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SOME THETA FUNCTION AND FALSE THETA FUNCTION IDENTITIES ASSOCIATED WITH RAMANUJAN

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Abstract. Ramanujan gave six identities involving θ-function expansion. We extended these identities by introducing a free parameter. We also give the interpretation of one of the identities using theory of partitions.

1. Introduction

In a series of papers, Andrews [1,2,3] discussed, in detail, certain group of identities, which he found in Ramanujan's papers while visiting Cambridge University. He termed these papers as Ramanujan's "Lost" Notebook. In the second paper of the series [2], he considered identities of Ramanujan relating to θ-functions and partial theta functions. For proving these identities he proved two lemmas and suggested that they may be placed in the standard hierarchy of basic hypergeometric series identities and the possible extension of (1.9), Andrews [2, p. 174]. We, Srivastava [5], subsequently placed his lemmas in the standard hierarchy of basic hypergeometric series and extended his lemmas and consequently the results in (1.9), as suggested by Andrews.

In the same paper, Andrews has proved six more identities and called them corollaries of the θ-expansions, Andrews [2, $(3.1)_R$ -$(3.6)_R$, pp. 178-180] and [(5.2)-(5.5), p.183]. In this paper we extend these identities of Ramanujan by introducing a free parameter. It is interesting to note that in these extended identities the term q^{n^2} in Ramanujan's identities comes out $q^{n^2+kn}, (k = \pm 1, \pm 2)$. We have given the interpretation of one of the identities (5.3) in the theory of partitions.

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Rogers introduced false θ-functions. He gave certain identities involving these false θ-functions. Andrews have obtained partition theorems from these identities. Ramanujan stated in the "Lost" Notebook certain identities for these false θ-functions. We have given more general identities involving these false θ-functions.

2. Notations

The following q-notation have been used :

For $\left|q^k\right| < 1$,

$$(a;q^k)_n = (1-a)(1-aq^k)......(1-aq^{k(n-1)}), \qquad n \geq 1$$
$$(a;q^k)_0 = 1,$$
$$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj}).$$

3. Corollaries of the θ-Expansions

We give below the identities given by Ramanujan and the identities given by us. We denote the identities of Ramanujan by a suffix R

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}} = \frac{\sum_{n=-\infty}^{\infty} q^{3n^2-n}}{\sum_{n=0}^{\infty}(-q)^{(n^2+n)/2}}; \qquad (3.1)_R$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2}{(q^2;q^2)_{2n}} = \frac{\sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2}}{\sum_{n=0}^{\infty} q^{(n^2+n)/2}}; \qquad (3.2)_R$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-n}}{\prod_{n=1}^{\infty}(1-q^n)} \prod_{n=1}^{\infty}(1-q^{18n-9}); \qquad (3.3)_R$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}}{\prod_{n=1}^{\infty}(1-q^{4n})}; \qquad (3.4)_R$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2+2n}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}} \prod_{n=0}^{\infty}(1+q^{12n+6}); \qquad (3.5)_R$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q^3;q^6)_n}{(q;q^2)_n^2(q^4;q^4)_n} = \frac{\sum_{n=-\infty}^{\infty} q^{4n^2+2n}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}} \prod_{n=0}^{\infty}(1-q^{12n+6}). \qquad (3.6)_R$$

4. Analogous identities

We shall prove the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-q;q^2)_n}{(q;q)_{2n}} = \frac{\sum_{n=-\infty}^{\infty} q^{n^2+n}}{\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}}; \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}(q;q^2)_n^2}{(q^2;q^2)_{2n}} = \frac{\sum_{n=-\infty}^{\infty} q^{3n(n+1)/2}}{\sum_{n=0}^{\infty} q^{n(n+1)/2}}; \tag{4.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} = q^{-3}\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2} e^{2ni\pi/3}}{(q^2;q^2)_\infty}$$

$$= q^{-3}\frac{(q^9;q^{18})_\infty \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}}{(q;q)_\infty}; \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{1}{q}\frac{\sum_{n=-\infty}^{\infty}(-1)^{n+1} q^{(3n^2-n)/2}}{(q^4;q^4)_\infty}; \tag{4.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n}}$$
$$= \frac{(-q^{12};q^{12})_\infty \sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2} + (-q^6;q^{12})_\infty \sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2+2n-1}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}}; \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-2n}(q^3;q^6)_n}{(q;q^2)_n^2(q^4;q^4)_n} = \frac{3(q^{12};q^{12})_\infty + \frac{1}{q}\sum_{n=-\infty}^{\infty} q^{2n^2+n}(q^6;q^{12})_\infty}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}}. \tag{4.6}$$

