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# A FUNCTIONAL EQUATION ON HYPERPLANES PASSING THROUGH THE ORIGIN

Jae-Hyeong Bae* and Won-Gil Park**

Abstract. In this paper, we obtain the general solution and the stability of the multi-dimensional Cauchy's functional equation

$$f(x_1+y_1,\cdots,x_n+y_n)=f(x_1,\cdots,x_n)+f(y_1,\cdots,y_n).$$

The function $f$ given by $f(x_1,\cdots,x_n)=a_1x_1+\cdots+a_nx_n$ is a solution of the above functional equation.

## 1. Introduction

Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be real numbers not all equal to 0. Then the set consisting of all vectors $(x_1,\cdots,x_n)$ in $\mathbb{R}^n$ such that

$$\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n=c \tag{1.1}$$

for $c$ a constant is a subspace of $\mathbb{R}^n$ called a hyperplane. Thus, for example, lines are hyperplanes in $\mathbb{R}^2$ and planes are hyperplanes in $\mathbb{R}^3$. If $c=0$ in (1.1), then the equation simplifies to

$$\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n=0$$

and we say that the hyperplane **passes through the origin** or the **linear** hyperplane.

More generally, a linear hyperplane is any codimension-1 vector subspace of a vector space. Equivalently, a linear hyperplane $V$ in a vector space $W$ is any subspace such that $W/V$ is one-dimensional. Equivalently, a linear hyperplane is the linear transformation kernel of any nonzero linear map from the vector space to the underlying field.

In this paper, let $X$ and $Y$ be real vector spaces. For a mapping $f: X^n\to Y$, consider the multi-dimensional Cauchy's functional equation:

$$f(x_1+y_1,\cdots,x_n+y_n)=f(x_1,\cdots,x_n)+f(y_1,\cdots,y_n) \tag{1.2}$$

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When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \cdots, x_n) = a_1x_1 + \cdots + a_nx_n$$

is a solution of (1.2).

For a mapping $g : X \to Y$, consider the Cauchy's functional equation:

$$g(x + y) = g(x) + g(y). \tag{1.3}$$

Recently, the authors investigeted various functional equations ([2], [3], [4]). In this paper, we investigate the relation between (1.2) and (1.3). And we find out the general solution and prove the generalized Hyers-Ulam stability of (1.2).

## 2. Results

The multi-dimensional vector-variable Cauchy's functional equation (1.2) induces the Cauchy's functional equation (1.3) as follows.

Theorem 2.1. Let $f : X^n \to Y$ be a mapping satisfying (1.2) and let $g : X \to Y$ be the mapping given by

$$g(x) := f(x, \cdots, x) \tag{2.1}$$

for all $x \in X$, then $g$ satisfies (1.3).

*Proof.* By (1.2) and (2.1),

$$\begin{aligned} g(x + y) &= 2f(x + y, \cdots, x + y) \\ &= f(x, \cdots, x) + f(y, \cdots, y) \\ &= g(x) + g(y) \end{aligned}$$

for all $x, y \in X$. □

The Cauchy's functional equation (1.3) induces the multi-dimensional Cauchy's functional equation (1.2) with an additional condition.

Theorem 2.2. Let $a_1, \cdots, a_n \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying (1.3). If $f : X^n \to Y$ is the mapping given by

$$f(x_1, \cdots, x_n) := a_1g(x_1) + \cdots + a_ng(x_n) \tag{2.2}$$

for all $x_1, \cdots, x_n \in X$, then $f$ satisfies (1.2). Furthermore, (2.1) holds if

$$a_1 + \cdots + a_n = 1.$$

*Proof.* By (1.3) and (2.2),

$$\begin{aligned} f(x_1+y_1,\cdots,x_n+y_n) &\\ &= a_1g(x_1+y_1)+\cdots+a_ng(x_n+y_n)\\ &= a_1[g(x_1)+g(y_1)]+\cdots+a_n[g(x_n)+g(y_n)]\\ &= [a_1g(x_1)+\cdots+a_ng(x_n)]+[a_1g(y_1)+\cdots+a_ng(y_n)]\\ &= f(x_1,\cdots,x_n)+f(y_1,\cdots,y_n)\end{aligned}$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_n \in X$. Furthermore, if $a_1+\cdots+a_n=1$,

