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LIMIT SETS OF POINTS WHOSE STABLE SETS HAVE NONEMPTY INTERIOR

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Abstract. In this paper, we show that if a homeomorphism has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the limit sets of wandering points whose stable sets have nonempty interior consist of single periodic orbit.

1. Introduction and preliminaries

Throughout this paper, let X be a compact metric space with a metric function d and f be a homeomorphism of X. Our purpose here is to study dynamical properties of points whose limit sets consist of single periodic orbit, together with the related concepts of wanderingness and the pseudo-orbit-tracing-property. In [4], Ruess and Summers studied the motions whose limit sets consist of a single periodic motion. In [3], Ombach gave necessary and sufficient conditions that a limit set of a point consists of a single periodic orbit under the condition that f is expansive homomorphism with the pseudo-orbit-tracing-property. Also, author studied stable points whose limit sets consist of single periodic orbit [2].

In this work, we show that if a homeomorphism has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the limit set of wandering point whose stable set have nonempty interior consists of single periodic orbit.

For x in X, $O_f(x)$ and $O_f^+(x)$ denote the f-orbit and positive f-orbit of x, respectively. Let $\omega_f(x)$ and $\alpha_f(x)$ denote the positive limit set and negative limit set of x for f, respectively, and let $\Omega(f)$ be the nonwandering set of f. A sequence of points $\{x_i\}_{i\in[a,b]}$, $(a < b)$ is called a δ-*pseudo-orbit* of

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f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in [a, b-1]$. A finite pseudo-orbit $\{x_0, x_1, \ldots, x_n\}$ is called a pseudo-orbit from x_0 to x_n. A sequence of points $\{x_i\}_{i\in[a,b]}$ is called ε-*traced* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for $i \in [a, b]$. We say that f has the *pseudo-orbit-tracing-property* if, for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by some point $x \in X$.

A subset M of X is called f-*minimal* if f-orbit of every point in M is dense in M. f is called *topologically transitive* if there is a point z in X whose orbit is dense in X.

Let $B(x, \varepsilon)$ denote $\{y \in X : d(x, y) < \varepsilon\}$.

Before we prove the main theorem, we prepare several lemmas. The following results are well known.

PROPOSITION 1.1. *If f has the pseudo-orbit-tracing-property, then the following properties hold:*

(1) *f^k has the pseudo-orbit-tracing-property for every integer $k \neq 0$;*
(2) *f restricted to its nonwandering set has the pseudo-orbit-tracing-property;*
(3) *$\Omega(f) = \Omega(f|_{\Omega(f)})$;*
(4) *if Y is an open and closed f-invariant subset of X, then f restricted to Y has the pseudo-orbit-tracing-property.*

PROPOSITION 1.2 [1]. *If X is a nontrivial connected f-minimal set, then f cannot have the pseudo-orbit-tracing-property.*

2. Main results

In this section, we assume that f has the pseudo-orbit-tracing-property and its nonwandering set is locally connected. Now, we give a decomposition theorem for nonwandering sets.

LEMMA 2.1. *There exists a decomposition of $\Omega(f)$ satisfying the following:*

(1) *There is a decomposition of $\Omega(f)$ into disjoint closed sets; $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ such that each Ω_i is f-invariant and f restricted to*

each Ω_i is topologically transitive. (Such the subsets Ω_i are called basic sets.)

(2) *Again, there is a decomposition of each Ω_i into connected components of $\Omega(f)$; $\Omega_i = \Omega_1^i \cup \Omega_2^i \cup \cdots \cup \Omega_{n_i}^i$ such that these connected components are permuted by f and the map f^{n_i} restricted to each Ω_j^i is topologically transitive.*

(3) *Each basic set Ω_i is contained in $\Omega(f^{n_i})$.*

proof. The proofs of (a) and (b) are given in [2].

(3) Suppose that x is in Ω_i. Let U be an open neighborhood of x and $N > 0$ be an arbitrary integer. Then, by (3) of Proposition 1.1, there is y in $U \cap \Omega(f)$ and an integer n with $n > N$ such that $f^n(y)$ is in U. Since $\{\Omega_j^i\}$ are permuted by f, n must be multiple of n_i. This shows that x is in $\Omega(f^{n_i})$. □

Remark. In the above lemma, each basic set is a chain component of nonwandering set of f (see [2]).

We recall the definitions of stable and unstable sets. We define the *stable* and *unstable* set of x for f by

$$W^s(x,f) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}, \quad \text{and}$$
$$W^u(x,f) = \{y \in X : \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\},$$

respectively.

Here, we introduce our main result.

Theorem 2.2. *If x is an wandering point of f with int $W^s(x,f) \neq \emptyset$, then $\omega_f(x)$ consists of one periodic orbit.*

Proof. Let x be an wandering point and *int* $W^s(x,f) \neq \emptyset$. Since each basic set is a chain component of f, it is not difficult to show that each limit set is contained in only one basic set of nonwandering set of f. Let $\omega_f(x)$ is contained in Ω_i and $\{\Omega_j^i\}$, $(1 \leq j \leq n_i)$ is the set of connected components of $\Omega(f)$ contained in Ω_i. We now let $f^{n_i} = g$, for convenience.