5. Proof of these identities

For proving these identities we shall mainly using our following result, Srivastava [4,(2.2), p.121] :

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-q/a;q^2)_n(b;q^2)_n(c;q^2)_n(dq^2;q^2)_n}{(d;q^2)_n(q^2;q^2)_{2n}} \left(\frac{1}{bc}\right)^n$$

$$= \prod\begin{bmatrix} q^2/b, q^2/c \\ q^2, 1/bc \end{bmatrix}; q^2\Bigg] \times [\{2 - \frac{1}{b} - \frac{1}{c} + \frac{d(1-b)(1-c)}{bc(1-d)}\}$$

$$+ \sum_{N=1}^{\infty} \frac{(a^N+a^{-N})q^{N^2}(b;q^2)_N(c;q^2)_N(dq^2;q^2)_N}{(d;q^2)_N(q^2/b;q^2)_N(q^2/c;q^2)_N}\left(\frac{1}{bc}\right)^N$$

$$\times\{1-(\tfrac{1}{b}+\tfrac{1}{c})q^{2N}+q^{4N}+dq^{2N}\frac{(1-bq^{2N})(1-cq^{2N})}{bc(1-dq^{2N})}\}]. \tag{5.1}$$

Putting $d = 0$ in the above, we get

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-q/a;q^2)_n(b;q^2)_n(c;q^2)_n}{(q^2;q^2)_{2n}} \left(\frac{1}{bc}\right)^n$$

$$= \prod \begin{bmatrix} q^2/b, q^2/c \\ q^2, 1/bc \end{bmatrix}; q^2 \times [\{2 - \tfrac{1}{b} - \tfrac{1}{c}\}$$

$$+ \sum_{n=1}^{\infty} \frac{(a^N + a^{-N}) q^{N^2} (b;q^2)_n (c;q^2)_n}{(q^2/b;q^2)_N (q^2/c;q^2)_N} \left(\frac{1}{bc}\right)^N$$

$$\times \{1 - \left(\tfrac{1}{b} + \tfrac{1}{c}\right) q^{2N} + q^{4N}\}]. \quad (5.2)$$

Proof of (4.1)
Taking $a = -e^{2iz}$ and letting $b \longrightarrow \infty$, $c \longrightarrow \infty$ in (5.2), we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}}{(q^2;q^2)_{2n}} \theta_{4;n}(z;q)$$
$$= 2 + 2\sum_{N=1}^{\infty} (-1)^N q^{3N^2-2N} (1 + q^{4N}) \cos 2Nz$$
$$= 2\sum_{N=-\infty}^{\infty} (-1)^N q^{3N^2-2N} \cos 2Nz. \quad (5.3)$$

Taking $z = \pi/4$,and writing $q^{1/2}$ for q in (5.3), we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-q;q^2)_n}{(q;q)_{2n}}$$
$$= \frac{2}{(q;q)_\infty} \sum_{N=-\infty}^{\infty} (-1)^N q^{(3N^2+2N)/2} \cos(N\pi/2),$$

when $N = 2m+1$ the summation on the right side vanishes. Hence when $N = 2m$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-q;q^2)_n}{(q;q)_{2n}} = \frac{2}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{6m^2+2m}$$

$$= \frac{2}{(q;q)_\infty} (q^4;q^{12})_\infty (q^8;q^{12})_\infty (q^{12};q^{12})_\infty$$

$$= \frac{2}{(q;q)_\infty} (q^4;q^4)_\infty$$
$$= 2\frac{(-q;q^2)_\infty (-q^2;q^2)_\infty (q^4;q^4)_\infty}{(q;q)_\infty (-q;q)_\infty}$$

$$= 2\frac{(-q^2;q^2)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty / (-q;q^2)_\infty}$$
$$= \frac{\sum_{n=-\infty}^{\infty} q^{n^2+n}}{\sum_{n=0}^{\infty} (-q)^{n(n+1)/2}}.$$