$$f(x,\cdots,x)=a_1g(x)+\cdots+a_ng(x)=g(x)$$

for all $x \in X$. □

EXAMPLE 2.1. *Let $X$ be the space of $m \times m$ real matrices. Consider the mapping $g : X \to \mathbb{R}$ given by $g(A) := tr(A)$ =the trace of $A$ for all $A \in X$. For $a_1,\cdots,a_n \in \mathbb{R}$, consider the function $f : X^n \to \mathbb{R}$ given by*

$$f(A_1,\cdots,A_n)=\sum_{i=1}^{n} a_i\ tr(A_i)$$

*for all $A_1,\cdots,A_n \in X$. By Theorem 2.2, the function $f$ satisfies* (1.2).

In the following theorem, we find out the general solution of the multi-dimensional Cauchy's functional equation (1.2).

THEOREM 2.3. *A mapping $f : X^n \to Y$ satisfies* (1.2) *if and only if there exist additive mappings $A_1,\cdots,A_n : X \to Y$ such that*

$$f(x_1,\cdots,x_n)=A_1(x_1)+\cdots+A_n(x_n)$$

*for all $x_1,\cdots,x_n \in X$.*

*Proof.* We first assume that $f$ is a solution of (1.2). Define $A_1,\cdots,A_n : X \to Y$ by

$$A_1(x):=f(x,0,\cdots,0),\cdots,A_n(x):=f(0,\cdots,0,x)$$

for all $x_1,\cdots,x_n \in X$. One can easily verify that $A_1,\cdots,A_n$ are additive mappings.

Letting $y_2=-x_2,\cdots,y_n=-x_n$ and $y_1=x_1$ in (1.2),

$$2f(x_1,0,\cdots,0)=f(x_1,\cdots,x_n)+f(x_1,-x_2,\cdots,-x_n)$$

for all $x_1, \cdots, x_n \in X$. Putting $y_2 = x_2, \cdots, y_n = x_n$ and $y_1 = -x_1$ in (1.2),

$$f(0, 2x_2, \cdots, 2x_n) = f(x_1, \cdots, x_n) + f(-x_1, x_2, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. Setting $y_1 = -x_1, \cdots, y_n = -x_n$ in (1.2),

$$f(-x_1, \cdots, -x_n) = -f(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. By the above three equalities,

$$f(x_1, \cdots, x_n) = f(x_1, 0, \cdots, 0) + \frac{1}{2} f(0, 2x_2, \cdots, 2x_n) \tag{2.3}$$

for all $x_1, \cdots, x_n \in X$. Letting $y_2 = x_2$, $y_3 = -x_3, \cdots, y_n = -x_n$ and $x_1 = y_1 = 0$ in (1.2),

$$2f(0, x_2, 0, \cdots, 0) = f(0, x_2, x_3, \cdots, x_n) + f(0, x_2, -x_3, \cdots, -x_n)$$

for all $x_2, \cdots, x_n \in X$. Putting $y_2 = -x_2$, $y_3 = x_3, \cdots, y_n = x_n$ and $x_1 = y_1 = 0$ in (1.2),

$$f(0, 0, 2x_3, \cdots, 2x_n) = f(0, x_2, x_3, \cdots, x_n) + f(0, -x_2, x_3, \cdots, x_n)$$

for all $x_2, \cdots, x_n \in X$. By the above two equalities,

$$f(0, x_2, \cdots, x_n) = f(0, x_2, 0, \cdots, 0) + \frac{1}{2} f(0, 0, 2x_3, \cdots, 2x_n) \tag{2.4}$$