Since $\omega_g(x) \subset \omega_f(x) \subset \Omega_i$, $\omega_g(x)$ intersects some connected component in Ω_i. Let

$$\Omega^i_{j_0} \cap \omega_g(x) \neq \emptyset \qquad \text{for some} \quad j_0.$$

Again, we now let $\Omega^i_{j_0} = K$, for convenience. In view of the part (2) of Lemma 2.1, $g(K) = K$ holds.

Here, we claim that K is a g-minimal set. To see this, assume, on the contrary, that K is not g-minimal. Then there is a proper subset M of K which is g-minimal. Take points $p,\ q,\ z$ such that

$$p \in \omega_g(x) \cap K, \quad q \in K \setminus M \qquad \text{and} \quad z \in M,$$

and let $d(q, M) = \varepsilon_0$. Since $W^s(x, f) \subset W^s(x, g)$ and so, *int* $W^s(x, g) \neq \emptyset$, we can take a point y and a positive number γ with $\gamma < \varepsilon_0/2$ such that

$$(1) \qquad y \in B(y, \gamma) \subset W^s(x, g).$$

Let ε be a positive number satisfying $\varepsilon < \min\{\varepsilon_0/4,\ \gamma\}$ and let $\delta = \delta(\varepsilon) < \varepsilon$ be a positive number with the property of the pseudo-orbit-tracing-property of g. By (1), there exists a positive integer N_1 such that

$$(2) \qquad d(g^i(x),\ g^i(y)) < \delta \qquad \text{for all} \quad i > N_1.$$

Also, since $p \in \omega_g(x)$, there is an integer N_2 with $N_2 > 2N_1$ such that

$$(3) \qquad d(g^{N_2}(x),\ p) < \delta.$$

By (2) and (3), the sequence of points defined by

$$(4) \quad \{a_i\}_{i=0}^{N_2} = \{y,\ g(y),\ g^2(y),\ \cdots,\ g^{N_1}(y),\ g^{N_1+1}(x)\ \cdots,\ g^{N_2-1}(x),\ p\}$$

is a δ-pseudo-orbit of g from y to p. By the part (3) of Lemma 2.1, we have $q \in \Omega(g)$ and so, we get a periodic δ-pseudo-orbit $\{q_0, q_1, \ldots, q_n\}$ of g from q to q. Also, since g restricted to K is topologically transitive there is a point in K whose g-orbit is dense in K. Therefore, we can find δ-pseudo-orbits of g from p to q and p to z. Let

$$\{b_0 = p, b_1, \cdots b_l = q\} \quad \text{and} \quad \{c_0 = p, c_1 \cdots, c_m = z\}$$

be δ-pseudo-orbits of g from p to q and p to g, respectively. Consider the following two sequences of points defined by

$$\{b_i\}_{i=0}^{\infty} = \{a_0, a_1, \cdots, a_{N_2}, b_1, \cdots, b_l, q_1, \cdots, q_n, q_1 \cdots q_n, q_1, \cdots\} \text{ and}$$

$$\{c_i\}_{i=0}^{\infty} = \{a_0, a_1, \cdots, a_{N_2}, c_1, \cdots, c_m, f(z), f^2(z), f^3(z), \cdots\}.$$

Then $\{b_i\}$ and $\{c_i\}$ are δ-pseudo-orbit of g. Hence, there are two points y_b and y_c, which are ε-tracing these two pseudo-orbit $\{b_i\}$ and $\{c_i\}$, respectively. In particular, we get

$$y_b, \quad y_c \in \ B(y, \varepsilon) \ \subset \ B(y, \gamma) \subset W^s(x, g).$$

Thus we have $d(f^n(y_b), f^n(y_c)) \longrightarrow 0$. So, there is an integer N_3 such that

$$d(f^i(y_b), \ f^i(y_c)) < \varepsilon \quad \text{for all} \quad i > N_3.$$

Let $N = \max\{N_2 + l, \ N_2 + m, \ N_3\}$. Then we can take an integer L with $L > N$ satisfying the following.

$$d(y_b, q) < \varepsilon, \quad d(y_b, y_c) < \varepsilon \quad \text{and} \quad d(y_c, f^{L-n-m-1}(z)) < \varepsilon.$$

Therefore, we have

$$d(q, \ f^{L-l-m-1}(z)) < d(q, y_b) + d(y_b, y_c) + d(y_c, f^{L-l-m-1}(z)) < 3\varepsilon.$$

Clearly, $d(q, M) \leq d(q, f^{L-l-m-1}(z))$. This shows that

$$d(q, M) < 3\varepsilon < \varepsilon_0$$

This contradicts the fact that $d(q, M) = \varepsilon_0$. We have shown that K is a compact g-minimal set.

g restricted to $\Omega(g)$ has the pseudo-orbit-tracing-property and K is a connected open and closed g-invariant subset of $\Omega(g)$ because $\Omega(g) \subset \Omega(f)$. Hence g restricted to K must have the pseudo-orbit-tracing-property. This implies, in view of Proposition 1.2, K must be singleton. Let $K = \{w\}$. Then $g(w) = w$ implies $f^{n_i}(w) = w$ and therefore $\omega_f(x) = \Omega_i$ consists of only one periodic orbit $\{w, f(w), f^2(w), \cdots, f^{n_i-1}(w)\}$. This completes the proof of this theorem. □

Applying the above theorem to the inverse of f we get immediately

Corollary 2.3. *If x is an wandering point of f with int $W^u(x, f) \neq \emptyset$, then $\alpha_f(x)$ consists of one periodic orbit.*

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