Proof of (4.2)
Taking $z = 0$ in (5.3), we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}(q;q^2)_n^2}{(q^2;q^2)_{2n}} = \frac{2}{(q^2;q^2)_\infty} \sum_{N=-\infty}^{\infty} (-1)^N q^{3N^2+2N}$$

$$= 2\frac{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$= 2\frac{(q^6;q^6)_\infty/(q^3;q^6)_\infty}{(q^2;q^2)_\infty/(q;q^2)_\infty}$$

$$= \frac{\sum_{n=-\infty}^{\infty} q^{3n(n+1)/2}}{\sum_{n=0}^{\infty} q^{n(n+1)/2}}.$$

Proof of (4.3)

Taking $q = e^{\pi i\tau}$, $\operatorname{Im}\tau > 0$ and making use of (17) and $\theta_4(-z+\pi\tau) = -q^{-1}e^{2iz}\ \theta_4(z)$, Rainville [4, pp. 318-319] in (5.3), we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}}{(q^2;q^2)_{2n}}\theta_{4;n}(z;q) = -2q^{-3}\cos 2z\ \theta_4(z;q^3)$$

Taking $z = \pi/3$, we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-2n}(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} = q^{-3}\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}e^{2ni\pi/3}}{(q^2;q^2)_\infty}$$

$$= q^{-3}\frac{(q^9;q^{18})_\infty\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}}{(q;q)_\infty},$$ by Andrews [2, p. 179].

(4.3) can be written as

$$1 + \sum_{n=1}^{\infty} \frac{q^{2n^2-2n}(1+q+q^2)(1+q^3+q^6)......(1+q^{2n-1}+q^{4n-2})}{(1-q^2)(1-q^4)......(1-q^{4n})}$$

$$= q^{-3}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}\prod_{m=1}^{\infty}\frac{(1-q^{18m-9})}{(1-q^m)}.$$

Proof of (4.4)

Putting $a = -e^{2iz}, b = q,\ c \longrightarrow \infty$ and $z = \pi/2$ in (5.2), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-2n}(-q;q^2)_n}{(q^4;q^4)_n}$$
$$= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left[(2-\tfrac{1}{q}) + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2-2n}(1-q^{2n-1}+q^{4n})\right]$$

$$= \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+2n} - \frac{1}{q}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right]$$

$$= -\frac{1}{q}\frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q^2;q^2)_\infty(-q^2;q^2)_\infty}$$

$$= -\frac{1}{q}\frac{(q;q)_\infty}{(q^4;q^4)_\infty},$$ the first term on the right side vanishes,

$$= \frac{1}{q}\frac{\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{(3n^2-n)/2}}{(q^4;q^4)_\infty},$$

by using Jacobi's triple product identity and observing that

$$\prod_{n=0}^{\infty}(1-q^{3n+1})(1-q^{3n+2})(1-q^{3n+3}) = \prod_{n=1}^{\infty}(1-q^n).$$

Proof of (4.5)

Putting $a = e^{2iz}, b = -q,\ c \longrightarrow \infty$ and $z = \pi/3$ in (5.2), we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2-2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n}}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[(2+\tfrac{1}{q}) + 2\sum_{n=1}^{\infty}q^{2n^2-2n}(1+q^{2n-1}+q^{4n})\cos\tfrac{2n\pi}{3}\right]$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[2\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\cos\tfrac{2n\pi}{3} + \tfrac{1}{q}\sum_{n=-\infty}^{\infty}q^{2n^2}\cos\tfrac{2n\pi}{3}\right].$$