for all $x_2, \cdots, x_n \in X$. By (2.3) and (2.4),

$$\begin{aligned} f(x_1, \cdots, x_n) &= f(x_1, 0, \cdots, 0) + f(0, x_2, 0, \cdots, 0) \\ &\quad + \frac{1}{2^2} f(0, 0, 2^2 x_3, \cdots, 2^2 x_n), \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Continuing this process, one can obtain that

$$\begin{aligned} f(x_1, \cdots, x_n) &= f(x_1, 0, \cdots, 0) + \cdots + \frac{1}{2^{n-1}} f(0, \cdots, 0, 2^{n-1} x_n) \\ &= f(x_1, 0, \cdots, 0) + \cdots + f(0, \cdots, 0, x_n) \\ &= A_1(x_1) + \cdots + A_n(x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

Conversely, we assume that there exist additive mappings $A_1, \cdots, A_n : X^n \to Y$ such that

$$f(x_1, \cdots, x_n) = A_1(x_1) + \cdots + A_n(x_n)$$

for all $x_1, \cdots, x_n \in X$. Since $A_1, \cdots, A_n$ are additive,

$$\begin{aligned} &f(x_1 + y_1, \cdots, x_n + y_n) \\ &\quad = 2A_1(x_1 + y_1) + \cdots + 2A_n(x_n + y_n) \\ &\quad = A_1(x_1) + A_1(y_1) + \cdots + A_n(x_n) + A_n(y_n)) \\ &\quad = f(x_1, \cdots, x_n) + f(y_1, \cdots, y_n) \end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$. □

Let $Y$ be complete and let $\varphi : X^{2n} \to [0, \infty)$ be a function satisfying

$$\begin{aligned} \tilde{\varphi}&(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ &:= \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \varphi(2^j x_1, \cdots, 2^j x_n, -2^j x_1, \cdots, -2^j x_n) < \infty \end{aligned} \tag{2.5}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$.

Theorem 2.4. Let $f : X^n \to Y$ be a mapping such that

$$\begin{aligned} &\|f(x_1 + y_2, \cdots, x_n + y_n) - f(x_1, \cdots, x_n) - f(y_1, \cdots, y_n)\| \\ &\quad \le \varphi(x_1, \cdots, x_n, y_1, \cdots, y_n) \end{aligned} \tag{2.6}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$. Then there exists a unique multi-dimensional Cauchy's mapping $F : X^n \to Y$ such that

$$\begin{aligned} &\|f(x_1, \cdots, x_n) - F(x_1, \cdots, x_n)\| \\ &\quad \le \tilde{\varphi}(x_1, \cdots, x_n, x_1, \cdots, x_n) + \tilde{\varphi}(-x_1, \cdots, -x_n, -2x_1, \cdots, -2x_n) \end{aligned} \tag{2.7}$$

for all $x_1, \cdots, x_n \in X$.

*Proof.* Letting $y_1 = -x_1, \cdots, y_n = -x_n$ in (2.6), we have

$$\|f(x_1, \cdots, x_n) + f(-x_1, \cdots, -x_n)\| \le \varphi(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$$

for all $x_1, \cdots, x_n \in X$. Replacing $x_1, \cdots, x_n, y_1, \cdots, y_n$ by $-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n$ respectively in (2.6), we have

$$\begin{aligned} &\|f(x_1, \cdots, x_n) - f(-x_1, \cdots, -x_n) - f(2x_1, \cdots, 2x_n)\| \\ &\quad \le \varphi(-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. By the above two inequalities,

$$\begin{aligned} &\|f(x_1, \cdots, x_n) - \frac{1}{2} f(2x_1, \cdots, 2x_n)\| \\ &\le \frac{1}{2} \big[\varphi(x_1, \cdots, x_n, -x_1, \cdots, -x_n) + \varphi(-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n)\big] \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Thus we obtain

