$
$$\|f_2(x) - Q(x) - f_2(0) - A(x) - A_1(x)\| \le M_2'\varepsilon\|x\|^p,$$
$$\|f_5(x) - Q(x) - f_5(0) - A(x) + A_1(x)\| \le M_2'\varepsilon\|x\|^p,$$
$$\|f_3(x) - Q(x) - f_3(0) - A(x) - A_2(x)\| \le M_2'\varepsilon\|x\|^p,$$
$$\|f_6(x) - Q(x) - f_6(0) - A(x) + A_2(x)\| \le M_2'\varepsilon\|x\|^p$$

where $M_1' = \frac{3}{2} + \frac{3^p+11}{2^p|2^p-4|} + \frac{2}{2^p} + \min(\frac{2}{2^p} + \frac{4}{4^p} + \frac{2(3^p+11)+4\cdot 2^p+2\cdot 4^p}{4^p|2^p-2|}, \frac{10+2^p}{4|2-2^p|})$, $M_2'' = \frac{3^p+11}{2^p|2^p-4|} + \min(1, \frac{3^p+5}{4^p}) + \min(\frac{2(4+2^p)}{4|2-2^p|}, \frac{1}{2} + \frac{8+2^p}{4|2-2^p|}, \frac{2(3^p+8)}{2^p|2^p-2|})$ *for all* $x \in V$. *Moreover, the functions* Q, A, A_1, A_2 *are given by the equalities in Theorem 2.6, where* p *is a positive real number.*

Corollary 2.8. *Let* $\varepsilon > 0$ *be a fixed real number. Suppose that the functions* $f_i : V \to X$, $i = 1, 2, \ldots, 6$, *satisfy*

$$\|f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z)\| \le \varepsilon$$

for all $x, y, z \in V \setminus \{0\}$.

Then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ *satisfying*

$$\|f_1(x) - Q(x) - f_1(0) - A(x) + A_1(x) + A_2(x)\| \le 17\varepsilon,$$
$$\|f_4(x) - Q(x) - f_4(0) - A(x) - A_1(x) - A_2(x)\| \le 17\varepsilon,$$
$$\|f_2(x) - Q(x) - f_2(0) - A(x) - A_1(x)\| \le \tfrac{28}{3}\varepsilon,$$
$$\|f_5(x) - Q(x) - f_5(0) - A(x) + A_1(x)\| \le \tfrac{28}{3}\varepsilon,$$
$$\|f_3(x) - Q(x) - f_3(0) - A(x) - A_2(x)\| \le \tfrac{28}{3}\varepsilon,$$
$$\|f_6(x) - Q(x) - f_6(0) - A(x) + A_2(x)\| \le \tfrac{28}{3}\varepsilon$$

for all $x \in V \setminus \{0\}$. *Moreover, the function* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f_k(2^n x) + f_k(-2^n x)}{2 \cdot 4^n}$$

for $i = 1, 2, 3, 4, 5, 6$ *and the functions* A, A_1, A_2 *are given by*

$$A(x) = \lim_{n\to\infty} \tfrac{f_1(2^n x)+f_4(2^n x)-f_1(-2^n x)-f_4(-2^n x)}{2^{n+2}},$$
$$A_1(x) = \lim_{n\to\infty} \tfrac{f_2(2^n x)-f_5(2^n x)-f_2(-2^n x)+f_5(-2^n x)}{2^{n+2}},$$
$$A_2(x) = \lim_{n\to\infty} \tfrac{f_3(2^n x)-f_6(2^n x)-f_3(-2^n x)+f_6(-2^n x)}{2^{n+2}}$$

for all $x \in V$.

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