Now

$$2\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\cos\tfrac{2n\pi}{3}$$
$$= \sum_{n=-\infty}^{\infty}q^{2n^2+2n}(e^{2in\pi/3}+e^{-2in\pi/3})$$
$$= \sum_{n=-\infty}^{\infty}q^{2n^2}(q^2e^{2i\pi/3})^n + \sum_{n=-\infty}^{\infty}q^{2n^2}(q^2e^{-2i\pi/3})^n$$
$$= (q^4;q^4)_\infty(-q^4e^{2i\pi/3};q^4)_\infty(-e^{-2i\pi/3};q^4)_\infty$$
$$+(q^4;q^4)_\infty(-q^4e^{-2i\pi/3};q^4)_\infty(-e^{2i\pi/3};q^4)_\infty$$
$$= (q^4;q^4)_\infty(-q^4e^{2i\pi/3};q^4)_\infty(-q^4e^{-2i\pi/3};q^4)_\infty(2+2\cos\tfrac{2\pi}{3})$$
$$= \frac{(q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(-q^4;q^4)_\infty}.$$

Hence

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}2\sum_{n=-\infty}^{\infty}(-1)^nq^{2n^2+2n}\cos\tfrac{2n\pi}{3}$$
$$= \frac{(-q;q^2)_\infty(q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q^2;q^2)_\infty(-q^4;q^4)_\infty}$$
$$= \frac{(q^2;q^4)_\infty(q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q;q)_\infty(-q^4;q^4)_\infty}$$
$$= \frac{(q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}$$
$$= (q^{12};q^{12})_\infty\frac{\sum_{n=-\infty}^{\infty}(-1)^nq^{4n^2}}{\sum_{n=0}^{\infty}(-1)^nq^{2n^2+n}}.$$

And

$$\sum_{n=-\infty}^{\infty} q^{2n^2} \cos \tfrac{2n\pi}{3} = \sum_{n=-\infty}^{\infty} q^{2n^2} (e^{2in\pi/3} + e^{-2in\pi/3})$$
$$= (q^4;q^4)_\infty (-q^2 e^{2i\pi/3};q^4)_\infty (-q^2 e^{-2i\pi/3};q^4)_\infty$$
$$= \tfrac{(q^4;q^4)_\infty (-q^6;q^{12})_\infty}{(-q^2;q^4)_\infty}.$$

Hence

$$\tfrac{(-q;q^2)_\infty}{q(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2} \cos \tfrac{2n\pi}{3} = \tfrac{(q^4;q^4)_\infty (-q^6;q^{12})_\infty}{q(q;q^4)_\infty (-q^2;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty}$$
$$= \tfrac{1}{q} \tfrac{\sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2+2n}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}} (-q^6;q^{12})_\infty.$$

Therefore

$$\sum_{n=0}^{\infty} \tfrac{q^{n^2-2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n}}$$
$$= \tfrac{(-q^{12};q^{12})_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2} + (-q^6;q^{12})_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2+2n-1}}{\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}}.$$

Proof of (4.6)
Putting $a = -e^{2iz}, b = -q,\ c \longrightarrow \infty$ and $z = 5\pi/3$ in (5.2), we have

$$\sum_{n=0}^{\infty} \tfrac{q^{n^2-2n}(q^3;q^6)_n}{(q;q^2)_n^2 (q^4;q^4)_n}$$
$$= \tfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \left[(2 + \tfrac{1}{q}) + 2 \sum_{n=1}^{\infty} (-1)^n q^{2n^2-2n} (1 + q^{2n-1} + q^{4n}) \cos \tfrac{2n\pi}{3} \right]$$
$$= \tfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} [2 \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+2n} \cos \tfrac{2n\pi}{3}$$
$$+ \tfrac{1}{q} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} \cos \tfrac{2n\pi}{3}]$$

Now

$$2 \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+2n} \cos \tfrac{2n\pi}{3}$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+2n} (e^{2in\pi/3} + e^{-2in\pi/3})$$
$$= (2 - 2\cos \tfrac{2n\pi}{3}) (q^4;q^4)_\infty (q^4 e^{2i\pi/3};q^4)_\infty (q^4 e^{-2i\pi/3};q^4)_\infty$$
$$= 3 \tfrac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q^4;q^4)_\infty}.$$