$$\begin{aligned}\left\|\frac{1}{2^j}f(2^jx_1,\cdots,2^jx_n)-\frac{1}{2^{j+1}}f(2^{j+1}x_1,\cdots,2^{j+1}x_n)\right\|\\ \le \frac{1}{2^{j+1}}[\varphi(2^jx_1,\cdots,2^jx_n,-2^jx_1,\cdots,-2^jx_n)\\ +\varphi(-2^jx_1,\cdots,-2^jx_n,2^{j+1}x_1,\cdots,2^{j+1}x_n)]\end{aligned}$$

for all $x_1, \cdots, x_n \in X$ and all $j$. For given integers $l, m (0 \le l < m)$, we get

$$\begin{aligned}&\left\|\frac{1}{2^l}f(2^lx_1,\cdots,2^lx_n)-\frac{1}{2^m}f(2^mx_1,\cdots,2^mx_n)\right\|\\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}[\varphi(2^jx_1,\cdots,2^jx_n,-2^jx_1,\cdots,-2^jx_n)\\ &\qquad +\varphi(-2^jx_1,\cdots,-2^jx_n,2^{j+1}x_1,\cdots,2^{j+1}x_n)]\end{aligned}\tag{2.8}$$

for all $x_1, \cdots, x_n \in X$. By (2.8), the sequence $\{\frac{1}{2^j}f(2^jx_1,\cdots,2^jx_n)\}$ is a Cauchy sequence for all $x_1, \cdots, x_n \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^j}f(2^jx_1,\cdots,2^jx_n)\}$ converges for all $x_1, \cdots, x_n \in X$. Define $F : X^n \to Y$ by

$$F(x_1,\cdots,x_n) := \lim_{j\to\infty}\frac{1}{2^j}f(2^jx_1,\cdots,2^jx_n)$$

for all $x_1, \cdots, x_n \in X$. By (2.6), we have

$$\begin{aligned}&\|f(2^j(x_1+y_1),\cdots,2^j(x_n+y_n))-f(2^jx_1,\cdots,2^jx_n)\\ &\quad -f(2^jy_1,\cdots,2^jy_n)\| \le \varphi(2^jx_1,\cdots,2^jx_n,2^jy_1,\cdots,2^jy_n)\end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$ and all $j$. Letting $j \to \infty$ and using (2.5), we see that $F$ satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (2.8), one can obtain the inequality (2.7). If $G : X^n \to Y$ is another multi-dimensional additive mapping satisfying (2.7), we obtain

$$\begin{aligned}
&\|F(x_1, \cdots, x_n) - G(x_1, \cdots, x_n)\| \\
&\quad = \frac{1}{2^n}\|F(2^n x_1, \cdots, 2^n x_n) - G(2^n x_1, \cdots, 2^n x_n)\| \\
&\quad \le \frac{1}{2^n}\|F(2^n x_1, \cdots, 2^n x_n) - f(2^n x_1, \cdots, 23^n x_n)\| \\
&\qquad + \frac{1}{2^n}\|f(2^n x_1, \cdots, 2^n x_n) - G(2^n x_1, \cdots, 2^n x_n)\| \\
&\quad \le \frac{2}{2^n}[\tilde{\varphi}(2^n x_1, \cdots, 2^n x_n, 2^n x_1, \cdots, 2^n x_n) \\
&\qquad + \tilde{\varphi}(-2^n x_1, \cdots, -2^n x_n, -2^{n+1} x_1, \cdots, -2^{n+1} x_n)] \\
&\quad \to 0 \;\text{ as }\; n \to \infty
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Hence the mapping $F$ is the unique multi-dimensional additive mapping, as desired. □

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Department of Mathematics and Applied Mathematics
Kyung Hee University
Yongin 449-701, Republic of Korea
*E-mail*: jhbae@khu.ac.kr

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National Institute for Mathematical Sciences
385-16 Doryong-dong, Yuseong-gu
Daejeon 305-340, Republic of Korea
*E-mail*: wgpark@nims.re.kr