Hence

$$2\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+2n}\cos\frac{2n\pi}{3} = 3\frac{(q^{12};q^{12})_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty}&(-1)^n q^{2n^2}\cos\frac{2n\pi}{3}\\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}(e^{2in\pi/3}+e^{-2in\pi/3})\\ &= (q^4;q^4)_\infty(q^2e^{2i\pi/3};q^4)_\infty(q^2e^{-2i\pi/3};q^4)_\infty\\ &= \frac{(q^4;q^4)_\infty(q^6;q^{12})_\infty}{(q^2;q^4)_\infty}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{1}{q}\sum_{n=-\infty}^{\infty}q^{2n^2}\cos\frac{2n\pi}{3} &= \frac{1}{q}\frac{(q^4;q^4)_\infty(q^6;q^{12})_\infty}{(q;q^4)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}\\ &= \frac{1}{q}\frac{\sum_{n=-\infty}^{\infty}(q^2)^{n(n+\frac{1}{2})}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}}(q^6;q^{12})_\infty\ ,\end{aligned}$$

so

$$\sum_{n=0}^{\infty}\frac{q^{n^2-2n}(q^3;q^6)_n}{(q;q^2)_n^2(q^4;q^4)_n} = \frac{3(q^{12};q^{12})_\infty+\frac{1}{q}\sum_{n=-\infty}^{\infty}q^{2n^2+n}(q^6;q^{12})_\infty}{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}}.$$

6. Implications in theory of partition

(4.3) can be written as

$$1+\sum_{n=1}^{\infty}\frac{q^{2n^2-2n}(1+q+q^2)(1+q^3+q^6)\ldots\ldots(1+q^{2n-1}+q^{4n-2})}{(1-q^2)(1-q^4)\ldots\ldots(1-q^{4n})}$$

$$= q^{-3}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}\prod_{m=1}^{\infty}\frac{(1-q^{18m-9})}{(1-q^m)}. \qquad (6.1)$$

Now $\frac{q^{2n^2-2n}}{(q^2;q^2)_n}$ is the generating function for partitions $b'_1+b'_2+\ldots.+b'_s$ with either $2n$ or $2n-1$ parts all even and $b'_1 > b'_2 \geq b'_3 > b'_4 \geq \ldots..$ And

$$\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})$$

is the generating function for partitions into odd parts each $\leq 2n-1$ and each appearing at most twice. Hence the identity (6.1) implies Theorem2 of Andrews [2, p. 175].

7. Identities for false theta functions

We can write Theorem 2 of Srivastava [4, p. 123] as

$$\sum_{n=0}^{\infty} \frac{(-\alpha;q^2)_{n+1}(-q^2/\alpha;q^2)_n(b;q^2)_n(c;q^2)_n(dq^2;q^2)_n}{(d;q^2)_n(q^2;q^2)_{2n+1}} \left(\frac{q^2}{bc}\right)^n$$

$$= \frac{(q^4/b;q^2)_\infty(q^4/c;q^2)_\infty}{(q^2/bc;q^2)_\infty(q^2;q^2)_\infty} \sum_{N=0}^{\infty} \frac{(\alpha^{1+N}+\alpha^{-N})q^{N^2+N}(b;q^2)_N(c;q^2)_N(dq^2;q^2)_N}{(d;q^2)_N(q^4/b;q^2)_N(q^4/c;q^2)_N} \left(\frac{q^2}{bc}\right)^N$$

$$\times \left[1 - \frac{q^{2N+2}}{b} - \frac{q^{2N+2}}{c} + q^{4N+2} + dq^{2N+2}\frac{(1-bq^{2N})(1-cq^{2N})}{bc(1-dq^{2N})}\right] \quad (7.1)$$

Case 1

Taking $d = 0, \alpha = e^{2iz}, b = q^2$ and letting $c \longrightarrow \infty$ in (7.1), we have after a little simplification

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n}}{(q^{2n+2};q^2)_{n+1}} \theta_{2,n}(z,q^2)$$
$$= \frac{(q^2;q^2)_\infty}{\cos z}[\cos z - 2\sin z \sum_{N=1}^{\infty}(-1)^N q^{2N^2} \sin 2Nz$$
$$-\tfrac{1}{2}q^{-\frac{1}{4}}\theta_{2,f}(z,q)], \quad (7.2)$$

where $\theta_{2,f}$ is the false theta function.

Case 2

Taking $z = \pi$ in (7.2), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n}}{(q^{2n+2};q^2)_{n+1}} \theta_{2,n}(\pi,q) = \tfrac{1}{2}q^{-\frac{1}{4}}(q^2;q^2)_\infty\, \theta_{2,f}(\pi,q). \quad (7.3)$$

Case 3

Writing q for q^2 and putting $a = e^{2iz}, b = q, z = \pi/6$ and letting $c \longrightarrow \infty$ in (7.1), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2-n)/2}(q^3;q^3)_n}{(q;q)_{2n+1}}$$
$$= \sum_{N=0}^{\infty} q^{9N^2-3N} - \sum_{N=0}^{\infty} q^{9N^2+3N}$$
$$+ \sum_{N=0}^{\infty} q^{9N^2-6N+1} - \sum_{N=1}^{\infty} q^{9N^2-6N+1}. \quad (7.4)$$

In simplifying we have used

$$\frac{(-1)^n \cos(2n+1)\frac{5\pi}{6}}{\cos\frac{5\pi}{6}} \begin{array}{lll} = 1 & \text{if} & n\equiv 0 \pmod 3 \\ = 0 & \text{if} & n\equiv 1 \pmod 3 \\ = -1 & \text{if} & n\equiv 2 \pmod 3 \end{array}$$

and

$$\frac{(-1)^n(\cos(2n+1)\frac{5\pi}{6}-\cos(2n-1)\frac{5\pi}{6})}{\cos\frac{5\pi}{6}} \begin{array}{llll} = & 0 & \text{if} & n\equiv 0 (\mathrm{mod}\, 3) \\ = & 1 & \text{if} & n\equiv 1 (\mathrm{mod}\, 3) \\ = & -1 & \text{if} & n\equiv 2 (\mathrm{mod}\, 3). \end{array}$$

Case 4

We prove

$$(1+q^2)\sum_{n=0}^{\infty}\frac{(-1)^n q^{-2n}}{(q^2;q^2)_{n+1}}\theta_{2,n}(z,q^2)$$
$$= q^{\frac{9}{4}}(q^2;q^2)_\infty \sum_{N=0}^{\infty}(-1)^N q^{N^2-N}(1+q^{4N+2})\cos(2N+1)z. \quad (7.5)$$

The left side is divergent. We see that if we put $b=-c$ in (7.1), there is singularity in the left side at $c=q$ and the series on the right side is divergent. To overcome this we define the left side as equal to limit of $c \longrightarrow q^-$.

Proof Let $q \longrightarrow q^2$, then put $b=-c=q^2$, $a=e^{-2iz}$ and simplify, we have (7.5).

8. Conclusion

Andrews [2] asserted that Ramanujan's results suggest possible extension of further investigation. We have tried to justify Andrews assertion. We feel that our results are not special cases of any known results. Moreover, these identities are of interest in number theory.

References

[1] G. E. Andrews, *Ramanujan's "Lost" Notebook I: Partial θ - Function Expansions,* Adv. in Math. **41** (1981), 137-172.

[2] G. E. Andrews, *Ramanujan's "Lost" Notebook II: θ- Function Expansions,* Adv. in Math. **41** (1981), 173-185.

[3] G. E. Andrews, *Ramanujan's "Lost" Notebook III: The Rogers-Ramanujan Continued Fraction,* Adv. in Math. **41** (1981), 186-208.

[4] E. D. Rainville, *Special Function, Chelsea Publishing Company,* Bronx, New York, 1960b.

[5] Bhaskar Srivastava, *Partial Theta Function Expansions,* Tohoku Math Journal **42** (1990), 119-125.

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